Fusion products of Kirillov-Reshetikhin modules and the $X = M$ conjecture

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Abstract

In this article, we show in the ADE case that the fusion product of Kirillov-Reshetikhin modules for a current algebra, whose character is expressed in terms of fermionic forms, can be constructed from one-dimensional modules by using Joseph functors. As a consequence, we obtain some identity between fermionic forms and Demazure operators. Since the same identity is also known to hold for one-dimensional sums of nonexceptional type, we can show from these results the $X = M$ conjecture for type $A_n^{(1)}$ and $D_n^{(1)}$.

1 Introduction

Let $\mathfrak{g}$ be an affine Kac-Moody Lie algebra with index set I, and $U_q'(\mathfrak{g})$ the corresponding quantum affine algebra without the degree operator. In [13, 12], it was conjectured that a certain subfamily of finite-dimensional $U_q'(\mathfrak{g})$-modules known as Kirillov-Reshetikhin (KR for short) modules $W^{r,\ell}$ has crystal bases $B^{r,\ell}$ called KR crystals. Here the index r corresponds to a node of $I_0 = I \setminus \{0\}$ where 0 is the node specified in [16], and ℓ is a positive integer. This conjecture has been confirmed in many cases, in particular when $\mathfrak{g}$ is nonexceptional [26].

Let $B = B^{r_p,\ell_p} \otimes \cdots \otimes B^{r_1,\ell_1}$ be a tensor product of KR crystals. In [13, 12], the authors defined the one-dimensional sum

$$X(B,\mu,q) = \sum_{\substack{b \in B \\ \widetilde{e}_i b = 0 \ (i \in I_0) \\ \mathrm{wt}(b) = \mu}} q^{D(b)} \in \mathbb{Z}[q,q^{-1}],$$

where μ is a dominant integral weight of the simple Lie subalgebra $\mathfrak{g}_0 \subseteq \mathfrak{g}$ whose Dynkin nodes are I_0, D is a certain $\mathbb{Z}$-function on B called the energy function, and $\widetilde{e}_i$ are Kashiwara operators. Then they conjectured that $X(B,\mu,q)$ has an explicit expression $M(\boldsymbol{\nu},\mu,q)$ called the fermionic form, where $\boldsymbol{\nu}$ is the sequence of elements of $I_0 \times \mathbb{Z}_{>0}$ corresponding to B. This conjecture is called the $X = M$ conjecture, which was confirmed in many instances [18, 28, 29], but a proof in full generality has not been available except for the type $\mathfrak{g} = A_n^{(1)}$ in [18]. It should be mentioned that this conjecture was recently settled for nonexceptional $\mathfrak{g}$ under the large rank hypothesis by combining the results in [18], [21] and [25], and for general $\mathfrak{g}$ if $\ell_1 = \cdots = \ell_p = 1$ [23].

Recently, an identity was proved in [24] which connects one-dimensional sums of nonexceptional type with Demazure operators. Let us recall the result briefly. For simplicity, we assume $\mathfrak{g}$ is of type $A_n^{(1)}$ or $D_n^{(1)}$ here. Denote by P

the weight lattice of $\mathfrak{g}$ and by D_i for $i \in I$ the Demazure operator on the group ring $\mathbb{Z}[P]$ defined by

$$D_i(f) = \frac{f - e^{-\alpha_i} \cdot s_i(f)}{1 - e^{-\alpha_i}},$$

where the simple reflection s_i acts on $\mathbb{Z}[P]$ by $s_i(e^\lambda) = e^{s_i(\lambda)}$. Let W be the Weyl group of $\mathfrak{g}$, $\widetilde{W}$ the extended affine Weyl group, and Σ the subgroup of the group of Dynkin automorphisms such that $\widetilde{W} \cong W \rtimes \Sigma$ (for the precise definition, see Section 3). Then for each $\tau \in \Sigma$ and $w \in W$ with a reduced expression $w = s_{i_k} \cdots s_{i_1}$, $D_{w\tau}$ is defined by $D_{w\tau} = D_{i_k} \cdots D_{i_1} \circ \tau$ (this definition does not depend on the choice of the expression). Let now $B = B^{r_p,\ell_p} \otimes \cdots \otimes B^{r_1,\ell_1}$ be a tensor product of KR crystals such that $\ell_1 \le \cdots \le \ell_p$, P_0^+ the set of dominant integral weights of $\mathfrak{g}_0$, and $V_{\mathfrak{g}_0}(\mu)$ the irreducible $\mathfrak{g}_0$-module with highest weight $\mu \in P_0^+$. Then the following identity was proved in [24], where C is some constant and we set $q = e^{-\delta}$:

$$\begin{aligned} &q^C e^{\ell_p \Lambda_0} \sum_{\mu \in P_0^+} X(B,\mu,q) \operatorname{ch} V_{\mathfrak{g}_0}(\mu) \qquad (1.1)\\ &\quad = D_{t_{w_0(\varpi_{r_p})}} \Big(e^{(\ell_p - \ell_{p-1})\Lambda_0} \cdots D_{t_{w_0(\varpi_{r_2})}} \big(e^{(\ell_2 - \ell_1)\Lambda_0} \cdot D_{t_{w_0(\varpi_{r_1})}} (e^{\ell_1 \Lambda_0}) \big) \cdots \Big). \end{aligned}$$

Here δ is the null root, Λ_0 is the fundamental weight of $\mathfrak{g}$ associated with the node 0, ϖ_i are the fundamental weights of $\mathfrak{g}_0$, w_0 is the longest element of the Weyl group of $\mathfrak{g}_0$, and $t_{w_0(\varpi_{r_j})} \in \widetilde{W}$ are the translations.

The goal of this article is to show the identity (1.1) with the one-dimensional sums replaced by the corresponding fermionic forms. Namely, we will prove the following result as a corollary of Theorem 1.2 stated below (Corollary 7.3):

Corollary 1.1. *Assume that $\mathfrak{g}$ is of nontwisted type and $\mathfrak{g}_0$ is of ADE type. Let $\boldsymbol{\nu} = \big((r_1,\ell_1),\dots,(r_p,\ell_p)\big)$ be a sequence of elements of $I_0 \times \mathbb{Z}_{>0}$ such that $\ell_1 \le \cdots \le \ell_p$. Then we have*

$$\begin{aligned} &q^C e^{\ell_p \Lambda_0} \sum_{\mu \in P_0^+} M(\boldsymbol{\nu},\mu,q) \operatorname{ch} V_{\mathfrak{g}_0}(\mu)\\ &\quad = D_{t_{w_0(\varpi_{r_p})}} \Big(e^{(\ell_p - \ell_{p-1})\Lambda_0} \cdots D_{t_{w_0(\varpi_{r_2})}} \big(e^{(\ell_2 - \ell_1)\Lambda_0} \cdot D_{t_{w_0(\varpi_{r_1})}} (e^{\ell_1 \Lambda_0}) \big) \cdots \Big) \end{aligned}$$

with some constant C, where we set $q = e^{-\delta}$.

Then as an immediate consequence of (1.1) and this corollary, the $X = M$ conjecture for $A_n^{(1)}$ and $D_n^{(1)}$ is settled (Theorem 7.5).

Our strategy of the proof of Corollary 1.1 is to show the isomorphism between two modules whose characters are equal to the left hand side and the right hand side respectively. Let us describe what these modules are. The module corresponding to the left hand side is the fusion product of Kirillov-Reshetikhin (KR) modules for the current algebra $\mathfrak{g}_0 \otimes \mathbb{C}[t]$. KR modules for $\mathfrak{g}_0 \otimes \mathbb{C}[t]$, which we denote by $KR^{r,\ell}$ ($r \in I_0$, $\ell \in \mathbb{Z}_{>0}$) in this article, were defined in [2, 4] in terms of generators and relations. They can also be obtained from $W^{r,\ell}$ (KR modules for $U_q'(\mathfrak{g})$) by specialization and restriction, which is why they are so named. The fusion product is a refinement of the usual tensor product defined in [7], which constructs a cyclic graded $(\mathfrak{g}_0 \otimes \mathbb{C}[t])$-module from some

such modules. It was proved by Di Francesco and Kedem in [5] that the fusion product of KR modules has the required (graded) character. Namely, they proved the following character formula for the fusion product of KR modules: for a sequence $\boldsymbol{\nu} = \bigl((r_1, \ell_1), \dots, (r_p, \ell_p)\bigr)$ of elements of $I_0 \times \mathbb{Z}_{>0}$, we have

$$\operatorname{ch} KR^{r_p,\ell_p} * \cdots * KR^{r_1,\ell_1} = \sum_{\mu \in P_0^+} M(\boldsymbol{\nu}, \mu, q) \operatorname{ch} V_{\mathfrak{g}_0}(\mu), \tag{1.2}$$

where $*$ denotes the fusion product.

To present the module corresponding to the right hand side, we recall the definition of Joseph functors introduced by Joseph [14]. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$, and $\mathfrak{p}_i$ for $i \in I$ the parabolic subalgebra associated with the set $\{i\}$ containing $\mathfrak{b}$. We denote by $\mathfrak{b}$-Mod (resp. $\mathfrak{p}_i$-Mod) the category of finite-dimensional $\mathfrak{h}$-semisimple left $\mathfrak{b}$-modules (resp. $\mathfrak{p}_i$-modules). Then the functor $\mathcal{D}_i \colon \mathfrak{b}\text{-Mod} \to \mathfrak{p}_i\text{-Mod}$ is defined as the left adjoint functor of the restriction functor $\mathfrak{p}_i$-Mod $\to \mathfrak{b}$-Mod. For $w \in W$ with a reduced expression $w = s_{i_k} \cdots s_{i_1}$, the Joseph functor $\mathcal{D}_w$ is defined by

$$\mathcal{D}_w = \mathcal{D}_{i_k} \cdots \mathcal{D}_{i_1} \colon \mathfrak{b}\text{-Mod} \to \mathfrak{b}\text{-Mod},$$

which does not depend on the choice of the expression. We also define $\mathcal{D}_{w\tau}$ for $w \in W$ and $\tau \in \Sigma$ by $\mathcal{D}_{w\tau} = \mathcal{D}_w \circ (\tau^{-1})^*$, where $(\tau^{-1})^*$ denotes the pull-back functor associated with the Lie algebra automorphism τ^{-1} of $\mathfrak{b}$. Then the $\mathfrak{b}$-module corresponding to the right hand side of Corollary 1.1 is constructed using these functors as follows:

$$\mathcal{D}_{t_{w_0(\varpi_{r_p})}}\Bigl(\mathbb{C}_{(\ell_p-\ell_{p-1})\Lambda_0} \otimes \cdots \otimes \mathcal{D}_{t_{w_0(\varpi_{r_2})}}\bigl(\mathbb{C}_{(\ell_2-\ell_1)\Lambda_0} \otimes \mathcal{D}_{t_{w_0(\varpi_{r_1})}} \mathbb{C}_{\ell_1\Lambda_0}\bigr)\cdots\Bigr).$$

From the results of [14] and [19], we can easily see that $\operatorname{ch} \mathcal{D}_w M = D_w \operatorname{ch} M$ holds for $w \in \widetilde{W}$ if a $\mathfrak{b}$-module M has a Demazure flag (see Definition 2.2). It is checked from this fact that the above module in fact has the required character.

Now, what we have to prove is the following isomorphism, which is our main theorem in this article (Theorem 6.1):

Theorem 1.2. *Assume that $\mathfrak{g}$ is of nontwisted type and $\mathfrak{g}_0$ is of type ADE. Let $KR^{r_1,\ell_1}, \dots, KR^{r_p,\ell_p}$ be a sequence of KR modules such that $\ell_1 \le \cdots \le \ell_p$. Then there exists an isomorphism of $\mathfrak{b}$-modules*

$$\begin{aligned}&\mathbb{C}_{\ell_p\Lambda_0 + C\delta} \otimes \bigl(KR^{r_p,\ell_p} * \cdots * KR^{r_2,\ell_2} * KR^{r_1,\ell_1}\bigr)\\ &\quad\cong \mathcal{D}_{t_{w_0(\varpi_{r_p})}}\Bigl(\mathbb{C}_{(\ell_p-\ell_{p-1})\Lambda_0} \otimes \cdots \otimes \mathcal{D}_{t_{w_0(\varpi_{r_2})}}\bigl(\mathbb{C}_{(\ell_2-\ell_1)\Lambda_0} \otimes \mathcal{D}_{t_{w_0(\varpi_{r_1})}} \mathbb{C}_{\ell_1\Lambda_0}\bigr)\cdots\Bigr)\end{aligned}$$

with some constant C, where the left hand side is naturally regarded as a $\mathfrak{b}$-module.

As explained above, Corollary 1.1 follows as a direct consequence of this theorem. It should be remarked that this theorem for $\mathfrak{g} = A_1^{(1)}$ was already proved by Feigin and Loktev in [7]. In the case $\ell_1 = \cdots = \ell_p$, this was proved by Fourier and Littelmann in [9].

The plan of this article is as follows. In Section 2, we review the results on Demazure modules and Joseph functors. In Section 3, we prepare some notation

and elementary lemmas concerning a nontwisted affine Lie algebra. In Section 4, we recall the definition of KR modules for a current algebra. For the later convenience, we define them as $\mathfrak{p}_{I_0}$-modules in this article, where $\mathfrak{p}_{I_0}$ denotes the parabolic subalgebra associated with I_0. In Section 5, we recall the definition of fusion products. Then we state our main theorem in Section 6, and explain in Section 7 how the $X = M$ conjecture for type $A_n^{(1)}$ and $D_n^{(1)}$ is deduced from this theorem. The last two sections are devoted to prove the main theorem. In Section 8, we introduce the $\mathfrak{b}$-fusion product, which is some modified version of the fusion product constructing a $\mathfrak{b}$-module from some $\mathfrak{b}$-modules. Then in Section 9, we give the proof of the main theorem using $\mathfrak{b}$-fusion products.

Acknowledgments: The author would like to express his gratitude to R. Kodera, S. Naito and Y. Saito for a lot of helpful discussions and comments.

2 Demazure modules and Joseph functors

Let $\mathfrak{g}$ be a complex symmetrizable Kac-Moody Lie algebra with index set I and Chevalley generators $\{e_i, f_i \mid i \in I\}$, $\mathfrak{b} \subseteq \mathfrak{g}$ its Borel subalgebra, and $\mathfrak{h} \subseteq \mathfrak{b}$ its Cartan subalgebra. Let $\alpha_i \in \mathfrak{h}^*$ $(i \in I)$ be the simple roots, $\alpha_i^\vee \in \mathfrak{h}$ $(i \in I)$ the simple coroots, and W the Weyl group of $\mathfrak{g}$ with simple reflections s_i $(i \in I)$. Denote by

$$P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for } i \in I\} \text{ and}$$
$$P^+ = \{\lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for } i \in I\}$$

the weight lattice and the set of dominant integral weights respectively. For a $\mathfrak{h}$-module M, we denote by M_λ for $\lambda \in \mathfrak{h}^*$ the weight space

$$M_\lambda = \{v \in M \mid hv = \langle \lambda, h \rangle v \text{ for } h \in \mathfrak{h}\}.$$

We call a vector v of M a *weight vector* if $v \in M_\lambda$ for some $\lambda \in \mathfrak{h}^*$.

Denote by $V(\lambda)$ the irreducible highest weight $\mathfrak{g}$-module with highest weight $\lambda \in P^+$. For $w \in W$, let $u_{w\lambda}$ be a nonzero vector of the one-dimensional weight space $V(\lambda)_{w\lambda}$.

Definition 2.1. For $\lambda \in P^+$ and $w \in W$, the $\mathfrak{b}$-submodule

$$V_w(\lambda) = U(\mathfrak{b})u_{w\lambda} \subseteq V(\lambda)$$

is called the *Demazure module* associated with λ and w.

For a subset $J \subseteq I$, we denote by $\mathfrak{p}_J$ the parabolic subalgebra associated with J, which is the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{b}$ and $\{f_i \mid i \in J\}$. Note that $\mathfrak{p}_\emptyset = \mathfrak{b}$. If $J = \{i\}$, we denote $\mathfrak{p}_J$ by $\mathfrak{p}_i$. In this article, we denote by $\mathfrak{p}_J$-Mod the category of finite-dimensional $\mathfrak{h}$-semisimple left $\mathfrak{p}_J$-modules. If $\langle w\lambda, \alpha_i^\vee \rangle \leq 0$, then the Demazure module $V_w(\lambda)$ is preserved by the action of f_i. Hence $V_w(\lambda)$ belongs to $\mathfrak{p}_J$-Mod if $\langle w\lambda, \alpha_i^\vee \rangle \leq 0$ holds for all $i \in J$.

Definition 2.2. Let $M \in \mathfrak{b}$-Mod. A filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M$ is called a *Demazure flag* of M if every successive quotient M_i/M_{i-1} is isomorphic to some Demazure module.

For every $i \in I$, it is known that the restriction functor $\mathfrak{p}_i$-Mod $\to$ $\mathfrak{b}$-Mod has the left adjoint functor $\mathcal{D}_i \colon \mathfrak{b}$-Mod $\to \mathfrak{p}_i$-Mod [14, 19]. We often regard $\mathcal{D}_i$ as a functor from $\mathfrak{b}$-Mod to $\mathfrak{b}$-Mod in the obvious way.

Proposition 2.3 ([19, Remark 8.1.18])**.** *Let $w \in W$, and $w = s_{i_k} \cdots s_{i_1}$ be its arbitrary reduced expression. Then the functor $\mathcal{D}_w = \mathcal{D}_{i_k} \cdots \mathcal{D}_{i_1} \colon \mathfrak{b}$-Mod $\to$ $\mathfrak{b}$-Mod does not depend on the choice of the expression.*

Definition 2.4. The functor $\mathcal{D}_w \colon \mathfrak{b}$-Mod $\to$ $\mathfrak{b}$-Mod is called the *Joseph functor* associated with $w \in W$.

Note that for every $i \in I$ and $M \in \mathfrak{b}$-Mod, the canonical $\mathfrak{b}$-module homomorphism $M \to \mathcal{D}_i M$ is defined as the image of the identity under the bijection $\mathrm{Hom}_{\mathfrak{p}_i\text{-Mod}}(\mathcal{D}_i M, \mathcal{D}_i M) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{b}\text{-Mod}}(M, \mathcal{D}_i M)$. The following lemma was proved in [14, Subsection 2.7 and Lemma 2.8 (iv)] for finite-dimensional $\mathfrak{g}$, and the proof goes without any change for general $\mathfrak{g}$:

Lemma 2.5. *Let $i \in I$.*
(i) *For every $M \in \mathfrak{p}_i$-Mod, the canonical $\mathfrak{b}$-module homomorphism $M \to \mathcal{D}_i M$ is an isomorphism. In particular, we have $\mathcal{D}_i^2 N \cong \mathcal{D}_i N$ for every $N \in \mathfrak{b}$-Mod.*
(ii) *Assume that $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of objects of $\mathfrak{b}$-Mod and M_3 is isomorphic to some $\mathfrak{b}$-submodule of a finite-dimensional $\mathfrak{p}_i$-module. Then the sequence $0 \to \mathcal{D}_i M_1 \to \mathcal{D}_i M_2 \to \mathcal{D}_i M_3 \to 0$ is exact.*

For $\lambda \in \mathfrak{h}^*$, we denote by $\mathbb{C}_\lambda$ the one-dimensional $\mathfrak{b}$-module spanned by a weight vector with weight λ on which e_i $(i \in I)$ acts trivially. Note that we have $V_{\mathrm{id}}(\lambda) = \mathbb{C}_\lambda$ for $\lambda \in P^+$. Now we recall the following theorem:

Theorem 2.6 ([19, Proposition 8.1.17 and Corollary 8.1.26])**.** *For every $\lambda \in P^+$ and $w \in W$, we have*

$$\mathcal{D}_w \mathbb{C}_\lambda \cong V_w(\lambda)$$

as $\mathfrak{b}$-modules.

Corollary 2.7. *Assume that $M \in \mathfrak{b}$-Mod has a Demazure flag. Then $\mathcal{D}_w M$ has a Demazure flag for every $w \in W$.*

Proof. It is enough to show the assertion for $w = s_i$. Let $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M$ be a Demazure flag of M. We show the assertion by the induction on k. For a Demazure module $V_w(\lambda)$, we have from Lemma 2.5 (i) and Theorem 2.6 that

$$\mathcal{D}_i V_w(\lambda) \cong \begin{cases} V_{s_i w}(\lambda) & \text{if } \ell(s_i w) = \ell(w) + 1, \\ V_w(\lambda) & \text{if } \ell(s_i w) = \ell(w) - 1, \end{cases} \tag{2.1}$$

where ℓ denotes the length function. Hence the assertion for $k = 1$ follows. Assume $k > 1$ and write $M_k / M_{k-1} \cong V_{w_k}(\lambda_k)$. Recall that $V_{w_k}(\lambda_k)$ is defined as a $\mathfrak{b}$-submodule of $V(\lambda_k)$. Hence this module is also a $\mathfrak{b}$-submodule of the $\mathfrak{p}_i$-module $U(\mathfrak{p}_i) V_{w_k}(\lambda_k) \subseteq V(\lambda_k)$, which is finite-dimensional since $V(\lambda_k)$ is integrable. Hence the sequence $0 \to \mathcal{D}_i M_{k-1} \to \mathcal{D}_i M \to \mathcal{D}_i V_{w_k}(\lambda_k) \to 0$ is exact by Lemma 2.5 (ii), and then the assertion follows from the induction hypothesis. □

Let $\mathbb{Z}[P]$ denote the group ring of P with basis e^λ ($\lambda \in P$), and define for $i \in I$ a linear operator D_i on $\mathbb{Z}[P]$ by

$$D_i(f) = \frac{f - e^{-\alpha_i} \cdot s_i(f)}{1 - e^{-\alpha_i}},$$

where s_i acts on $\mathbb{Z}[P]$ by $s_i(e^\lambda) = e^{s_i(\lambda)}$. We call D_i the *Demazure operator* associated with i. The following lemma is proved by elementary calculations such as $s_i(fg) = s_i(f)s_i(g)$.

Lemma 2.8 ([8, Lemma 2])**.** *Assume that $f \in \mathbb{Z}[P]$ is s_i-invariant. Then we have*

$$D_i(fg) = fD_i(g) \quad \text{for every } g \in \mathbb{Z}[P].$$

In particular, we have $D_i^2 = D_i$.

For a finite-dimensional semisimple $\mathfrak{h}$-module M such that $\{\lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0\} \subseteq P$, we denote the character of M by

$$\mathrm{ch}\, M = \sum_{\lambda \in P} \dim M_\lambda \cdot e^\lambda \in \mathbb{Z}[P].$$

For every reduced expression $w = s_{i_k} \cdots s_{i_1}$ of $w \in W$, the operator $D_w = D_{i_k} \cdots D_{i_1}$ on $\mathbb{Z}[P]$ is independent of the choice of the expression [19, Corollary 8.2.10], and it is known that the character of a Demazure module is expressed as follows:

Theorem 2.9 ([19, Theorem 8.2.9])**.** *For every Demazure module $V_w(\lambda)$, we have*

$$\mathrm{ch}\, V_w(\lambda) = D_w(e^\lambda).$$

Corollary 2.10. *Assume that $M \in \mathfrak{b}$-Mod has a Demazure flag. Then for every $w \in W$, we have*

$$\mathrm{ch}\, \mathcal{D}_w M = D_w \mathrm{ch}\, M.$$

Proof. By Corollary 2.7, it is enough to show the assertion for $w = s_i$. Let $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k = M$ be a Demazure flag of M. We show the assertion by the induction on k. Assume $k = 1$, and write $M \cong V_w(\lambda)$. If $\ell(s_i w) = \ell(w) + 1$, we have

$$\mathrm{ch}\, \mathcal{D}_i V_w(\lambda) = \mathrm{ch}\, V_{s_i w}(\lambda) = D_{s_i w}(e^\lambda) = D_i \mathrm{ch}\, V_w(\lambda) \tag{2.2}$$

by Theorem 2.6 and Theorem 2.9. If $\ell(s_i w) = \ell(w) - 1$, we have $\mathrm{ch}\, \mathcal{D}_i V_w(\lambda) = \mathrm{ch}\, V_w(\lambda)$ by (2.1), and

$$D_i \mathrm{ch}\, V_w(\lambda) = D_i^2 \mathrm{ch}\, V_{s_i w}(\lambda) = D_i \mathrm{ch}\, V_{s_i w}(\lambda) = \mathrm{ch}\, V_w(\lambda)$$

by (2.2) and Lemma 2.8. Hence the assertion for $k = 1$ follows. Assume $k > 1$ and write $M_k / M_{k-1} \cong V_{w_k}(\lambda_k)$. Then as proved in the proof of Corollary 2.7, the sequence $0 \to \mathcal{D}_i M_{k-1} \to \mathcal{D}_i M \to \mathcal{D}_i V_{w_k}(\lambda_k) \to 0$ is exact. Hence the assertion follows from the induction hypothesis. □

The following theorem is obtained by taking the classical limit of [15, Theorem 5.22]:

Theorem 2.11. *Assume that $\mathfrak{g}$ is symmetric (i.e. the Cartan matrix of $\mathfrak{g}$ is symmetric). Then for every $\lambda, \mu \in P^+$ and $w \in W$, the $\mathfrak{b}$-module $\mathbb{C}_\lambda \otimes V_w(\mu)$ has a Demazure flag.*

Remark 2.12. In [15, Theorem 5.22], a given Kac-Moody Lie algebra is assumed to be simply laced. This assumption, however, is used only in [15, Lemma 3.14] to apply a positivity result of Lusztig, and we can check that the proof of this positivity result in [22, §22.1.7] goes without any change for every symmetric Kac-Moody Lie algebra. Hence [15, Theorem 5.22] holds for symmetric Kac-Moody Lie algebras, and so does the above theorem.

From the theorem, we have the following corollary since the functor $\mathbb{C}_\lambda \otimes -$ is exact:

Corollary 2.13. *Assume that $\mathfrak{g}$ is symmetric. If $M \in \mathfrak{b}$-Mod has a Demazure flag, then $\mathbb{C}_\lambda \otimes M$ for $\lambda \in P^+$ has a Demazure flag.*

3 Nontwisted affine Lie algebra

From this section, we assume that $\mathfrak{g}$ is a nontwisted affine Lie algebra with $I = \{0, 1, \dots, n\}$ unless stated otherwise. Let Γ be the Dynkin diagram of $\mathfrak{g}$, $A = (a_{ij})_{i,j \in I}$ the Cartan matrix of $\mathfrak{g}$, $\Delta \subseteq \mathfrak{h}^*$ the root system of $\mathfrak{g}$, and $\Delta^+ \subseteq \Delta$ the set of positive roots corresponding to $\mathfrak{b}$. In this article, we use the Kac's labeling of nodes of Γ in [16, Section 4.8]. Let $(a_0, \dots, a_n)$ (resp. $(a_0^\vee, \dots, a_n^\vee)$) be the unique sequence of relatively prime positive integers satisfying

$$\sum_{j \in I} a_{ij} a_j = 0 \quad \text{for all } i \in I \ (\text{resp.} \sum_{i \in I} a_i^\vee a_{ij} = 0 \quad \text{for all } j \in I).$$

Let $d \in \mathfrak{h}$ be the degree operator, which is any element satisfying $\langle \alpha_i, d \rangle = \delta_{0i}$ for $i \in I$, $K = \sum_{i \in I} a_i^\vee \alpha_i^\vee \in \mathfrak{h}$ the canonical central element, and $\delta = \sum_{i \in I} a_i \alpha_i \in \mathfrak{h}^*$ the null root. For each $i \in I$, let $\Lambda_i \in P^+$ be the fundamental weight, which satisfies

$$\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij} \quad \text{for } j \in I \quad \text{and} \quad \langle \Lambda_i, d \rangle = 0.$$

Note that we have

$$P = \sum_{i \in I} \mathbb{Z} \Lambda_i + \mathbb{C}\delta \quad \text{and} \quad P^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \Lambda_i + \mathbb{C}\delta.$$

Let $(\ ,\ )$ be the W-invariant symmetric bilinear form on $\mathfrak{h}^*$ defined by

$$(\alpha_i, \alpha_j) = a_i^\vee a_i^{-1} a_{ij}, \quad (\alpha_i, \Lambda_0) = \delta_{0i} \quad \text{for } i, j \in I \quad \text{and} \quad (\Lambda_0, \Lambda_0) = 0.$$

We denote by $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) the Lie subalgebra of $\mathfrak{g}$ generated by $\{e_i \mid i \in I\}$ (resp. $\{f_i \mid i \in I\}$), and by $\mathfrak{g}_\alpha$ for $\alpha \in \Delta$ the root space of $\mathfrak{g}$.

Let $I_0 = I \setminus \{0\}$, and $\mathfrak{g}_0 \subseteq \mathfrak{g}$ be the simple Lie subalgebra generated by $\{e_i, f_i \mid i \in I_0\}$ with Cartan subalgebra $\mathfrak{h}_0 \subseteq \mathfrak{h}$ and Weyl group $W_0 \subseteq W$. Let $\Delta_0 \subseteq \Delta$ be the root system of $\mathfrak{g}_0$, $\Delta_0^+ = \Delta_0 \cap \Delta^+$ the set of positive roots, $P_0 \subseteq \mathfrak{h}_0^*$ the weight lattice of $\mathfrak{g}_0$, $P_0^+ \subseteq P_0$ the set of dominant integral weights, $Q_0 \subseteq P_0$ the root lattice of $\mathfrak{g}_0$, and $Q_0^+ = \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \alpha_i$. Denote by $\varpi_i \in \mathfrak{h}_0^*$ and $\varpi_i^\vee \in \mathfrak{h}_0$ $(i \in I_0)$ the fundamental weights and the fundamental coweights of $\mathfrak{g}_0$

respectively. For the notational convenience, we set $\varpi_0 = 0$ and $\varpi_0^\vee = 0$. We often regard $\mathfrak{h}_0^*$ as a subspace of $\mathfrak{h}^*$ by setting $\langle \mathfrak{h}_0^*, K\rangle = \langle \mathfrak{h}_0^*, d\rangle = 0$. Then we have $\mathfrak{h}^* = \mathfrak{h}_0^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$. We denote by $\theta = \sum_{i\in I_0} a_i\alpha_i$ the highest root of $\mathfrak{g}_0$, and by w_0 the longest element of W_0. Let $\mathfrak{n}_0^+$ (resp. $\mathfrak{n}_0^-$) be the Lie subalgebra of $\mathfrak{g}_0$ generated by $\{e_i \mid i \in I_0\}$ (resp. $\{f_i \mid i \in I_0\}$), and $\mathfrak{b}_0 = \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$. Set $e_{\alpha_i} = e_i$ and $e_{-\alpha_i} = f_i$ for $i \in I_0$, and for each $\alpha \in \Delta_0^+ \setminus \{\alpha_i \mid i \in I_0\}$ fix nonzero vectors $e_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$, $\alpha^\vee \in \mathfrak{h}_0$ so that

$$[e_\alpha, e_{-\alpha}] = \alpha^\vee, \quad [\alpha^\vee, e_{\pm\alpha}] = \pm 2e_{\pm\alpha}.$$

Recall that there exists a unique Lie algebra isomorphism

$$\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

satisfying

$$e_i \mapsto e_i \otimes 1,\ f_i \mapsto f_i \otimes 1\ (i \in I_0),\ e_0 \mapsto e_{-\theta} \otimes t,\ f_0 \mapsto e_\theta \otimes t^{-1},\ d \mapsto d.$$

The Lie algebra structure of $\mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$ is defined by

$$\begin{aligned}[x \otimes t^m + a_1K + b_1d, y \otimes t^n + a_2K + b_2d]& \\ = [x, y] \otimes t^{m+n} + nb_1y \otimes t^n - mb_2x \otimes t^m + m\delta_{m,-n}(x, y)K.&\end{aligned}$$

In the sequel, we always identify these two Lie algebras via the above isomorphism. It should be noted that we have

$$\mathfrak{p}_{I_0} = \mathfrak{g}_0 \otimes \mathbb{C}[t] \oplus \mathbb{C}K \oplus \mathbb{C}d.$$

Set $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$,

$$\mathfrak{p}'_J = \mathfrak{p}_J \cap \mathfrak{g}' \ \text{ for } J \subseteq I, \quad \text{and} \quad \mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}'.$$

Note that we have $\mathfrak{p}_J = \mathfrak{p}'_J \oplus \mathbb{C}d$ and $\mathfrak{h}' = \mathfrak{h}_0 \oplus \mathbb{C}K$. Let $\mathrm{cl}\colon \mathfrak{h}^* \to (\mathfrak{h}')^* = \mathfrak{h}^*/\mathbb{C}\delta$ denote the canonical projection, and set $P_{\mathrm{cl}} = \mathrm{cl}(P)$. Since W fixes δ, W acts on $\mathfrak{h}^*/\mathbb{C}\delta$ and P_{cl}. For $\ell \in \mathbb{Z}$, we denote by P_{cl}^ℓ the subset $\{\lambda \in P_{\mathrm{cl}} \mid \langle \lambda, K\rangle = \ell\}$ of P_{cl}.

As [16, (6.5.2)], we define for $\lambda \in P_0$ an endomorphism t_λ of $\mathfrak{h}^*$ by

$$t_\lambda(\mu) = \mu + \langle \mu, K\rangle\lambda - \big((\mu, \lambda) + \frac{1}{2}(\lambda, \lambda)\langle \mu, K\rangle\big)\delta. \tag{3.1}$$

The map $\lambda \mapsto t_\lambda$ defines an injective group homomorphism from P_0 to the group of linear automorphisms of $\mathfrak{h}^*$ orthogonal with respect to $(\ ,\ )$. Let $c_i = a_i/a_i^\vee$ for $i \in I_0$, and define the sublattices M and $\widetilde{M}$ of P_0 by

$$M = \sum_{w\in W_0} \mathbb{Z}w(\theta), \quad \widetilde{M} = \bigoplus_{i\in I_0} \mathbb{Z}c_i\varpi_i.$$

Let $T(M)$ and $T(\widetilde{M})$ be the subgroups of $\mathrm{GL}(\mathfrak{h}^*)$ defined by

$$T(M) = \{t_\lambda \mid \lambda \in M\}, \quad T(\widetilde{M}) = \{t_\lambda \mid \lambda \in \widetilde{M}\}.$$

It is known that $W \cong W_0 \ltimes T(M)$ [16, Proposition 6.5]. Define the subgroup $\widetilde{W}$ of $\mathrm{GL}(\mathfrak{h}^*)$ by

$$\widetilde{W} = W_0 \ltimes T(\widetilde{M}),$$

which is called the *extended affine Weyl group*. The action of $\widetilde{W}$ preserves Δ, and $w \in W_0$ and $\lambda \in \widetilde{M}$ satisfy

$$wt_\lambda w^{-1} = t_{w(\lambda)}.$$

By $\mathrm{Aut}\,(\Gamma)$ we denote the group of automorphisms of the Dynkin diagram Γ, that is, the group of permutations τ of I satisfying $a_{ij} = a_{\tau(i)\tau(j)}$ for all $i, j \in I$. Let $\mathfrak{h}^*_{\mathbb{R}} = \sum_{i\in I} \mathbb{R}\alpha_i + \mathbb{R}\Lambda_0 \subseteq \mathfrak{h}^*$, $C = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid (\lambda, \alpha_i) \geq 0 \text{ for all } i \in I\}$ be the fundamental chamber, and $\Sigma \subseteq \widetilde{W}$ the subgroup consisting of elements preserving C. Then we have

$$\widetilde{W} \cong W \rtimes \Sigma.$$

Since $\tau \in \Sigma$ preserves the set of simple roots, τ induces a permutation of I (also denoted by τ) by $\tau(\alpha_i) = \alpha_{\tau(i)}$, which belongs to $\mathrm{Aut}\,(\Gamma)$ since $(\ ,\ )$ is τ-invariant. By abuse of notation, we denote by Σ both the subgroups of $\widetilde{W}$ and $\mathrm{Aut}\,(\Gamma)$.

Lemma 3.1. *Let τ be an arbitrary element of Σ, and $\bar{\tau}$ the unique element of W_0 such that $\tau \in \bar{\tau} \cdot T(\widetilde{M})$. Then we have*

$$\tau(\lambda + a\delta) = \bar{\tau}(\lambda) + \big(a + \langle \lambda, \varpi^\vee_{\tau^{-1}(0)}\rangle\big)\delta \quad \textit{for } \lambda \in P_0 \textit{ and } a \in \mathbb{C}.$$

Proof. As $t_\mu(\alpha_i) \equiv \alpha_i \bmod \mathbb{Z}\delta$ for every $i \in I$ and $\mu \in \widetilde{M}$, we have $\tau(\alpha_i) \equiv \bar{\tau}(\alpha_i) \bmod \mathbb{Z}\delta$. This forces $\tau(\alpha_i) = \bar{\tau}(\alpha_i) + \delta_{\tau(i),0}\delta$ for $i \in I_0$ since τ preserves $\{\alpha_0, \ldots, \alpha_n\}$ and $\bar{\tau} \in W_0$. Now the assertion follows since

$$\begin{aligned}
\tau(\lambda + a\delta) &= \sum_{i\in I_0} \langle \lambda, \varpi_i^\vee\rangle \tau(\alpha_i) + a\delta \\
&= \sum_{i\in I_0} \langle \lambda, \varpi_i^\vee\rangle \bar{\tau}(\alpha_i) + \big(a + \langle \lambda, \varpi^\vee_{\tau^{-1}(0)}\rangle\big)\delta \\
&= \bar{\tau}(\lambda) + \big(a + \langle \lambda, \varpi^\vee_{\tau^{-1}(0)}\rangle\big)\delta.
\end{aligned}$$

□

We define an action of Σ on $\mathfrak{g}$ by letting $\tau \in \Sigma$ act as a Lie algebra automorphism defined by

$$\tau(e_i) = e_{\tau(i)}, \quad \tau(\alpha_i^\vee) = \alpha^\vee_{\tau(i)}, \quad \tau(f_i) = f_{\tau(i)} \quad \text{and} \quad \tau(d) = d + \varpi^\vee_{\tau(0)}.$$

This action obviously preserves $\mathfrak{b}$ and $\mathfrak{b}'$. For a module M, we denote by τ^*M the pull-back of M with respect to τ. The image of $v \in M$ under the canonical linear isomorphism $M \to \tau^*M$ is denoted by τ^*v. Note that this isomorphism maps the weight space M_λ onto $(\tau^*M)_{\tau^{-1}(\lambda)}$.

We prepare some notation here. Let $w \in \widetilde{W}$ be an arbitrary element, and take unique elements $w' \in W$ and $\tau \in \Sigma$ so that $w = w'\tau$. Then define a functor $\mathcal{D}_w \colon \mathfrak{b}\text{-Mod} \to \mathfrak{b}\text{-Mod}$ by

$$\mathcal{D}_w = \mathcal{D}_{w'} \circ \big(\tau^{-1}\big)^*,$$

and a linear operator $D_w \colon \mathbb{Z}[P] \to \mathbb{Z}[P]$ by

$$D_w = D_{w'} \circ \tau,$$

where τ acts on $\mathbb{Z}[P]$ by $\tau(e^\lambda) = e^{\tau(\lambda)}$. We set

$$V_w(\Lambda) = V_{w'}\big(\tau(\Lambda)\big) \quad \text{for } \Lambda \in P^+.$$

Then we have the following lemma:

Lemma 3.2. (i) *Let $\tau \in \Sigma$, $\Lambda \in P^+$ and $w \in W$. Then we have*

$$\big(\tau^{-1}\big)^* V_w(\Lambda) \cong V_{\tau w}(\Lambda).$$

(ii) *Assume $M \in \mathfrak{b}$-Mod has a Demazure flag. Then for every $w \in \widetilde{W}$, $\mathcal{D}_w M$ has a Demazure flag and we have*

$$\operatorname{ch} \mathcal{D}_w M = D_w \operatorname{ch} M.$$

Proof. Since $\big(\tau^{-1}\big)^* V(\Lambda)$ is an integrable highest weight $\mathfrak{g}$-module with highest weight $\tau(\Lambda)$, we have $\big(\tau^{-1}\big)^* V(\Lambda) \cong V\big(\tau(\Lambda)\big)$. Moreover, we see that $\big(\tau^{-1}\big)^* V_w(\Lambda)$ is the $\mathfrak{b}$-submodule of $\big(\tau^{-1}\big)^* V(\Lambda)$ generated by the weight space with weight $\tau w(\Lambda)$. Hence the assertion (i) follows. Then, since the functor $\big(\tau^{-1}\big)^*$ is exact, the first assertion of (ii) follows from (i) and Corollary 2.7. Since we have $\operatorname{ch}\big(\tau^{-1}\big)^* M = \tau \operatorname{ch} M$, the second one follows from Corollary 2.10. □

Let $\mathbb{Z}[P_{\mathrm{cl}}]$ denote the group algebra of P_{cl} with basis e^λ ($\lambda \in P_{\mathrm{cl}}$), and by cl we also denote the projection from $\mathbb{Z}[P]$ to $\mathbb{Z}[P_{\mathrm{cl}}]$ defined by $e^\lambda \mapsto e^{\mathrm{cl}(\lambda)}$. For each $w \in W$, a linear operator $\overline{D}_w$ on $\mathbb{Z}[P_{\mathrm{cl}}]$ is defined by

$$\mathrm{cl} \circ D_w = \overline{D}_w \circ \mathrm{cl}.$$

Lemma 3.3 ([8])**.** *Let $f \in \mathbb{Z}[P_{\mathrm{cl}}^0]$, and assume f is W_0-invariant. Then we have*

$$\overline{D}_w(fg) = f\overline{D}_w(g) \quad \textit{for every } g \in \mathbb{Z}[P_{\mathrm{cl}}] \textit{ and } w \in W.$$

Proof. Since $f \in \mathbb{Z}[P_{\mathrm{cl}}^0]$, we have $s_0(f) = s_\theta(f) = f$, where s_θ denotes the reflection associated with θ. Hence f is s_i-invariant for all $i \in I$, and then the assertion is easily proved from Lemma 2.8. □

We need the following elementary lemmas later:

Lemma 3.4. *Let $M \in \mathfrak{b}$-Mod. For every $w \in \widetilde{W}$ and $C \in \mathbb{C}$, we have*

$$\mathcal{D}_w(\mathbb{C}_{C\delta} \otimes M) \cong \mathbb{C}_{C\delta} \otimes \mathcal{D}_w M.$$

Proof. It is enough to show the assertions for $w = \tau \in \Sigma$ and $w = s_i$. The first case is obvious, and the second one follows since we have

$$\begin{aligned} &\operatorname{Hom}_{\mathfrak{b}\text{-Mod}}(\mathbb{C}_{C\delta} \otimes M, M') \cong \operatorname{Hom}_{\mathfrak{b}\text{-Mod}}(M, \mathbb{C}_{-C\delta} \otimes M') \\ &\cong \operatorname{Hom}_{\mathfrak{p}_i\text{-Mod}}(\mathcal{D}_i M, \mathbb{C}_{-C\delta} \otimes M') \cong \operatorname{Hom}_{\mathfrak{p}_i\text{-Mod}}(\mathbb{C}_{C\delta} \otimes \mathcal{D}_i M, M') \end{aligned}$$

for every $M' \in \mathfrak{p}_i$-Mod. □

Lemma 3.5. *Let M_1 be an object of $\mathfrak{b}$-Mod which is generated by a weight vector v_1, M_2 an object of $\mathfrak{b}$-Mod, and Φ a homomorphism of $\mathfrak{b}'$-modules from M_1 to M_2 which maps v_1 to a weight vector v_2.*
(i) *For some $C \in \mathbb{C}$, Φ induces a homomorphism of $\mathfrak{b}$-modules from $\mathbb{C}_{C\delta} \otimes M_1$ to M_2.*
(ii) *Assume further that M_2 extends to an object of $\mathfrak{p}_i$-Mod for some $i \in I$. Then there exists a homomorphism $\widetilde{\Phi}$ of $\mathfrak{p}_i'$-modules from $\mathcal{D}_i M_1$ to M_2 satisfying $\Phi = \widetilde{\Phi} \circ \iota$, where $\iota\colon M_1 \to \mathcal{D}_i M_1$ is the canonical homomorphism of $\mathfrak{b}$-modules.*

Proof. Let $\lambda_1, \lambda_2 \in \mathfrak{h}^*$ be the respective weights of v_1, v_2, and $C = \langle \lambda_2 - \lambda_1, d \rangle$. It is easily seen that the induced map $\Phi'\colon \mathbb{C}_{C\delta} \otimes M_1 \to M_2$ preserves the weights, and hence it is a homomorphism of $\mathfrak{b}$-modules. The assertion (i) is proved. Then under the assumption of (ii), there exists a homomorphism $\widetilde{\Phi}'\colon \mathcal{D}_i(\mathbb{C}_{C\delta} \otimes M_1) \to M_2$ of $\mathfrak{p}_i$-modules such that $\Phi' = \widetilde{\Phi}' \circ \iota$ since $\mathcal{D}_i$ is left adjoint to the restriction functor $\mathfrak{p}_i$-Mod $\to \mathfrak{b}$-Mod. Since $\mathcal{D}_i(\mathbb{C}_{C\delta} \otimes M_1) \cong \mathbb{C}_{C\delta} \otimes \mathcal{D}_i M_1$ holds by Lemma 3.4, required $\widetilde{\Phi}$ is obtained by restricting $\widetilde{\Phi}'$ to $\mathfrak{p}_i'$. □

4 Kirillov-Reshetikhin modules

Following [4], we define the following $\mathfrak{p}_{I_0}$-modules:

Definition 4.1. For $r \in I_0$ and $\ell \in \mathbb{Z}_{>0}$, let $KR^{r,\ell}$ be the $\mathfrak{p}_{I_0}$-module generated by a nonzero vector $v_{r,\ell}$ with relations

$$(\mathfrak{n}_0^+ \otimes \mathbb{C}[t])v_{r,\ell} = 0, \quad (\mathfrak{h}_0 \otimes t\mathbb{C}[t])v_{r,\ell} = 0, \quad hv_{r,\ell} = \langle \ell\varpi_r, h \rangle v_{r,\ell} \ \text{ for } h \in \mathfrak{h},$$
$$f_r^{\ell+1} v_{r,\ell} = (f_r \otimes t)v_{r,\ell} = 0 \quad \text{and} \quad f_i v_{r,\ell} = 0 \ \text{ for } i \in I_0 \setminus \{r\}.$$

We call $KR^{r,\ell}$ the *Kirillov-Reshetikhin module* (KR module for short) for $\mathfrak{p}_{I_0}$ associated with r and ℓ.

Remark 4.2. (i) Kirillov-Reshetikhin modules were originally defined in [4] as $(\mathfrak{g}_0 \otimes \mathbb{C}[t])$-modules. Since we would like to consider $KR^{r,\ell}$ as a $\mathfrak{b}$-module later, we adopt the above definition in this article. It is obvious that the restriction of $KR^{r,\ell}$ to $\mathfrak{g}_0 \otimes \mathbb{C}[t]$ coincides with the original one.
(ii) As the name indicates, $KR^{r,\ell}$ has strong connections with the Kirillov-Reshetikhin module $W^{r,\ell}$ for the quantum affine algebra $U_q'(\mathfrak{g})$ [2, 4, 5]. We return to this topic in Section 7.

It is easily seen that the $\mathfrak{g}_0$-submodule $U(\mathfrak{g}_0)v_{r,\ell} \subseteq KR^{r,\ell}$ is isomorphic to the irreducible module with highest weight $\ell\varpi_r$. For each $w \in W_0 \setminus \{\mathrm{id}\}$, take and fix a nonzero vector $v_{r,\ell}^w$ of this $\mathfrak{g}_0$-submodule whose weight is $w(\ell\varpi_r)$, and set $v_{r,\ell}^{\mathrm{id}} = v_{r,\ell}$.

Lemma 4.3. (i) *Let $w \in W_0$. For $\alpha \in Q_0^+$ and $k \in \mathbb{Z}$, we have*

$$KR^{r,\ell}_{\ell w(\varpi_r) + w(\alpha) + k\delta} = \begin{cases} \mathbb{C}v_{r,\ell}^w & \text{if } \alpha = 0 \text{ and } k = 0, \\ \{0\} & \text{otherwise.} \end{cases}$$

(ii) *As a $\mathfrak{b}$-module, $KR^{r,\ell}$ is generated by $v_{r,\ell}^{w_0}$.*

Proof. The assertion (i) for $w = \mathrm{id}$ follows by definition. Since $KR^{r,\ell}$ is finite-dimensional [3, Theorem 1.2.2], its weight set and dimensions of weight spaces are W_0-invariant. From this, (i) for general w is proved. Then (i) for $w = w_0$ implies (ii). □

The following theorem, which easily follows from [9, Theorem 4], connects KR modules with Demazure modules.

Theorem 4.4. *Let r be an arbitrary element of I_0 and ℓ a positive integer satisfying $c_r^{-1}\ell \in \mathbb{Z}$. Then there exists an isomorphism*

$$\mathbb{C}_{c_r^{-1}\ell\Lambda_0} \otimes KR^{r,\ell} \xrightarrow{\sim} V_{t_{c_r w_0(\varpi_r)}}(c_r^{-1}\ell\Lambda_0)$$

of $\mathfrak{p}'_{I_0}$-modules which maps $u_{c_r^{-1}\ell\Lambda_0} \otimes v_{r,\ell}^{w_0}$ to $u_{t_{c_r w_0(\varpi_r)}(c_r^{-1}\ell\Lambda_0)}$.

Remark 4.5. If $\mathfrak{g}_0$ is of ADE type, then $c_r = 1$ holds for all $r \in I_0$. Hence the above isomorphism follows for all KR modules in this case.

5 Fusion product

The fusion product was defined in [7] as a construction of a graded cyclic $(\mathfrak{g}_0 \otimes \mathbb{C}[t])$-module. Here we slightly reformulate it as a construction of a $\mathfrak{p}_{I_0}$-module. Note that $U(\mathfrak{p}'_{I_0})$ has a natural grading defined by

$$U(\mathfrak{p}'_{I_0})^k = \{X \in U(\mathfrak{p}'_{I_0}) \mid [d, X] = kX\},$$

from which we define a natural filtration on $U(\mathfrak{p}'_{I_0})$ by

$$U(\mathfrak{p}'_{I_0})^{\le k} = \bigoplus_{q \le k} U(\mathfrak{p}'_{I_0})^q.$$

Let M be a cyclic $\mathfrak{p}'_{I_0}$-module with a generator v, and denote by $F_v^k(M)$ for $k \in \mathbb{Z}_{\ge -1}$ the subspace $U(\mathfrak{p}'_{I_0})^{\le k}v$ of M. (Note that $F_v^{-1}(M) = 0$.) Then the associated $\mathfrak{p}_{I_0}$-module $\mathrm{gr}_{F_v}(M)$ is defined by

$$\mathrm{gr}_{F_v}(M) = \bigoplus_{k \ge 0} F_v^k(M)/F_v^{k-1}(M),$$

where d acts on $F_v^k(M)/F_v^{k-1}(M)$ as multiplication by k.

Now we recall the definition of fusion products. Let $M^1, \dots, M^p$ be a sequence of objects of $\mathfrak{p}_{I_0}$-Mod such that each M^j is generated by a weight vector v_j, and $c_1, \dots, c_p$ pairwise distinct complex numbers. Note that each M^j is $\mathbb{Z}$-graded by the action of d. For each $1 \le j \le p$, define $\mathfrak{p}'_{I_0}$-module $M^j_{c_j}$ by the pullback $\varphi_{c_j}^* M^j$, where φ_c is an automorphism of $\mathfrak{p}'_{I_0}$ defined by

$$x \otimes t^k \mapsto x \otimes (t + c)^k \text{ for } x \in \mathfrak{g}_0, \quad K \mapsto K.$$

For $w \in M^j$, we denote by w' its image under the canonical map $M^j \to M^j_{c_j}$. As shown in [7, Proposition 1.4], $M^1_{c_1} \otimes \cdots \otimes M^p_{c_p}$ is a cyclic $\mathfrak{p}'_{I_0}$-module generated by $v'_1 \otimes \cdots \otimes v'_p$, and we define a $\mathfrak{p}_{I_0}$-module $M^1_{c_1,v_1} * \cdots * M^p_{c_p,v_p}$ by

$$M^1_{c_1,v_1} * \cdots * M^p_{c_p,v_p} = \mathrm{gr}_{F_{v'_1 \otimes \cdots \otimes v'_p}}(M^1_{c_1} \otimes \cdots \otimes M^p_{c_p}).$$

When the parameters or generators are clear from the context, we write simply as $M^1_{c_1} * \cdots * M^p_{c_p}$, $M^1_{v_1} * \cdots * M^p_{v_p}$ or $M^1 * \cdots * M^p$.

Definition 5.1 ([7])**.** The $\mathfrak{p}_{I_0}$-module $M^1 * \cdots * M^p$ is called the *fusion product*.

Lemma 5.2. (i) *As* $(\mathfrak{g}_0 \oplus \mathbb{C}K)$*-modules,*

$$M^1 * \cdots * M^p \cong M^1 \otimes \cdots \otimes M^p.$$

(ii) *Let* $w_1, \dots, w_p$ *be weight vectors of* $M_1, \dots, M_p$ *respectively, and assume that*

$$U(\mathfrak{g}_0)(w_1 \otimes \cdots \otimes w_p) = U(\mathfrak{g}_0)(v_1 \otimes \cdots \otimes v_p).$$

Then we have

$$M^1_{w_1} * \cdots * M^p_{w_p} \cong M^1_{v_1} * \cdots * M^p_{v_p}.$$

(iii) *For any* $c \in \mathbb{C}$*, we have*

$$M^1_{c_1+c} * \cdots * M^p_{c_p+c} \cong M^1_{c_1} * \cdots * M^p_{c_p}.$$

(iv) $M^1_{c_1,v_1}$ (*the fusion product of a single module* M^1) *is isomorphic to* M^1 *as a* $\mathfrak{p}'_{I_0}$*-module.*

Proof. The assertions (i) and (ii) easily follow from the definition. As the automorphism φ_c preserves the subspace $U(\mathfrak{p}'_{I_0})^{\le k}$, we have

$$\varphi_c\big(U(\mathfrak{p}'_{I_0})^{\le k}\big)(v'_1 \otimes \cdots \otimes v'_p) = U(\mathfrak{p}'_{I_0})^{\le k}(v'_1 \otimes \cdots \otimes v'_p).$$

Since the left hand side is equal to the filtration of $M^1_{c_1+c} \otimes \cdots \otimes M^p_{c_p+c}$, (iii) is proved. When $c_1 = 0$, (iv) obviously follows. Then the assertion follows in general by (iii). □

6 Statement of the main theorem

Now we state the main theorem of this article. This is a generalization of [7, Theorem 2.5] where the case $\mathfrak{g}_0 = \mathfrak{sl}_2$ is proved, and [9, Theorem 4 and Corollary 5] where the case $\ell_1 = \cdots = \ell_p$ is proved.

Theorem 6.1. *Assume that* $\mathfrak{g}_0$ *is of ADE type. Let* $KR^{r_1,\ell_1}, \dots, KR^{r_p,\ell_p}$ *be a sequence of KR modules such that* $\ell_1 \le \cdots \le \ell_p$*, and set* $v_j = v_{r_j,\ell_j} \in KR^{r_j,\ell_j}$ *for* $1 \le j \le p$*. Then for arbitrary pairwise distinct complex numbers* $c_1, \dots, c_p$*, there exists an isomorphism of* $\mathfrak{b}$*-modules*

$$\begin{aligned} &\mathbb{C}_{\ell_p\Lambda_0 + C\delta} \otimes \Big(KR^{r_p,\ell_p}_{c_p,v_p} * \cdots * KR^{r_2,\ell_2}_{c_2,v_2} * KR^{r_1,\ell_1}_{c_1,v_1}\Big) \qquad (6.1)\\ &\cong \mathcal{D}_{t_{w_0(\varpi_{r_p})}}\Big(\mathbb{C}_{(\ell_p-\ell_{p-1})\Lambda_0} \otimes \cdots \otimes \mathcal{D}_{t_{w_0(\varpi_{r_2})}}\big(\mathbb{C}_{(\ell_2-\ell_1)\Lambda_0} \otimes \mathcal{D}_{t_{w_0(\varpi_{r_1})}}\mathbb{C}_{\ell_1\Lambda_0}\big)\cdots\Big) \end{aligned}$$

with some constant C*.*

We postpone the proof of this theorem to the latter part of this article. We see from Corollary 2.13 and Lemma 3.2 (ii) that the right hand side of (6.1) has a Demazure flag. Hence we can prove inductively using Lemma 3.2 (ii) that the following equation holds:

Corollary 6.2. *Under the notation and the assumptions of Theorem 6.1, we have*

$$e^{\ell_p\Lambda_0+C\delta}\mathrm{ch}\,KR^{r_p,\ell_p}*\cdots*KR^{r_2,\ell_2}*KR^{r_1,\ell_1}$$
$$=D_{t_{w_0(\varpi_{r_p})}}\Big(e^{(\ell_p-\ell_{p-1})\Lambda_0}\cdots D_{t_{w_0(\varpi_{r_2})}}\big(e^{(\ell_2-\ell_1)\Lambda_0}\cdot D_{t_{w_0(\varpi_{r_1})}}(e^{\ell_1\Lambda_0})\big)\cdots\Big).$$

Remark 6.3. The right hand side of (6.1) also appeared in [20]. In the article, it was proved that this module, which was called a *generalized Demazure module*, is isomorphic to the space of global sections of a certain line bundle on a *Bott-Samelson variety*.

7 $X = M$ conjecture

In this section, we give an important application of Theorem 6.1, the proof of the $X = M$ conjecture for type $A_n^{(1)}$ and $D_n^{(1)}$. Here, we assume that $\mathfrak{g}$ is a general (possibly twisted) affine Lie algebra.

For a sequence $\boldsymbol{\nu} = \big((r_1,\ell_1),\dots,(r_p,\ell_p)\big)$ of elements of $I_0\times\mathbb{Z}_{>0}$ and $\mu\in P_0^+$, denote by $M(\boldsymbol{\nu},\mu,q)\in\mathbb{Z}[q^{-1}]$ the associated *fermionic form* (see [12, 13] for definition, in which the fermionic form is denoted by $M(W,\mu,q)$ with $W=\bigotimes_{1\le j\le p}W^{r_j,\ell_j}$). The most important result concerning fermionic forms in this article is the following theorem proved by Di Francesco and Kedem using the result of [1]:

Theorem 7.1 ([5])**.** *Assume that $\mathfrak{g}$ is of nontwisted type. For a sequence $\boldsymbol{\nu} = \big((r_1,\ell_1),\dots,(r_p,\ell_p)\big)$ of elements of $I_0\times\mathbb{Z}_{>0}$ and pairwise distinct complex numbers $c_1,\dots,c_p$, we have*

$$\mathrm{ch}\,KR_{c_p}^{r_p,\ell_p}*\cdots*KR_{c_2}^{r_2,\ell_2}*KR_{c_1}^{r_1,\ell_1}=\sum_{\mu\in P_0^+}M(\boldsymbol{\nu},\mu,q)\mathrm{ch}\,V_{\mathfrak{g}_0}(\mu),$$

where we set $q=e^{-\delta}$ and denote by $V_{\mathfrak{g}_0}(\mu)$ the irreducible $\mathfrak{g}_0$-module with highest weight μ.

Remark 7.2. In [5], the above theorem was proved under the assumption that the dimension of each KR^{r_j,ℓ_j} is equal to that of the corresponding KR-module W^{r_j,ℓ_j} for the quantum affine algebra $U_q'(\mathfrak{g})$. This assumption was proved to hold for nonexceptional types in [2, 4], and we can see from the pentagon of identities in [17, Subsection 1.2] that this holds in general.

From Corollary 6.2 and Theorem 7.1, we have the following corollary:

Corollary 7.3. *Assume that $\mathfrak{g}$ is of nontwisted type and $\mathfrak{g}_0$ is of ADE type. Let $\boldsymbol{\nu} = \big((r_1,\ell_1),\dots,(r_p,\ell_p)\big)$ be a sequence of elements of $I_0\times\mathbb{Z}_{>0}$ such that $\ell_1\le\cdots\le\ell_p$. Then we have*

$$q^Ce^{\ell_p\Lambda_0}\sum_{\mu\in P_0^+}M(\boldsymbol{\nu},\mu,q)\mathrm{ch}\,V_{\mathfrak{g}_0}(\mu)$$
$$=D_{t_{w_0(\varpi_{r_p})}}\Big(e^{(\ell_p-\ell_{p-1})\Lambda_0}\cdots D_{t_{w_0(\varpi_{r_2})}}\big(e^{(\ell_2-\ell_1)\Lambda_0}\cdot D_{t_{w_0(\varpi_{r_1})}}(e^{\ell_1\Lambda_0})\big)\cdots\Big)$$

with some constant C, where we set $q=e^{-\delta}$.

Next we recall the definition of one-dimensional sums. Assume that $\mathfrak{g}$ is of nonexceptional type, and denote by $B^{r,\ell}$ the *Kirillov-Reshetikhin crystal* (KR crystal for short) associated with $r \in I_0$ and $\ell \in \mathbb{Z}_{>0}$. Note that the existence of KR crystals was proven in [26], and their explicit structure was given in [10] for nonexceptional types. It is known that $B^{r,\ell}$ is perfect if and only if $\ell \in c_r\mathbb{Z}$ [11]. Let $B = B^{r_p,\ell_p} \otimes \cdots \otimes B^{r_1,\ell_1}$ be a tensor product of KR crystals, and $D = D_B \colon B \to \mathbb{Z}$ the *energy function* defined on B. For the definitions of these objects, see [12, 13]. The *one-dimensional sum* $X(B,\mu,q) \in \mathbb{Z}[q,q^{-1}]$ for $\mu \in P_0^+$ is defined by

$$X(B,\mu,q) = \sum_{\substack{b \in B \\ \widetilde{e}_i b = 0 \ (i \in I_0) \\ \mathrm{wt}(b) = \mu}} q^{D(b)},$$

where $\widetilde{e}_i$ are Kashiwara operators. In [24, Corollary 7.3], the following proposition was proved:

Proposition 7.4. *Assume that $\mathfrak{g}$ is of nonexceptional type. Let $B = B^{r_p,c_{r_p}\ell_p} \otimes \cdots \otimes B^{r_1,c_{r_1}\ell_1}$ be a tensor product of perfect KR crystals such that $\ell_1 \le \cdots \le \ell_p$. Then we have*

$$q^{C'} e^{\ell_p \Lambda_0} \sum_{\mu \in P_0^+} X(B,\mu,q) \operatorname{ch} V_{\mathfrak{g}_0}(\mu) = D_{t_{c_{r_p} w_0(\varpi_{r_p})}} \Big(e^{(\ell_p - \ell_{p-1})\Lambda_0} \cdots \\ \cdots D_{t_{c_{r_2} w_0(\varpi_{r_2})}} \big(e^{(\ell_2 - \ell_1)\Lambda_0} \cdot D_{t_{c_{r_1} w_0(\varpi_{r_1})}} (e^{\ell_1 \Lambda_0}) \big) \cdots \Big)$$

with some constant C', where we set $q = e^{-\delta}$.

Now we show the following theorem from the above results, which is the $X = M$ conjecture presented in [12, 13]. This result for $D_n^{(1)}$ is new. This has already been proved for $A_n^{(1)}$ in [18], but our approach is quite different from theirs:

Theorem 7.5. *Assume that $\mathfrak{g}$ is of nontwisted, nonexceptional type and $\mathfrak{g}_0$ is of ADE type (i.e. $\mathfrak{g} = A_n^{(1)}$ or $D_n^{(1)}$). Let $\boldsymbol{\nu} = \big((r_1,\ell_1),\dots,(r_p,\ell_p)\big)$ be a sequence of elements of $I_0 \times \mathbb{Z}_{>0}$, and $B = B^{r_p,\ell_p} \otimes \cdots \otimes B^{r_1,\ell_1}$. Then for every $\mu \in P_0^+$, we have*

$$q^{-D(u(B))} X(B,\mu,q) = M(\boldsymbol{\nu},\mu,q),$$

where $u(B)$ denotes the unique element of B whose weight is $\sum_{1 \le j \le p} \ell_j \varpi_j$.

Proof. Recall that energy functions and one-dimensional sums are invariant under reordering of the given sequence by [27, Proposition 2.15], and so are fermionic forms by definition. Hence we may assume $\ell_1 \le \cdots \le \ell_p$. Then as $c_r = 1$ holds for all $r \in I_0$, we have from Corollary 7.3 and Proposition 7.4 that

$$q^{C''} \sum_{\mu \in P_0^+} X(B,\mu,q) \operatorname{ch} V_{\mathfrak{g}_0}(\mu) = \sum_{\mu \in P_0^+} M(\boldsymbol{\nu},\mu,q) \operatorname{ch} V_{\mathfrak{g}_0}(\mu) \tag{7.1}$$

with some constant C'', which implies

$$q^{C''} X(B,\mu,q) = M(\boldsymbol{\nu},\mu,q)$$

for every μ since the characters of irreducible $\mathfrak{g}_0$-modules are linearly independent. It remains to show $C'' = -D\big(u(B)\big)$. Let $\lambda = \sum_{1\le j\le p} \ell_j \varpi_j$. Since

$$\sum_{\mu\in P_0^+} X(B,\mu,q)\mathrm{ch}\, V_{\mathfrak{g}_0}(\mu) = \sum_{b\in B} q^{D(b)} e^{\mathrm{wt}(b)}$$

holds by definition, the coefficient of e^λ in the left hand side of (7.1) is equal to $q^{C''+D(u(B))}$. On the other hand, we easily see from Theorem 7.1 that the coefficient of e^λ in the right hand side is 1. Hence we have $q^{C''+D(u(B))} = 1$, which implies $C'' = -D\big(u(B)\big)$. The theorem is proved. □

8 $\mathfrak{b}$-fusion product

We devote the rest of this article to prove Theorem 6.1. In this section, we introduce a construction of a $\mathfrak{b}$-module, which we call the $\mathfrak{b}$-fusion product, defined by modifying the definition of the fusion product in Section 5. This construction is essentially used in the proof of the theorem. Here we only assume that $\mathfrak{g}$ is a nontwisted affine Lie algebra (that is, $\mathfrak{g}_0$ is allowed to be of type $BCFG$) since the definition of $\mathfrak{b}$-fusion products makes sense in this setting.

Let $M^1,\dots,M^p$ be a sequence of objects of $\mathfrak{p}_{I_0}$-Mod such that each M^j is generated (as a $\mathfrak{p}_{I_0}$-module) by a weight vector v_j, and N an object of $\mathfrak{b}$-Mod which is generated (as a $\mathfrak{b}$-module) by a weight vector u. Note that M^j and N are $\mathbb{Z}$-graded by the action of d. For pairwise distinct *nonzero* complex numbers $c_1,\dots,c_p$, define a $\mathfrak{p}'_{I_0}$-module $M^1_{c_1}\otimes\cdots\otimes M^p_{c_p}$ as in Section 5.

Lemma 8.1. *$N\otimes M^1_{c_1}\otimes\cdots\otimes M^p_{c_p}$ is generated by the vector $u\otimes v'_1\otimes\cdots\otimes v'_p$ as a $\mathfrak{b}'$-module.*

Proof. The proof is similar to that of [6, Proposition 2.8]. Let

$$\mathfrak{b}'' = \mathfrak{b}'\cap\big(\mathfrak{g}_0\otimes\mathbb{C}[t]\big) = \mathfrak{b}_0\oplus\mathfrak{g}_0\otimes\mathbb{C}[t].$$

Since N and M^j are finite-dimensional and $\mathbb{Z}$-graded, there exists a sufficiently large positive integer L such that $\mathfrak{g}_0\otimes t^L\mathbb{C}[t]$ acts trivially on them. It suffices to show that the Lie algebra homomorphism

$$\mathfrak{b}''\to\mathfrak{b}''/\big(\mathfrak{g}_0\otimes(t^L)\big)\oplus\bigoplus_{1\le j\le p}\mathfrak{g}_0\otimes\mathbb{C}[t]/\big((t-c_j)^L\big) \tag{8.1}$$

is surjective, where (f) denotes the ideal of $\mathbb{C}[t]$ generated by $f\in\mathbb{C}[t]$. In fact since $\mathfrak{g}_0\otimes\big((t-c_j)^L\big)$ (resp. $\mathfrak{g}_0\otimes\big(t^L\big)$) annihilates $M^j_{c_j}$ (resp. N), this implies

$$\begin{aligned}U(\mathfrak{b}'')(u\otimes v'_1\otimes\cdots\otimes v'_p) &= U\Big(\mathfrak{b}''/\mathfrak{g}_0\otimes\big(t^L\big)\Big)u\otimes U\Big(\mathfrak{g}_0\otimes\mathbb{C}[t]/\big((t-c_1)^L\big)\Big)v'_1\otimes\\ &\qquad\cdots\otimes U\Big(\mathfrak{g}_0\otimes\mathbb{C}[t]/\big((t-c_p)^L\big)\Big)v'_p\\ &= N\otimes M^1_{c_1}\otimes\cdots\otimes M^p_{c_p},\end{aligned}$$

and hence the lemma follows. Since $t^L,(t-c_1)^L,\dots,(t-c_p)^L$ are relatively prime, the Lie algebra homomorphism

$$\mathfrak{g}_0\otimes\big(t^L\big)\to\bigoplus_{1\le j\le p}\mathfrak{g}_0\otimes\mathbb{C}[t]/\big((t-c_j)^L\big)$$

is surjective. From this, we easily see the surjectivity of the homomorphism (8.1)

□

Define the subspace $U(\mathfrak{b}')^{\leq k} \subseteq U(\mathfrak{b}')$ for $k \in \mathbb{Z}$ similarly as $U(\mathfrak{p}'_{I_0})^{\leq k}$. By considering the filtration

$$\widetilde{F}^k_{u\otimes v'_1\otimes\cdots\otimes v'_p}(N \otimes M^1_{c_1} \otimes \cdots \otimes M^p_{c_p}) = U(\mathfrak{b}')^{\leq k}(u \otimes v'_1 \otimes \cdots \otimes v'_p) \tag{8.2}$$

of $N \otimes M^1_{c_1} \otimes \cdots \otimes M^p_{c_p}$, we define $\left[N_u * M^1_{c_1,v_1} * \cdots * M^p_{c_p,v_p}\right]_{\mathfrak{b}}$ as the associated $\mathfrak{b}$-module

$$\left[N_u * M^1_{c_1,v_1} * \cdots * M^p_{c_p,v_p}\right]_{\mathfrak{b}} = \mathrm{gr}_{\widetilde{F}_{u\otimes v'_1\otimes\cdots\otimes v'_p}}(N \otimes M^1_{c_1} \otimes \cdots \otimes M^p_{c_p}),$$

which we call the $\mathfrak{b}$-*fusion product*. We sometimes omit the parameters or the generators when they are clear from the context. It should be noted that in the definition of the $\mathfrak{b}$-fusion product, only the leftmost module is allowed to be a $\mathfrak{b}$-module, and the others are assumed to be $\mathfrak{p}_{I_0}$-modules. By definition, it is easily seen for every $\ell \in \mathbb{C}$ that

$$\mathbb{C}_{\ell\Lambda_0} \otimes \left[N * M^1 * \cdots * M^p\right]_{\mathfrak{b}} \cong \left[(\mathbb{C}_{\ell\Lambda_0} \otimes N) * M^1 * \cdots * M^p\right]_{\mathfrak{b}}. \tag{8.3}$$

In some special cases, an original fusion product is connected to a certain $\mathfrak{b}$-fusion product by the following lemma:

Lemma 8.2. *Assume that each generator v_j of M^j is annihilated by $\mathfrak{n}_0^-$. Then we have the following isomorphisms of $\mathfrak{b}$-modules:*
(i)

$$M^1_0 * M^2_{c_2} * \cdots * M^p_{c_p} \cong \left[M^1 * M^2_{c_2} * \cdots * M^p_{c_p}\right]_{\mathfrak{b}}, \text{ and}$$

(ii)

$$M^1_{c_1} * \cdots * M^p_{c_p} \cong \left[\mathbb{C}_{\mathrm{triv}} * M^1_{c_1} * \cdots * M^p_{c_p}\right]_{\mathfrak{b}},$$

where $\mathbb{C}_{\mathrm{triv}}$ denotes the trivial module.

Proof. From the assumption, M^1 is generated by v_1 as a $\mathfrak{b}$-module. Hence the right hand side of the isomorphism (i) makes sense. The isomorphisms easily follow from the definition since we have

$$U(\mathfrak{p}'_{I_0})^{\leq k}(v'_1 \otimes v'_2 \otimes \cdots \otimes v'_p) = U(\mathfrak{b}')^{\leq k}(v'_1 \otimes v'_2 \otimes \cdots \otimes v'_p)$$

by the assumption and the Poincaré-Birkhoff-Witt theorem. □

Remark 8.3. The isomorphism (ii) of the above lemma does not hold in general. For example, let M^1 be a finite-dimensional irreducible $\mathfrak{g}_0$-module with a highest weight vector v_1, which is considered as a $\mathfrak{p}_{I_0}$-module via the evaluation map $\mathfrak{p}_{I_0} \to \mathfrak{g}_0 \colon x \otimes t^k \mapsto \delta_{0,k}x,\ K, d \mapsto 0$. Then by Lemma 5.2 (iv), the fusion product $M^1_{c_1,v_1}$ is isomorphic to M^1. However, we easily see that the degree 0 space of $\left[\mathbb{C}_{\mathrm{triv}} * M^1_{c_1,v_1}\right]_{\mathfrak{b}}$ is one-dimensional, and hence they are not isomorphic unless M^1 is trivial. On the other hand, $\left[\mathbb{C}_{\mathrm{triv}} * M^1\right]_{\mathfrak{b}}$ is isomorphic to M^1 if a lowest weight vector of M^1 is chosen as a generator. As seen from this example, the $\mathfrak{b}$-fusion product is sensitive to the choice of generators.

Lemma 8.4. *Let $i \in I$. The $\mathfrak{b}$-module $\big[N * M^1 * \cdots * M^p\big]_{\mathfrak{b}}$ extends to a $\mathfrak{p}_i$-module if N extends to a $\mathfrak{p}_i$-module and either of the following conditions is satisfied:*
(i) *$i \in I_0$ and all v_j and u are annihilated by f_i, or*
(ii) *$i = 0$, K acts trivially on $M^1, \dots, M^p$, and each v_j (resp. u) is annihilated by $e_\theta \otimes \mathbb{C}[t]$ (resp. $e_\theta \otimes t^{-1}$).*

Proof. The case (i) is easily proved since $N \otimes M^1_{c_1} \otimes \cdots \otimes M^p_{c_p}$ is a $\mathfrak{p}'_i$-module and we have

$$U(\mathfrak{b}')^{\le k}(u \otimes v'_1 \otimes \cdots \otimes v'_p) = U(\mathfrak{p}'_i)^{\le k}(u \otimes v'_1 \otimes \cdots \otimes v'_p) \tag{8.4}$$

for each k. Let us prove the case (ii). Since K acts trivially, a $\mathfrak{p}'_0$-module structure is defined on each $M^j_{c_j}$ by letting f_0 act by

$$\big(e_\theta \otimes t^{-1}\big)w' = -\sum_{k\ge 0}(-c_j)^{-k-1}\big((e_\theta \otimes t^k)w\big)' \quad \text{for } w \in M^j.$$

Note that the above sum is finite since $M^j \in \mathfrak{p}_{I_0}$-Mod. Hence $N \otimes M^1_{c_1} \otimes \cdots \otimes M^p_{c_p}$ extends to a $\mathfrak{p}'_0$-module. Moreover the equality (8.4) also holds in this case from the assumption. Hence the assertion also follows in this case. □

Take arbitrary vectors $w_j \in M^j$ $(1 \le j \le p)$ and $z \in N$. Let k be the unique integer such that

$$z \otimes w'_1 \otimes \cdots \otimes w'_p \in \widetilde{F}^k_{u\otimes\cdots\otimes v'_p}(N \otimes M^1_{c_1} \otimes \cdots \otimes M^p_{c_p}) \setminus \widetilde{F}^{k-1}_{u\otimes\cdots\otimes v'_p}(N \otimes M^1_{c_1} \otimes \cdots \otimes M^p_{c_p}),$$

and denote by $z * w_1 * \cdots * w_p$ the vector of $\big[N * M^1 * \cdots * M^p\big]_{\mathfrak{b}}$ which is the image of $z \otimes w'_1 \otimes \cdots \otimes w'_p$ under the projection

$$\widetilde{F}^k(N \otimes \cdots \otimes M^p_{c_p}) \twoheadrightarrow \widetilde{F}^k(N \otimes \cdots \otimes M^p_{c_p})/\widetilde{F}^{k-1}(N \otimes \cdots \otimes M^p_{c_p}).$$

Note that $u * v_1 * \cdots * v_p$ is a generator of $\big[N * M^1 * \cdots * M^p\big]_{\mathfrak{b}}$. The following lemma, which obviously follows by definition, is important for the later arguments.

Lemma 8.5. *Let $X \in U(\mathfrak{b}')^k$. Then X annihilates $u * v_1 * \cdots * v_p$ if and only if there exists some $Y \in U(\mathfrak{b}')^{\le k-1}$ satisfying*

$$(X - Y)(u \otimes v'_1 \otimes \cdots \otimes v'_p) = 0.$$

9 Proof of the main theorem

Now, we begin the proof of Theorem 6.1. Assume that $\mathfrak{g}$ is a nontwisted affine Lie algebra and $\mathfrak{g}_0$ is of type ADE. For a given sequence of KR modules $KR^{r_1,\ell_1}, \dots, KR^{r_p,\ell_p}$, we set $M^j = KR^{r_j,\ell_j}$ and $v_j = v_{r_j,\ell_j} \in M^j$ for $1 \le j \le p$ for short, and write $v^w_j = v^w_{r_j,\ell_j} \in M^j$ for $w \in W_0$ (defined in Section 4).

We shall show the theorem by the induction on p. The assertion of the theorem for $p = 1$ follows from Lemma 5.2 (iv), Theorem 2.6, Theorem 4.4, and Lemma 3.5 (i). Assume $p > 1$. By Lemma 5.2 (iii), we may (and do) assume $c_p = 0$, which implies $c_1, \dots, c_{p-1}$ are nonzero. First we show the following lemma:

Lemma 9.1. *We have the following isomorphisms of $\mathfrak{b}'$-modules:*

$$\mathbb{C}_{\ell_p\Lambda_0} \otimes \left(M^p_{v_p} * M^{p-1}_{v_{p-1}} * \cdots * M^1_{v_1}\right) \cong \left[V_{t_{w_0(\varpi_{r_p})}}(\ell_p\Lambda_0)_{u_{t_{w_0(\varpi_{r_p})}(\ell_p\Lambda_0)}} * M^{p-1}_{v^{w_0}_{p-1}} * \cdots * M^1_{v^{w_0}_1}\right]_{\mathfrak{b}}, \quad (9.1)$$

and

$$\mathcal{D}_{t_{w_0(\varpi_{r_p})}}\left(\mathbb{C}_{(\ell_p-\ell_{p-1})\Lambda_0} \otimes \mathcal{D}_{t_{w_0(\varpi_{r_{p-1}})}}\left(\mathbb{C}_{(\ell_{p-1}-\ell_{p-2})\Lambda_0} \otimes \cdots \otimes \mathcal{D}_{t_{w_0(\varpi_{r_1})}}\mathbb{C}_{\ell_1\Lambda_0}\right)\right) \cong \mathcal{D}_{t_{w_0(\varpi_{r_p})}}\left[\mathbb{C}_{\ell_p\Lambda_0} * M^{p-1}_{v^{w_0}_{p-1}} * \cdots * M^1_{v^{w_0}_1}\right]_{\mathfrak{b}}. \quad (9.2)$$

Proof. Since

$$U(\mathfrak{g}_0)(v^{w_0}_p \otimes \cdots \otimes v^{w_0}_1) = U(\mathfrak{g}_0)(v_p \otimes \cdots \otimes v_1)$$

holds, we have from Lemma 5.2 (ii) that

$$M^p_{0,v_p} * M^{p-1}_{c_{p-1},v_{p-1}} * \cdots * M^1_{c_1,v_1} \cong M^p_{0,v^{w_0}_p} * M^{p-1}_{c_{p-1},v^{w_0}_{p-1}} * \cdots * M^1_{c_1,v^{w_0}_1},$$

whose right hand side is isomorphic to $\left[M^p_{v^{w_0}_p} * M^{p-1}_{v^{w_0}_{p-1}} * \cdots * M^1_{v^{w_0}_1}\right]_{\mathfrak{b}}$ by Lemma 8.2 (i). Hence we have using (8.3) and Theorem 4.4 that

$$\mathbb{C}_{\ell_p\Lambda_0} \otimes \left(M^p_{v_p} * \cdots * M^1_{v_1}\right) \cong \left[\left(\mathbb{C}_{\ell_p\Lambda_0} \otimes M^p\right)_{u_{\ell_p\Lambda_0}\otimes v^{w_0}_p} * M^{p-1}_{v^{w_0}_{p-1}} * \cdots * M^1_{v^{w_0}_1}\right]_{\mathfrak{b}} \cong \left[V_{t_{w_0(\varpi_{r_p})}}(\ell_p\Lambda_0)_{u_{t_{w_0(\varpi_{r_p})}(\ell_p\Lambda_0)}} * M^{p-1}_{v^{w_0}_{p-1}} * \cdots * M^1_{v^{w_0}_1}\right]_{\mathfrak{b}}.$$

The isomorphism (9.1) is proved. Let us prove (9.2). By the induction hypothesis, there exists an isomorphism

$$\mathcal{D}_{t_{w_0(\varpi_{r_{p-1}})}}\left(\mathbb{C}_{(\ell_{p-1}-\ell_{p-2})\Lambda_0} \otimes \cdots \otimes \mathcal{D}_{t_{w_0(\varpi_{r_1})}}\mathbb{C}_{\ell_1\Lambda_0}\right) \cong \mathbb{C}_{\ell_{p-1}\Lambda_0} \otimes \left(M^{p-1}_{v_{p-1}} * \cdots * M^1_{v_1}\right),$$

whose right hand side is isomorphic to

$$\mathbb{C}_{\ell_{p-1}\Lambda_0} \otimes \left(M^{p-1}_{v^{w_0}_{p-1}} * \cdots * M^1_{v^{w_0}_1}\right) \cong \mathbb{C}_{\ell_{p-1}\Lambda_0} \otimes \left[\mathbb{C}_{\mathrm{triv}} * M^{p-1}_{v^{w_0}_{p-1}} * \cdots * M^1_{v^{w_0}_1}\right]_{\mathfrak{b}}$$

by Lemma 5.2 (ii) and Lemma 8.2 (ii). Hence we have using (8.3) that

$$\begin{aligned}\big(&\text{left hand side of (9.2)}\big) \\ &\cong \mathcal{D}_{t_{w_0(\varpi_{r_p})}}\left(\mathbb{C}_{\ell_p\Lambda_0} \otimes \left[\mathbb{C}_{\mathrm{triv}} * M^{p-1}_{v^{w_0}_{p-1}} * \cdots * M^1_{v^{w_0}_1}\right]_{\mathfrak{b}}\right) \\ &\cong \mathcal{D}_{t_{w_0(\varpi_{r_p})}}\left[\mathbb{C}_{\ell_p\Lambda_0} * M^{p-1}_{v^{w_0}_{p-1}} * \cdots * M^1_{v^{w_0}_1}\right]_{\mathfrak{b}}.\end{aligned}$$

The isomorphism (9.2) is proved. □

By Lemmas 9.1 and 3.5 (i), in order to prove the theorem it suffices to show the following isomorphism of $\mathfrak{b}'$-modules:

$$\left[V_{t_{w_0(\varpi_{r_p})}}(\ell_p\Lambda_0)_{u_{t_{w_0(\varpi_{r_p})}(\ell_p\Lambda_0)}} * M^{p-1}_{v^{w_0}_{p-1}} * \cdots * M^1_{v^{w_0}_1}\right]_{\mathfrak{b}} \cong \mathcal{D}_{t_{w_0(\varpi_{r_p})}}\left[\mathbb{C}_{\ell_p\Lambda_0} * M^{p-1}_{v^{w_0}_{p-1}} * \cdots * M^1_{v^{w_0}_1}\right]_{\mathfrak{b}}. \quad (9.3)$$

Let $w \in W$ and $\tau \in \Sigma$ be unique elements satisfying $w\tau = t_{w_0(\varpi_{r_p})}$, and $w = s_{i_k} \cdots s_{i_1}$ a reduced expression. For $0 \le q \le k$, let $w^q = s_{i_q} \cdots s_{i_1}\tau \in \widetilde{W}$, and $\overline{w}^q$ be the unique element of W_0 satisfying $w^q \in \overline{w}^q \cdot T(\widetilde{M})$ (note that $\overline{w}^k = \mathrm{id}$). We also write $\overline{\tau}$ for $\overline{w}^0$. Then since

$$V_{t_{w_0(\varpi_{r_p})}}(\ell_p\Lambda_0) \cong \mathcal{D}_{t_{w_0(\varpi_{r_p})}}\mathbb{C}_{\ell_p\Lambda_0} \cong \mathcal{D}_w\mathbb{C}_{\ell_p\Lambda_{\tau(0)}}$$

holds, we see that the isomorphism (9.3) is deduced by the induction on q from the following two propositions, and hence Theorem 6.1 is established:

Proposition 9.2. *We have*

$$\begin{aligned}&\Big[\mathbb{C}_{\ell_p\Lambda_{\tau(0)}} * M^{p-1}_{c_{p-1},v_{p-1}^{\overline{\tau}w_0}} * \cdots * M^1_{c_1,v_1^{\overline{\tau}w_0}}\Big]_{\mathfrak{b}}\\ &\qquad\cong (\tau^{-1})^*\Big[\mathbb{C}_{\ell_p\Lambda_0} * M^{p-1}_{c_{p-1},v_{p-1}^{w_0}} * \cdots * M^1_{c_1,v_1^{w_0}}\Big]_{\mathfrak{b}}\end{aligned}$$

as $\mathfrak{b}'$-modules.

Proposition 9.3. *For each $1 \le q \le k$, we have*

$$\begin{aligned}&\Big[V_{w^q}(\ell_p\Lambda_0)_{u_{w^q(\ell_p\Lambda_0)}} * M^{p-1}_{v_{p-1}^{\overline{w}^q w_0}} * \cdots * M^1_{v_1^{\overline{w}^q w_0}}\Big]_{\mathfrak{b}}\\ &\qquad\cong \mathcal{D}_{i_q}\Big[V_{w^{q-1}}(\ell_p\Lambda_0)_{u_{w^{q-1}(\ell_p\Lambda_0)}} * M^{p-1}_{v_{p-1}^{\overline{w}^{q-1}w_0}} * \cdots * M^1_{v_1^{\overline{w}^{q-1}w_0}}\Big]_{\mathfrak{b}}\end{aligned}$$

as $\mathfrak{b}'$-modules.

To show Proposition 9.2, we need to prepare several lemmas. The following one is proved similarly as [16, Lemma 3.8]:

Lemma 9.4. *Let M be a finite-dimensional $\mathfrak{p}'_{I_0}$-module. Then for every $w \in W_0$, there exists a linear automorphism η_w of M satisfying*

$$\begin{aligned}\mathrm{Ad}(\eta_w)(h \otimes t^s) =& w(h) \otimes t^s \quad \text{for } h \in \mathfrak{h}_0, s \in \mathbb{Z}_{\ge 0}, \quad \mathrm{Ad}(\eta_w)(K) = K,\\ \mathrm{Ad}(\eta_w)(e_\alpha \otimes t^s) &= a_w(\alpha)e_{w(\alpha)} \otimes t^s \quad \text{for } \alpha \in \Delta_0, s \in \mathbb{Z}_{\ge 0}, \quad \text{and}\\ &\eta_w(M_\lambda) = M_{w(\lambda)} \text{ for } \lambda \in \mathfrak{h}^*/\mathbb{C}\delta,\end{aligned}$$

where $a_w(\alpha)$ are some nonzero complex numbers which do not depend on M.

By applying $\mathrm{Ad}(\eta_{w_0})$ given in Lemma 9.4 to the defining relations of $KR^{r,\ell}$ in Definition 4.1, the following lemma is proved:

Lemma 9.5. *The annihilating ideal of $v^{w_0}_{r,\ell} \in KR^{r,\ell}$ in $U\big(\mathfrak{p}'_{I_0}\big)$ is generated by*

$$\begin{aligned}&\mathfrak{n}_0^- \otimes \mathbb{C}[t], \quad \mathfrak{h}_0 \otimes t\mathbb{C}[t], \quad h - \langle w_0(\ell\varpi_r), h\rangle \quad (h \in \mathfrak{h}'),\\ &e_{\bar r}^{\ell+1}, \quad e_{\bar r} \otimes t, \quad \text{and} \quad e_i \quad (i \in I_0 \setminus \{\bar r\}),\end{aligned}$$

where $\bar r$ is the node of I_0 such that $w_0(\alpha_r) = -\alpha_{\bar r}$.

Lemma 9.6. *For $c \in \mathbb{C}$, the annihilating ideal of $v^{w_0\prime}_{r,\ell} \in KR^{r,\ell}_c$ in $U(\mathfrak{b}')$ is generated by*

$$\begin{aligned}&\mathfrak{n}_0^- \otimes t\mathbb{C}[t], \quad \mathfrak{h}_0 \otimes (t-c)\mathbb{C}[t], \quad h - \langle w_0(\ell\varpi_r), h\rangle \quad (h \in \mathfrak{h}'),\\ &e_{\bar r}^{\ell+1}, \ e_{\bar r} \otimes (t-c), \quad \text{and} \quad e_i \quad (i \in I_0 \setminus \{\bar r\}).\end{aligned}$$

Proof. Let I be the subspace of $U(\mathfrak{b}')$ spanned by the above vectors. From Lemma 9.5, we see that the annihilating ideal of $v_{r,\ell}^{w_0\prime}$ in $U\bigl(\mathfrak{p}'_{I_0}\bigr)$ is equal to $U\bigl(\mathfrak{p}'_{I_0}\bigr)\bigl(I+\mathfrak{n}_0^-\bigr)$. We have to prove that

$$U\bigl(\mathfrak{p}'_{I_0}\bigr)\bigl(I+\mathfrak{n}_0^-\bigr)\cap U(\mathfrak{b}')\subseteq U(\mathfrak{b}')I.$$

Since $U\bigl(\mathfrak{p}'_{I_0}\bigr)=U(\mathfrak{b}')\oplus U\bigl(\mathfrak{p}'_{I_0}\bigr)\mathfrak{n}_0^-$ holds by the Poincaré-Birkhoff-Witt theorem, it suffices to show that

$$U(\mathfrak{p}'_{I_0})I\subseteq U(\mathfrak{b}')I\oplus U\bigl(\mathfrak{p}'_{I_0}\bigr)\mathfrak{n}_0^-.$$

Then since we have $U(\mathfrak{p}'_{I_0})=U(\mathfrak{b}')U(\mathfrak{n}_0^-)$ and $U(\mathfrak{n}_0^-)$ is generated by $\{f_i \mid i\in I_0\}$, it is enough to prove that $f_iI\subseteq I\oplus U\bigl(\mathfrak{p}'_{I_0}\bigr)\mathfrak{n}_0^-$ holds for $i\in I_0$, which is proved by elementary calculations. □

Lemma 9.7. *There exists a nonzero complex number b satisfying the following statement: for every KR module $KR^{r,\ell}$ and $c\in\mathbb{C}$, there exists a homomorphism*

$$KR_c^{r,\ell}\to\tau^*KR_{bc}^{r,\ell}$$

of $\mathfrak{b}'$-modules which maps $v_{r,\ell}^{w_0\prime}$ to $\tau^*v_{r,\ell}^{\overline{\tau}w_0\prime}$.

Proof. Since $KR_c^{r,\ell}$ is generated by $v_{r,\ell}^{w_0\prime}$ as a $\mathfrak{b}'$-module, it suffices to show for suitable $b\in\mathbb{C}^*$ that if $X\in U(\mathfrak{b}')$ annihilates $v_{r,\ell}^{w_0\prime}\in KR_c^{r,\ell}$, then $\tau(X)$ annihilates $v_{r,\ell}^{\overline{\tau}w_0\prime}\in KR_{bc}^{r,\ell}$. Since the $\mathfrak{h}'$-weights of $v_{r,\ell}^{w_0\prime}\in KR_c^{r,\ell}$ and $\tau^*v_{r,\ell}^{\overline{\tau}w_0\prime}\in\tau^*KR_{bc}^{r,\ell}$ coincide, we may assume $X\in U(\mathfrak{n}^+)$. Let $i_0=\tau^{-1}(0)\in I$, and $\psi_{i_0}\colon U(\mathfrak{b}')\to U\bigl(\mathfrak{p}'_{I_0}\bigr)$ be an algebra homomorphism defined by

$$\begin{aligned}&\psi_{i_0}(x\otimes t^s)=x\otimes t^{s+\langle\alpha,\varpi_{i_0}^\vee\rangle}\quad\text{for}\ \ x\in\mathfrak{g}_\alpha\ (\alpha\in\Delta_0),\\&\psi_{i_0}(h\otimes t^s)=h\otimes t^s\quad\text{for}\ \ h\in\mathfrak{h}_0,\quad\psi_{i_0}(K)=K.\end{aligned}$$

Using Lemma 9.6, we easily check that $\psi_{i_0}(X)$ also annihilates $v_{r,\ell}^{w_0\prime}\in KR_c^{r,\ell}$. Let $\eta=\eta_{\overline{\tau}}$ be the linear automorphism of $KR_c^{r,\ell}$ given in Lemma 9.4. Then since $\eta(v_{r,\ell}^{w_0\prime})\in\mathbb{C}^*v_{r,\ell}^{\overline{\tau}w_0\prime}$ holds by Lemma 4.3, $\mathrm{Ad}(\eta)\circ\psi_{i_0}(X)$ annihilates $v_{r,\ell}^{\overline{\tau}w_0\prime}\in KR_c^{r,\ell}$, which is equivalent to that $\varphi_c\circ\mathrm{Ad}(\eta)\circ\psi_{i_0}(X)$ annihilates $v_{r,\ell}^{\overline{\tau}w_0}\in KR^{r,\ell}$ (φ_c is defined in Section 5). It is easy to see from Lemma 3.1 that there exists some $b_i\in\mathbb{C}^*$ for each $i\in I$ such that

$$\mathrm{Ad}(\eta)\circ\psi_{i_0}(e_{\tau^{-1}(i)})=b_ie_i.$$

Define a linear automorphism H on $KR^{r,\ell}$ by

$$H(u)=\prod_{i\in I}b_i^{-\langle\lambda,\Lambda_i^\vee\rangle}\cdot u\quad\text{if } u\in KR_\lambda^{r,\ell}\ \ (\lambda\in P),$$

where $\Lambda_i^\vee\in\mathfrak{h}$ are the fundamental coweights of $\mathfrak{g}$. $\mathrm{Ad}(H)\circ\varphi_c\circ\mathrm{Ad}(\eta)\circ\psi_{i_0}(X)$ annihilates $v_{r,\ell}^{\overline{\tau}w_0}$ since it is a weight vector. Set $b=\prod_{i\in I}b_i^{a_i}$. It is easily checked that

$$\mathrm{Ad}(H)\circ\varphi_c\circ\mathrm{Ad}(\eta)\circ\psi_{i_0}(e_{\tau^{-1}(i)})=\begin{cases}e_{-\theta}\otimes t+bce_{-\theta} & \text{if } i=0,\\ e_i & \text{otherwise,}\end{cases}$$

which implies

$$\mathrm{Ad}(H)\circ\varphi_c\circ\mathrm{Ad}(\eta)\circ\psi_{i_0}=\varphi_{bc}\circ\tau\quad\text{on}\ \ U(\mathfrak{n}^+).$$

Hence we see that $\varphi_{bc}\circ\tau(X)$ annihilates $v_{r,\ell}^{\overline{\tau}w_0}$, which is equivalent to that $\tau(X)$ annihilates $v_{r,\ell}^{\overline{\tau}w_0\prime}\in KR_{bc}^{r,\ell}$. The assertion is proved. □

Now, we give the proof of Proposition 9.2:

Proof of Proposition 9.2. Note that the right hand side of Theorem 6.1 does not depend on the parameters $c_1,\dots,c_p$. Hence from the induction hypothesis on p and the proof of (9.2), we see that the $\mathfrak{b}$-module $\Big[\mathbb{C}_{\ell_p\Lambda_0}*M^{p-1}_{c_{p-1},v_{p-1}^{w_0}}*\cdots*M^1_{c_1,v_1^{w_0}}\Big]_{\mathfrak{b}}$ also does not depend on the parameters, and in particular we have

$$\Big[\mathbb{C}_{\ell_p\Lambda_0}*M^{p-1}_{c_{p-1},v_{p-1}^{w_0}}*\cdots*M^1_{c_1,v_1^{w_0}}\Big]_{\mathfrak{b}}\cong\Big[\mathbb{C}_{\ell_p\Lambda_0}*M^{p-1}_{b^{-1}c_{p-1},v_{p-1}^{w_0}}*\cdots*M^1_{b^{-1}c_1,v_1^{w_0}}\Big]_{\mathfrak{b}},$$

where b is the complex number given in Lemma 9.7. Hence the proposition is equivalent to the following isomorphism of $\mathfrak{b}'$-modules:

$$\Big[\mathbb{C}_{\ell_p\Lambda_{\tau(0)}}*M^{p-1}_{c_{p-1},v_{p-1}^{\overline{\tau}w_0}}*\cdots*M^1_{c_1,v_1^{\overline{\tau}w_0}}\Big]_{\mathfrak{b}}\\ \cong(\tau^{-1})^*\Big[\mathbb{C}_{\ell_p\Lambda_0}*M^{p-1}_{b^{-1}c_{p-1},v_{p-1}^{w_0}}*\cdots*M^1_{b^{-1}c_1,v_1^{w_0}}\Big]_{\mathfrak{b}}.$$

Let us prove this. Let $u_1\in\mathbb{C}_{\ell_p\Lambda_0}$ and $u_2\in\mathbb{C}_{\ell_p\Lambda_{\tau(0)}}$ be nonzero vectors. Since dimensions of two modules are equal, it suffices to show there exists a surjective homomorphism of $\mathfrak{b}'$-modules from the right hand side to the left hand side mapping $(\tau^{-1})^*(u_1*v_{p-1}^{w_0}*\cdots*v_1^{w_0})$ to $u_2*v_{p-1}^{\overline{\tau}w_0}*\cdots*v_1^{\overline{\tau}w_0}$, which is equivalent to show that if $X\in U(\mathfrak{b}')$ annihilates $u_1*v_{p-1}^{w_0}*\cdots*v_1^{w_0}$, then $\tau(X)$ annihilates $u_2*v_{p-1}^{\overline{\tau}w_0}*\cdots*v_1^{\overline{\tau}w_0}$. We may assume $X\in U(\mathfrak{b}')^s_\gamma$ for some $\gamma\in Q_0$ and $s\in\mathbb{Z}_{\geq 0}$, where we set

$$U(\mathfrak{b}')^s_\gamma=\{Z\in U(\mathfrak{b}')^s\mid[h,Z]=\langle\gamma,h\rangle Z\ \text{for}\ h\in\mathfrak{h}_0\}.$$

By Lemma 8.5, there exists $Y\in U(\mathfrak{b}')^{\leq s-1}$ such that

$$(X-Y)(u_1\otimes v_{p-1}^{w_0\prime}\otimes\cdots\otimes v_1^{w_0\prime})=0,\tag{9.4}$$

and we may assume $Y\in U(\mathfrak{b}')^{\leq s-1}_{\overline{\gamma}}$. We see from Lemma 9.7 that there exists a homomorphism of $\mathfrak{b}'$-modules

$$\mathbb{C}_{\ell_p\Lambda_0}\otimes M^{p-1}_{b^{-1}c_{p-1}}\otimes\cdots\otimes M^1_{b^{-1}c_1}\to\tau^*\Big(\mathbb{C}_{\ell_p\Lambda_{\tau(0)}}\otimes M^{p-1}_{c_{p-1}}\otimes\cdots\otimes M^1_{c_1}\Big)$$

which maps $u_1\otimes v_{p-1}^{w_0\prime}\otimes\cdots\otimes v_1^{w_0\prime}$ to $\tau^*\big(u_2\otimes v_{p-1}^{\overline{\tau}w_0\prime}\otimes\cdots\otimes v_1^{\overline{\tau}w_0\prime}\big)$. From this and (9.4), we have

$$\tau(X-Y)(u_2\otimes v_{p-1}^{\overline{\tau}w_0\prime}\otimes\cdots\otimes v_1^{\overline{\tau}w_0\prime})=0.$$

Moreover, we have $\tau(X)\in U(\mathfrak{b}')^{s+\langle\gamma,\varpi_{i_0}^\vee\rangle}_{\overline{\tau}(\gamma)}$ and $\tau(Y)\in U(\mathfrak{b}')^{\leq s+\langle\gamma,\varpi_{i_0}^\vee\rangle-1}_{\overline{\tau}(\gamma)}$ from Lemma 3.1, where we set $i_0=\tau^{-1}(0)$. Hence by Lemma 8.5, $\tau(X)$ annihilates

$u_2 * v_{p-1}^{\overline{\tau}w_0} * \cdots * v_1^{\overline{\tau}w_0}$. The assertion is proved. □

It remains to prove Proposition 9.3:

Proof of Proposition 9.3. We abbreviate $u^q = u_{w^q(\ell_p\Lambda_0)}$, $V_q = V_{w^q}(\ell_p\Lambda_0)$ and $v_j^q = v_j^{\overline{w}^q w_0}$. Fix $1 \le q \le k$, and assume when $q > 1$ that the assertion of the proposition holds for $q' < q$. Then we see that $\left[(V_{q-1})_{u^{q-1}} * M^{p-1}_{v^{q-1}_{p-1}} * \cdots * M^1_{v^{q-1}_1}\right]_{\mathfrak{b}}$ has a Demazure flag from the induction hypothesis on p, Corollary 2.13, Lemma 3.2 (ii), and Proposition 9.2.

Recall that, by the definition of Demazure modules, there exists a canonical embedding $V_{q-1} \hookrightarrow V_q$. Let z^{q-1} denote the image of u^{q-1} under this embedding. It should be noted that $z^{q-1} * v_{p-1}^{q-1} * \cdots * v_1^{q-1} \in \left[(V_q)_{u^q} * M^{p-1}_{v^q_{p-1}} * \cdots * M^1_{v^q_1}\right]_{\mathfrak{b}}$ is not a generator.

Claim 1. There exists a homomorphism of $\mathfrak{b}'$-modules

$$\Phi\colon \left[(V_{q-1})_{u^{q-1}} * M^{p-1}_{v^{q-1}_{p-1}} * \cdots * M^1_{v^{q-1}_1}\right]_{\mathfrak{b}} \to \left[(V_q)_{u^q} * M^{p-1}_{v^q_{p-1}} * \cdots * M^1_{v^q_1}\right]_{\mathfrak{b}}$$

which maps $u^{q-1} * v_{p-1}^{q-1} * \cdots * v_1^{q-1}$ to $z^{q-1} * v_{p-1}^{q-1} * \cdots * v_1^{q-1}$.

It suffices to show that, if $X \in U(\mathfrak{b}')$ annihilates $u^{q-1} * v_{p-1}^{q-1} * \cdots * v_1^{q-1}$, then X also annihilates $z^{q-1} * v_{p-1}^{q-1} * \cdots * v_1^{q-1}$. We may assume that $X \in U(\mathfrak{b}')^s$ for some $s \in \mathbb{Z}_{\ge 0}$. Then there exists some $Y \in U(\mathfrak{b}')^{\le s-1}$ satisfying

$$(X - Y)(u^{q-1} \otimes {v_{p-1}^{q-1}}' \otimes \cdots \otimes {v_1^{q-1}}') = 0$$

by Lemma 8.5. Obviously,

$$(X - Y)(z^{q-1} \otimes {v_{p-1}^{q-1}}' \otimes \cdots \otimes {v_1^{q-1}}') = 0 \tag{9.5}$$

also holds. Let N be the unique integer such that

$$z^{q-1} \otimes {v_{p-1}^{q-1}}' \otimes \cdots \otimes {v_1^{q-1}}' \notin U(\mathfrak{b}')^{\le N-1}(u^q \otimes {v_{p-1}^q}' \otimes \cdots \otimes v_1^{q\prime}) \text{ and}$$
$$z^{q-1} \otimes {v_{p-1}^{q-1}}' \otimes \cdots \otimes {v_1^{q-1}}' \in U(\mathfrak{b}')^{\le N}(u^q \otimes {v_{p-1}^q}' \otimes \cdots \otimes v_1^{q\prime}),$$

and $Z_N \in U(\mathfrak{b}')^N$ and $Z_{\le N-1} \in U(\mathfrak{b}')^{\le N-1}$ be vectors such that

$$(Z_N + Z_{\le N-1})(u^q \otimes {v_{p-1}^q}' \otimes \cdots \otimes v_1^{q\prime}) = z^{q-1} \otimes {v_{p-1}^{q-1}}' \otimes \cdots \otimes {v_1^{q-1}}'.$$

Then we have from (9.5) that

$$\begin{aligned} &XZ_N(u^q \otimes {v_{p-1}^q}' \otimes \cdots \otimes v_1^{q\prime}) \\ &\quad = X(z^{q-1} \otimes {v_{p-1}^{q-1}}' \otimes \cdots \otimes {v_1^{q-1}}') - XZ_{\le N-1}(u^q \otimes {v_{p-1}^q}' \otimes \cdots \otimes v_1^{q\prime}) \\ &\quad = \big(Y(Z_N + Z_{\le N-1}) - XZ_{\le N-1}\big)(u^q \otimes {v_{p-1}^q}' \otimes \cdots \otimes v_1^{q\prime}). \end{aligned}$$

Since

$$XZ_N \in U(\mathfrak{b}')^{s+N} \text{ and } Y(Z_N + Z_{\le N-1}) - XZ_{\le N-1} \in U(\mathfrak{b}')^{\le s+N-1},$$

$XZ_N(u^q * v^q_{p-1} * \cdots * v^q_1) = 0$ holds by Lemma 8.5. On the other hand, we have by definition that

$$Z_N(u^q * v^q_{p-1} * \cdots * v^q_1) = z^{q-1} * v^{q-1}_{p-1} * \cdots * v^{q-1}_1.$$

Hence

$$X(z^{q-1} * v^{q-1}_{p-1} * \cdots * v^{q-1}_1) = 0$$

holds, and Claim 1 is proved.

Set $i = i_q$. By Lemma 8.4, the $\mathfrak{b}$-module $\Big[(V_q)_{u^q} * M^{p-1}_{v^q_{p-1}} * \cdots * M^1_{v^q_1}\Big]_{\mathfrak{b}}$ extends to a $\mathfrak{p}_i$-module. Then by Lemma 3.5 (ii), there exists a homomorphism of $\mathfrak{p}'_i$-modules

$$\widetilde{\Phi}\colon \mathcal{D}_i\Big[(V_{q-1})_{u^{q-1}} * M^{p-1}_{v^{q-1}_{p-1}} * \cdots * M^1_{v^{q-1}_1}\Big]_{\mathfrak{b}} \to \Big[(V_q)_{u^q} * M^{p-1}_{v^q_{p-1}} * \cdots * M^1_{v^q_1}\Big]_{\mathfrak{b}}$$

which makes the following diagram commutative:

$$\begin{array}{ccc} \Big[(V_{q-1})_{u^{q-1}} * \cdots * M^1_{v^{q-1}_1}\Big]_{\mathfrak{b}} & \xrightarrow{\Phi} & \Big[(V_q)_{u^q} * \cdots * M^1_{v^q_1}\Big]_{\mathfrak{b}}, \\ \downarrow & \nearrow_{\widetilde{\Phi}} & \\ \mathcal{D}_i\Big[(V_{q-1})_{u^{q-1}} * \cdots * M^1_{v^{q-1}_1}\Big]_{\mathfrak{b}} & & \end{array}$$

where the vertical map is the canonical one.

Claim 2. The homomorphism $\widetilde{\Phi}$ is surjective.

It suffices to show the image of $\widetilde{\Phi}$ contains the generator $u^q * v^q_{p-1} * \cdots * v^q_1$, whose $\mathfrak{h}'$-weight λ is equal to

$$\lambda = \mathrm{cl}\Big(w^q(\ell_p\Lambda_0) + \sum_{1\le j\le p-1} \overline{w}^q w_0(\ell_j\varpi_{r_j})\Big) \in P_{\mathrm{cl}}.$$

Note that the image of $\widetilde{\Phi}$ contains $z^{p-1} * v^{q-1}_{p-1} * \cdots * v^{q-1}_1$, whose $\mathfrak{h}'$-weight μ is

$$\mu = \mathrm{cl}\Big(w^{q-1}(\ell_p\Lambda_0) + \sum_{1\le j\le p-1} \overline{w}^{q-1} w_0(\ell_j\varpi_{r_j})\Big) = s_i(\lambda).$$

As the image is a $\mathfrak{p}'_i$-module, its weight set contains $s_i(\mu) = \lambda$. Since $\Big[(V_q)_{u^q} * \cdots * M^1_{v^q_1}\Big]_{\mathfrak{b}}$ is isomorphic to $V_q \otimes \cdots \otimes M^1$ as a $(\mathfrak{g}_0 \oplus \mathbb{C}K)$-module, it is easily checked that the weight space with weight λ is one-dimensional. Hence the image contains the generator, and Claim 2 is proved.

Now, the following claim completes the proof of the proposition:

Claim 3. The dimensions of the both sides of $\widetilde{\Phi}$ are equal.

The dimension of the right hand side is equal to

$$\dim V_q \times \prod_{1\le j\le p-1} \dim M^j. \tag{9.6}$$

Let us calculate the dimension of the left hand side. As stated at the beginning of this proof, $\left[V_{q-1} * M^{p-1} * \cdots * M^1\right]_{\mathfrak{b}}$ has a Demazure flag. Hence by Corollary 2.10, the character of the left hand side of $\widetilde{\Phi}$ is equal to

$$D_i \operatorname{ch} \left[V_{q-1} * M^{p-1} * \cdots * M^1\right]_{\mathfrak{b}}.$$

For a $\mathfrak{h}'$-semisimple module M whose $\mathfrak{h}'$-weight set is contained in P_{cl}, denote by $\overline{\operatorname{ch}}\, M \in \mathbb{Z}[P_{\mathrm{cl}}]$ the $\mathfrak{h}'$-character of M. Since each M^j is a finite-dimensional $(\mathfrak{g}_0 \oplus \mathbb{C}K)$-module on which K acts trivially, $\overline{\operatorname{ch}}\, M^j$ belongs to $\mathbb{Z}[P_{\mathrm{cl}}^0]$ and is W_0-invariant. Hence we have from Lemma 3.3 that

$$\begin{aligned}
\operatorname{cl} \circ D_i \operatorname{ch} \left[V_{q-1} * M^{p-1} * \cdots * M^1\right]_{\mathfrak{b}} &= \overline{D}_i\, \overline{\operatorname{ch}} \left[V_{q-1} * M^{p-1} * \cdots * M^1\right]_{\mathfrak{b}} \\
= \overline{D}_i \left(\overline{\operatorname{ch}}\, V_{q-1} \times \prod_{1 \le j \le p-1} \overline{\operatorname{ch}}\, M^j\right) &= \prod_{1 \le j \le p-1} \overline{\operatorname{ch}}\, M^j \times \overline{D}_i\, \overline{\operatorname{ch}}\, V_{q-1} \\
&= \prod_{1 \le j \le p-1} \overline{\operatorname{ch}}\, M^j \times \operatorname{cl} \circ D_i \operatorname{ch} V_{q-1}.
\end{aligned}$$

Since $D_i \operatorname{ch} V_{q-1} = \operatorname{ch} V_q$ by Theorem 2.9, we see that the dimension of the left hand side is equal to (9.6). Hence Claim 3 is proved, and the proof of the proposition is complete. □

As stated above, Theorem 6.1 is now established from Propositions 9.2 and 9.3.

Remark 9.8. In Theorem 6.1, it is assumed that $\mathfrak{g}_0$ is of *ADE* type. The author, however, expects the theorem to be true for general types if all given KR modules satisfy the assumption of Theorem 4.4. In fact, all the proof of the theorem can also be applied in this case, except for the Joseph's theorem (Theorem 2.11) which is needed in the final step of the proof. Hence to prove the theorem for non-simply laced type by our approach, it is needed to prove the Joseph's theorem for this type. Since Proposition 7.4 has already been proved for nonexceptional type, this would also imply the $X = M$ conjecture for perfect KR crystals of type $B_n^{(1)}$ and $C_n^{(1)}$.

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