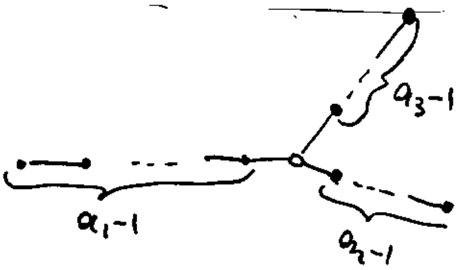
Affine Artin groups – 12.09. 2013

1. Introduction.

$$F(s,x) = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} + s_{01} + \frac{1}{Qe^{s_{02}}} x_1 x_2 x_3 + \sum_{\substack{1 \le i \le 3 \\ 1 \le e \le a_i - 1}} s_{ie} x_i^e$$

$$\chi := \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 1 > 0$$



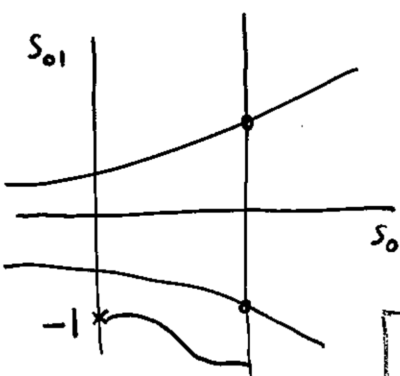
$s \in M := \mathbb{C}^\mu$

$\mu = a_1 + a_2 + a_3 - 1$

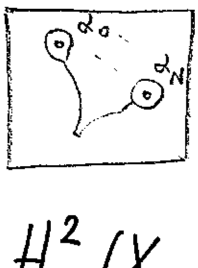
$\omega = \dfrac{dx}{Qe^{s_{02}}}$, $dx = dx_1 \wedge dx_2 \wedge dx_3$

primitive $\Rightarrow$ quantum cohom. of $\mathbb{P}^1_{a_1,a_2,a_3}$.

Put $X_{s,\lambda} = \{ x \in \mathbb{C}^3 : F(s,x) = \lambda \}$, $X_s = X_{s,0}$



discr. $= \{ s \in M : X_s$ is singular $\}$



simple affine roots

$\mathfrak{h}_{\mathbb{R}/\mathbb{C}} = H_2(X_{0,1}; \mathbb{R} \text{ or } \mathbb{C}) \supset \Delta$ vanishing cycles

$(\cdot \mid \cdot) = -$ (inters. pairing)

$\leadsto$ that $(\alpha|\alpha) = 2 \quad \forall \alpha \in \Delta$.

Δ is affine root system!

$$V_{\mathbb{R} \text{ or } \mathbb{C}} = \mathfrak{h}^\vee_{\mathbb{R} \text{ or } \mathbb{C}} = H^2(X_{0,1}; \mathbb{R} \text{ or } \mathbb{C})$$

$$\rho : \pi_1(M \setminus \text{discr.}) \longrightarrow GL(V_{\mathbb{R}})$$

$\uparrow$ gener. by $\gamma_0, \gamma_1, \dots, \gamma_N$

According to Picard-Lefschetz theory

$\mathrm{Im}(\rho) = W_{aff}$ afine Weyl group : $\rho(\gamma_i) = s_{\alpha_i}$

$$s_{\alpha_i}(x) = x - \langle x, \alpha_i \rangle \alpha_i^\vee \quad , \quad x \in V_{\mathbb{R}} \quad , \quad \alpha_i \in \Delta$$

$\rho(\gamma_i) = s_{\alpha_i} =: \sigma_i$, α_i : cycle vanishing along γ_i

W_{aff} is a <u>Coxeter</u> group : m_{ij} = order of $\sigma_i \sigma_j$

$$W_{aff} \cong \langle \sigma_0, \dots, \sigma_N \rangle / \langle \underset{\forall i}{\sigma_i^2 = 1} , \underbrace{\sigma_i \sigma_j \sigma_i \cdots}_{m_{ij}} = \underbrace{\sigma_j \sigma_i \sigma_j \cdots}_{m_{ij}} \rangle$$

m_{ij} factors m_{ij}

$\forall i, j$, s.t. $m_{ij} < \infty$

<u>Goal:</u> $\pi_1(M \setminus \text{discr.}) \cong \langle \gamma_0, \dots, \gamma_N \rangle / \langle \gamma_i \gamma_j \gamma_i \cdots = \gamma_j \gamma_i \gamma_j \cdots$

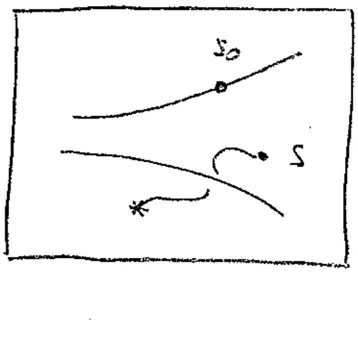
$\forall i,j$ s.t. $m_{ij} < \infty$

Affine Artin group

<u>Corollary:</u> $\mathrm{Ker}\,\rho$ = the normal subgroup gener. by γ_i^2 .

2. The period map.

M



$$I : M \setminus \text{discr.} \longrightarrow V_{\mathbb{C}} / W_{aff}$$

$$\langle I(s), \alpha \rangle = -\frac{1}{2\pi} \int_{\alpha_s \in H_2(X_s; \mathbb{C})} \frac{\omega}{dF}$$

$$\Gamma_\varepsilon = \{ |x_1| = |x_2| = 1 \ , |x_3| = \varepsilon \} \subset \mathbb{C}^3 \setminus X_{-1}$$

for $\varepsilon \gg 0$

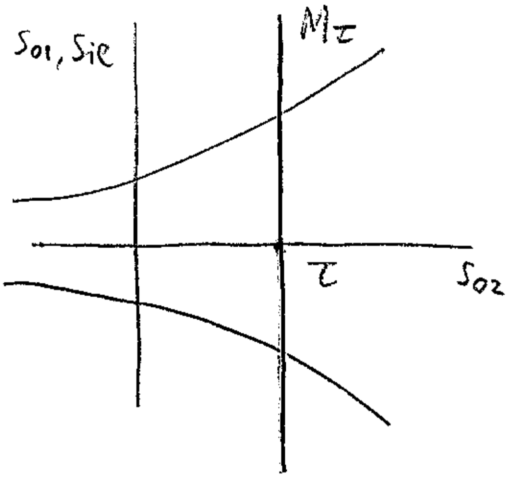
$$H_2(X_{0,1}; \mathbb{Z}) \cong H_3(\mathbb{C}^3 \setminus X_{0,1}; \mathbb{Z})$$

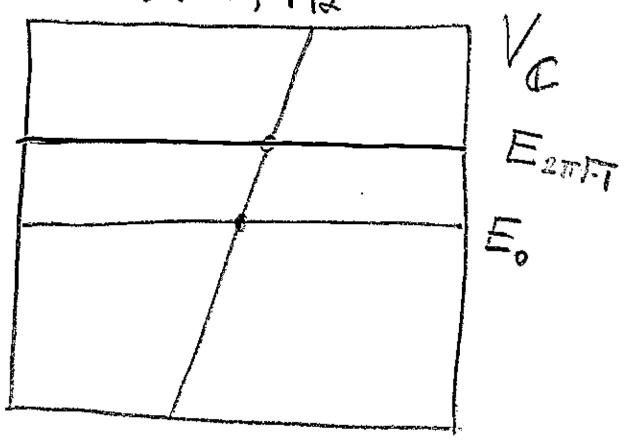
φ corresp. to $[\Gamma_\varepsilon]$

(1) $\langle I(s), \varphi \rangle = 2\pi\sqrt{-1}$

(2) φ is the imaginary root of Δ $\Rightarrow$ $(\alpha|\varphi)=0 \ \forall \alpha\in\Delta$

$\alpha\in\Delta$, $M_\alpha^{\mathbb{C}} = \{x : \langle x,\alpha\rangle = 0\}$



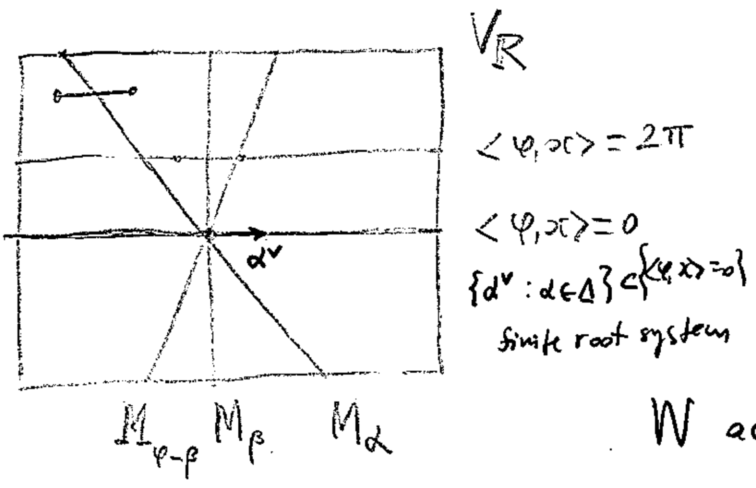


$E_\xi = \{x\in V_{\mathbb{C}} : \langle x,\varphi\rangle = \xi\}$

- $M_\tau \setminus$ discr. is a deform. retract of $M \setminus$ discr.
- $I : M_\tau \setminus \text{discr.} \xrightarrow{\cong} \left(E_{2\pi\sqrt{-1}} \setminus \{\bigcup_{\alpha\in\Delta} M_\alpha^{\mathbb{C}}\}\right)/W$

<u>Rem.</u> π_1 of this space was comp. by Nguyen Viet Dung

3. Chamber structure.



$H = \{x\in V_{\mathbb{R}} : \langle\varphi,x\rangle > 0\}$

$K(\mathcal{M},H)$ = conn. comp. of $H \setminus \{\bigcup_{\alpha\in\Delta} M_\alpha\}$

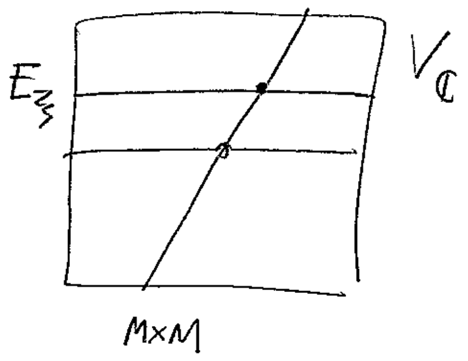
$\mathcal{M} = \{M_\alpha : \alpha\in\Delta\}$

W acts simply trans. on $K(\mathcal{M},H)$

i.e. $K(\mathcal{M},H) = \bigcup_{w\in W} w(C)$

Van der Lek:

$$Y = V \times H \setminus \left(\bigcup_{M \in \mathcal{M}} (M \times M) \right) \quad \subset \quad V_{\mathbb{C}} := V \times V$$



Let $\xi \in \mathbb{C}^*$, $\mathrm{Im}(\xi) > 0$

$$E_\xi = \{ x \in V_{\mathbb{C}} : \langle \varphi, x \rangle = \xi \}$$

$$Y_\xi = E_\xi \setminus \left(\bigcup_{M \in \mathcal{M}} (M \times M) \right)$$

W_{aff}-equivariant

$\Psi_s : Y \longrightarrow Y$, $\quad \Psi_s(z) = (1-s)z + \frac{s \cdot \xi}{\langle \varphi, z \rangle} \cdot z$

$0 \leq s \leq 1$ $\quad \because \quad \mathrm{Im} \langle \varphi, \Psi_s(z) \rangle = \mathrm{Im} \left((1-s) \langle \varphi, z \rangle + s \cdot \xi \right) > 0$

$\Psi_0 = id$, $\Psi_1 : Y \longrightarrow Y_\xi$ s.t. $\Psi_1 | Y_\xi = id_{Y_\xi}$

i.e. Ψ_s is a deform. retract

$\Rightarrow \quad \pi_1(Y) = \pi_1(Y_\xi)$ and $\pi_1(Y/W_{aff}) = \pi_1(Y_\xi / W_{aff})$.

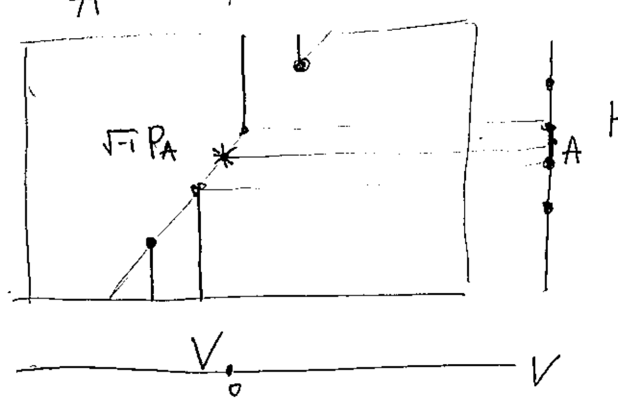
3. The fundamental groupoid of Y.

$V \supset H$, $\mathcal{M}$: set of hyperplanes w/ locally finite int. in H

$A \in K(\mathcal{M}) = K(\mathcal{M}, H)$: chambers of $H \setminus \bigcup_{M \in \mathcal{M}} M$

$$Y_A = V \times H \setminus \left(\bigcup_{M \in \mathcal{M}} M \times D_M^-(A) \right)$$

$H \setminus M = D_M^+(A) \sqcup D_M^-(A)$



open contractible subsets of Y.

$$Y = \bigcup_{A \in K(\mathcal{M})} Y_A$$

Proposition.

$$\left\{ \begin{array}{c} \text{Conn. comp. of} \\ Y_{A_1} \cap \cdots \cap Y_{A_n} \end{array} \right\} \text{ are in 1-to-1 corresp. w/ } \left\{ K(\mathcal{M}(A_1, \ldots, A_n)) \right\}$$

$\uparrow$ contractible!

hyperplanes $M \in \mathcal{M}$, s.t., M sep. A_i and A_j for some i and j

Pgal: groupoid w/ objects

$\mathrm{Obj} := K(\mathcal{M}, H)$ pregaleries

$$\mathrm{Mor}(A,B) = \left\{ G := C_0 \, {}^{U_1} C_1 \cdots {}^{U_n} C_n \; : \; C_0 = A, \; C_n = B, \; U_i \in K(\mathcal{M}(C_{i-1}, C_i)) \right\}$$

Π : groupoid w/ $\mathrm{Obj} = \sqrt{-1}\, P_A$, $A \in K(\mathcal{M})$

$\mathrm{Mor}(A,B) =$ homotopy classes of maps.

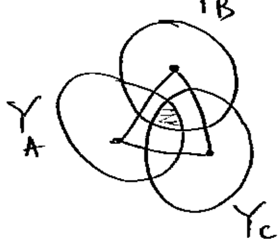
$\Rightarrow \; \mathrm{Pgal} \to \Pi$

$$\mathrm{Pgal}/\sim_R \;\cong\; \Pi$$

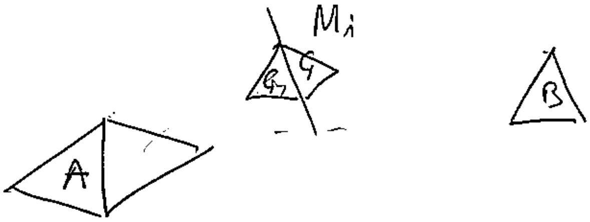
where $\sim_R$ is equiv. relation on Mor of Pgal gener. by

1) $A^{U'} B^{U''} C \sim_R A^U C$ for $U' \cap U'' \cap U \neq \emptyset$,

2) $A^V A = 1_A$.



4. Galeries



Tits galery:

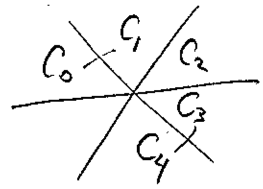
$A = C_0, C_1, \dots, C_n = B$ s.t. $|\mathcal{M}(C_{i-1}, C_i)| = 1$

i.e. $\mathcal{M}(C_{i-1}, C_i) = \{M_i\}$

$M_i \in \mathcal{M}$.

direct Tits galery: $M_i \neq M_j$ for $i \neq j$

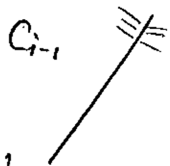
Tits galery, s.t.



Gal: groupoid w/ $\mathrm{Obj} = K(\mathcal{M}, H)$

$\mathrm{Mor}(A,B) = \{$pregalery $A \,{}^{U_1} C_1 \cdots {}^{U_{n-1}} C_{n-1} \,{}^{U_n} B$ s.t. $(A, C_1, \dots, C_{n-1}, B)$ is a Tits galery$\}$

called also galeries



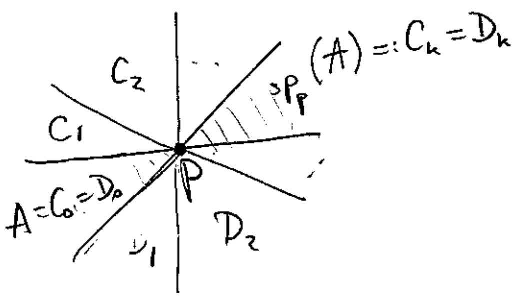
Remark. Since $\mathcal{M}(C_{i-1}, C_i) = \{M_i\}$

$U_i \in K(\{M_i\}) = \{D^+_{M_i}(C_{i-1}), D^-_{M_i}(C_{i-1})\}$ $\Rightarrow$ can replace U_i w/ + or −.

The plinth relation. $A \in K(\mathcal{M})$, chamber

codim. 1 facets are called walls

codim. 2 facets are called plinths



A, $sp_P(A)$ plinthed chambers

$\exists$ exactly two direct galeries $G_{A,P} = C_0^+ C_1^+ \cdots C_k^+$ and $G'_{A,P} = D_0^+ D_1^+ \cdots D_k^+$

from A to $sp_P(A)$

Let $\sim$ equivalence relation on Mor of Gal generated by $\forall$ $A \in K(M)$ and P plinth of A,

(1) $G \sim G'$

(2) $A^+ B^- A \sim 1_A$, $A^- B^+ A \sim 1_A$.

Thm [Van der Lek] If $G, G' \in \mathrm{Gal}_1 \dots$, then

$$G \underset{R}{\sim} G' \quad \text{iff} \quad G \sim G'.$$

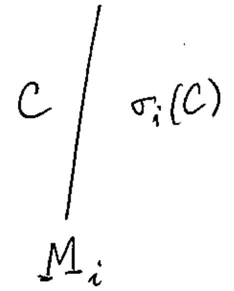
5. Computing $\pi_1(Y/W)$ in the affine case.

Y

$\downarrow \pi$ covering

$X := Y/W_{\mathrm{aff}}$

C : fundamental chamber, σ_i, $i = 0, 1, \dots, N$ refl. w.r.t. walls of C



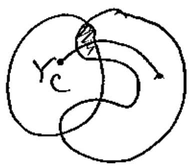
$S_i = C^+ \sigma_i(C) \in \mathrm{Gal}$

$\downarrow$

Pgal

$\downarrow$

Π



path $\tilde{\gamma}_i$ in Y

Put $\gamma_i = \pi_*(\tilde{\gamma}_i)$

May assume

$P_{w(C)} = w(P_C)$, $\forall w \in W_{\mathrm{aff}}$.

Assume that

$$\gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_k} = 1$$

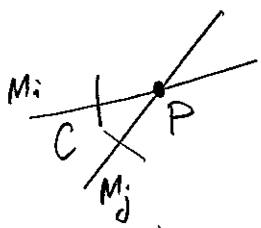
$$G = C^+ \sigma_{i_k}(C)^+ \sigma_{i_k}\sigma_{i_{k-1}}(C)^+ \cdots ^+ \sigma_{i_k} \cdots \sigma_{i_1}(C) = C$$

is a galery lift of $\gamma_{i_1} \cdots \gamma_{i_k}$

$\pi_* : \pi_1(Y) \longrightarrow \pi_1(X)$ is injective $\Longrightarrow G \underset{R}{\sim} 1_C$

i.e. $G \sim 1_C$

$$\begin{cases} G_0 = G \\ \vdots \\ G_i = \cdots \; C_0^+ C_1^+ \cdots ^+ C_k \cdots \\ G_{i+1} = \cdots \; D_0^+ D_1^+ \cdots ^+ D_k \cdots \\ \vdots \\ G_m = C^+ \sigma_i(C)^- C = 1_C \end{cases}$$



$C, \sigma_i(C), \sigma_i\sigma_j(C), \sigma_i\sigma_j\sigma_i(C), \dots$

$C, \sigma_j(C), \sigma_j\sigma_i(C), \sigma_j\sigma_i\sigma_j(C), \dots$

$\Downarrow$ via π

$\cdots \gamma_i \gamma_j \gamma_i = \cdots \gamma_j \gamma_i \gamma_j$

If $\sigma_i\sigma_j$ has order $m_{ij} < \infty$ then

$$\sigma_i\ ,\ \sigma_i\sigma_j\ ,\ \sigma_i\sigma_j\sigma_i\ ,\ \cdots\ ,\ \underbrace{\sigma_i\sigma_j\sigma_i\cdots}_{m_{ij}}$$

$$\sigma_j\ ,\ \sigma_j\sigma_i\ ,\ \sigma_j\sigma_i\sigma_j\ ,\ \cdots\ ,\ \underbrace{\sigma_j\sigma_i\sigma_j\cdots}_{m_{ij}}$$

are minimal (~~max~~ reduced) expressions

Easy to prove using the exchange condition.

$$\Rightarrow \quad C_0 + \cdots + C_k = C + \sigma_i(C) \cdots + (\sigma_i\sigma_j\sigma_i\cdots)(C)$$

$$D_0 + \cdots + D_k = C + \sigma_j(C) \cdots + (\sigma_j\sigma_i\sigma_j\cdots)(C)$$

$k = m_{ij}$

$$\pi_*[G_i] = \cdots (\underbrace{\gamma_i\gamma_j\gamma_i \cdots}_{k \text{ factors}}) \cdots$$

$$\pi_*[G_{i+1}] = \cdots (\underbrace{\gamma_j\gamma_i\gamma_j \cdots}_{k})$$

$$\pi_*[G_0] = \gamma_{i_1} \cdots \gamma_{i_k} \quad , \quad \pi_*[G_m] = 1 \ . \quad \boxtimes$$