Index Theory in Alg. K-Theory & Geometry

Joint w/ O. Braunling & M. Groechenig

k - field

$k((t))$ - k v.s. w/ t-adic topology

$k((t))$ is the basic example of a Tate v.s.

Tate v.s. arise naturally in geometry:

e.g. X - curve $/k$.

$x \in X$ closed point

$\Rightarrow \operatorname{Frac}(\widehat{\mathcal{O}}_{X,x})$ is a Tate v.s $/k$.

Def: A lattice L in a Tate v.s. V is a closed subspace s.t. $V/L \cong \bigoplus_I k$ as topological v.s.

E.g. $k[[t]] \subseteq k((t))$ has quotient $t^{-1}k[t^{-1}]$

M. Sato - integrable hierarchies (e.g. KP, KdV)

$\updownarrow$

flows on Grassmannian $Gr(k((t)))$

$\{\text{Points of } Gr(k((t)))\} = \{\text{lattices in } k((t))\}$

Fact: • $Gr(k((t)))$ is an ∞-dim'l Ind-scheme.

• $Gr(k((t))) \simeq Aut(k((t)))/Stab(k[[t]])$

In addition to connection to PDE,
$Gr(k((t)))$ also important for

- moduli of curves
- loop groups.

Goal of talk:

1) Discuss a dictionary due to Sato, Segal-Wilson
linear alg of $Gr(k((t)))$ $\longleftrightarrow$ classical functional analysis $/\mathbb{C}$

2) Present recent work w/ Braunling & Groechenig
extending this dictionary

3) Applications.

§1 Review of Fredholm Theory & Dictionary

H_i - Hilbert space $/\mathbb{C}$.

Def: A bounded linear operator $H_1 \xrightarrow{A} H_0$ is Fredholm if $\dim \ker A < \infty$ & $\dim \operatorname{coker} < \infty$

The index of A is the graded v.s.

$$\chi(H_1 \xrightarrow{A} H_0) := \ker A \oplus \operatorname{coker} A$$

Example: • (M,g) compact Riemannian mfld,

$\Rightarrow d + d^* : \Omega^{2*}(M) \longrightarrow \Omega^{2*+1}(M)$ is Fredholm

$$\operatorname{Index}(d+d^*) = H^{ev}(M) \oplus H^{odd}(M)$$

• (M,ω) - compact Kahler mfld

$\Rightarrow \bar\partial + \bar\partial^* : \Omega^{0,*}(M) \longrightarrow \Omega^{0,*}(M)$ is Fredholm

$$\operatorname{Index}(\bar\partial + \bar\partial^*) = H^{ev}_{\bar\partial}(M) \oplus H^{odd}_{\bar\partial}(M)$$

$\mathrm{Fred}(\mathcal{H})$ – space of Fredholm operators

$K^{top}_{\mathbb{C}}$ – (∞-loop) space representing topological complex K-theory

<u>Thm:</u> (Atiyah-Janich)
The assignment $A \longmapsto \mathrm{Index}(A)$ extends to a homotopy equivalence

$$\mathrm{Fred}(\mathcal{H}) \xrightarrow[\simeq]{\mathrm{Index}} K^{top}_{\mathbb{C}}$$

AJ gives starting point for Atiyah-Singer Index Thm.

k	$\mathbb{C}$
Tate v.s. $k((t))$	Hilbert space $L^2(S^1, \mathbb{C})$
Lattice $k[[t]] \subseteq k((t))$	Polarization $L^2_+(S^1, \mathbb{C}) \subseteq L^2(S^1, \mathbb{C})$ fns which extend holomorphically to $\mathbb{D}$
Sato Grassmannian $\mathrm{Gr}(k((t)))$ $\circlearrowright \uparrow$ $\mathrm{Aut}(k((t)))$? $\vdots$ $\mathbb{K}_k$	Segal-Wilson Grassmannian $\mathrm{Gr}(\mathcal{H}, \mathcal{H}^+)$ $\circlearrowright \uparrow \simeq$ "$GL_{res}(\mathcal{H}, \mathcal{H}^+)$" $\simeq \mathrm{Fred}(\mathcal{H})$ $\wr\wr \downarrow$ $K^{top}_{\mathbb{C}}$

§2. The Index Map

Dictionary is very strong.

Q: How are lattices $L \subseteq k((t))$ like Fredholm operators?

A: $\forall\, L \in Gr(k((t)))$, $L \xrightarrow{\pi_{k[\![t]\!]}} k[\![t]\!]$ has finite dim$^{\text{l}}$ kernel & cokernel.

Similarly, $\forall\, g \in Aut(k((t)))$, $g\,k[\![t]\!]$ is a lattice $\Rightarrow$ $g\,k[\![t]\!] \xrightarrow{\pi_{k[\![t]\!]}} k[\![t]\!]$ is Fredholm.

$\rightsquigarrow$

Q: algebraic analogue of AJ?

As (simplicial) sets, map should be

$$Aut(k((t))) \to Gr(k((t))) \longrightarrow \mathbb{K}_k$$

$$g \longmapsto g\,k[\![t]\!] \longmapsto \chi\left(g\,k[\![t]\!] \xrightarrow{\pi_{k[\![t]\!]}} k[\![t]\!]\right)$$

But, we want:

1) map which respects geometry, i.e. map of (simplicial) presheaves

2) a map which is an equivalence as w/ AJ.

So, let's do better.

For 1), we need to consider families (of Tate v.s. & their automorphisms)

X - scheme / k.

Using ideas of Beilinson & Drinfeld, can define category

$\mathrm{Tate}(X)$ - Tate vector bundles on X.

$\begin{pmatrix} E \\ \downarrow \\ X \end{pmatrix} \in \mathrm{Tate}(X) \Rightarrow E|_{\mathrm{Spec}(R) \in X}$ is a top^{l} direct summand of $P \oplus Q^{\vee}$

where $P, Q \in \mathrm{Proj}(R)$ & $Q^{\vee} = \mathrm{Hom}(Q, R)$

Category $\mathrm{Tate}(X)$ has K-theory

$\mathbb{K}_{\mathrm{Tate}(X)}$

Construction of K-Theory guarantees that

$\forall E \in \mathrm{Tate}(X)$, $\exists$ a canonical map

$$\mathrm{Aut}(E) \longrightarrow \Omega \mathbb{K}_{\mathrm{Tate}(X)}$$

<u>Thm</u>: (BGW)

X - scheme. $\exists$ a natural equivalence

$$\Omega \mathbb{K}_{\mathrm{Tate}(X)} \xrightarrow{\text{Index}} \mathbb{K}_X$$

s.t. $E \in \mathrm{Tate}(X)$, $L \in \mathrm{Gr}(E)$, we have

$$\mathrm{Aut}(E) \longrightarrow \Omega \mathbb{K}_{\mathrm{Tate}(X)} \longrightarrow \mathbb{K}_X$$

$$g \longmapsto \chi(gL \xrightarrow{\pi_L} L)$$

Rmk: The inverse $\mathbb{K}_X \longrightarrow \Omega\mathbb{K}_{\mathrm{Tate}(X)}$ also admits a natural description.

We have a square

$$\begin{tikzcd} \mathbb{K}_X \arrow[r] \arrow[d] & \mathbb{K}_{\mathrm{Ind}(X)} \arrow[d] \\ \mathbb{K}_{\mathrm{Pro}(X)} \arrow[r] & \mathbb{K}_{\mathrm{Tate}(X)} \end{tikzcd}$$

Eilenberg Swindle $\Rightarrow$ $\mathbb{K}_{\mathrm{Pro}(X)} \simeq \mathbb{K}_{\mathrm{Ind}(X)} \simeq *$

$\Rightarrow$ square determines $\mathbb{K}_X \xrightarrow{ES} \Omega\mathbb{K}_{\mathrm{Tate}(X)}$

Thm: (S. Saito '13)

ES is an equivalence.

§3. Applications

Now that we have an analogue of Atiyah-Janich, what can we do?

3 applications for today:

1) Cohomology of moduli space
2) Representation theory of n-fold loop groups

$\Big\}$ early stages

3) Reciprocity laws.

1) The Index map

$$\mathrm{Gr}(k((t))) \longrightarrow \mathbb{K}_k$$

gives rise to cohomology classes on $\mathrm{Gr}(k((t)))$.

Example: $V \longmapsto \Lambda^{top}(V)\,[\dim V]$

f. dim^l v.s$/k$

induces $\mathbb{K}_k \xrightarrow{\mathrm{Det}} \mathrm{Pic}_k^{\mathbb{Z}}$

groupoid of graded lines

$$\mathrm{Gr}(k((t))) \longrightarrow \mathbb{K}_k \xrightarrow{\mathrm{Det}} \mathrm{Pic}_k^{\mathbb{Z}}$$

determines a graded line bundle

$$\mathcal{L}_{\mathrm{Det}} \longrightarrow \mathrm{Gr}(k((t))) \qquad (\text{Sato–Sato '83})$$

$c_1(\mathcal{L}_{\mathrm{Det}}) \in H^*(\mathrm{Gr}(k((t))))$.

Other classes from c_i for $i > 1$.

For $\mathrm{Gr}(\mathcal{H}, \mathcal{H}^+)$, these generate the cohomology ring:

$$H^*(\mathrm{Gr}(\mathcal{H}, \mathcal{H}^+)) \cong \mathbb{Z}[c_i], \quad |c_i| = 2i.$$

Q: Same true for $\mathrm{Gr}(k((t)))$?

May be hard.

Warm up: study c_i by restricting to subschemes.

(Krichever)

$$(C, p, t) \longmapsto \hat{\mathcal{O}}_{C,p} \subseteq \mathrm{Frac}(\hat{\mathcal{O}}_{C,p})$$

curve, point, formal coord at p

$$\begin{array}{ccc} \hat{\mathcal{O}}_{C,p} & \subseteq & \mathrm{Frac}(\hat{\mathcal{O}}_{C,p}) \\ \downarrow \wr t & & \downarrow \wr t \\ k_p[[t]] & \subseteq & k_p((t)) \end{array}$$

lattice in $k((t))$ as Tate v.s.

Extends to

$$\left(\mathcal{M}_{g,1,t} \xleftrightarrow{\kappa} \mathrm{Gr}(k((t))) \right) \quad \left(\text{equiv. w.r.t. Virasoro}\right)$$

Analogue for Riemann surfaces $\leadsto \mathrm{Gr}(\mathcal{H}, \mathcal{H}^+)$

<u>Thm</u>: (Lion, Schwarz (3))

Over $\mathbb{C}$, $\quad \kappa^*(H^*_{Vir}(\mathrm{Gr}(\mathcal{H}, \mathcal{H}^+), \mathbb{C})) =$ Tautological ring of $\mathcal{M}_{g,n}$

Analogous statement over k?

2) (M. Sato)

$$\mathrm{Aut}(k((t))) \; - \; \infty\text{-dim}^{l} \text{ general linear group.}$$

$$= \left\{ g \in GL_k(k((t))) \;\middle|\; [g, \pi_{k[[t]]}] \text{ is finite rank} \right\}$$

$\mathrm{Aut}(k((t))) \circlearrowright \mathrm{Gr}(k((t)))$ does <u>not</u> lift to $\mathcal{L}_{Det} \longrightarrow \mathrm{Gr}(k((t)))$.

$$\widetilde{\mathrm{Aut}}(k((t))) := \left\{ \text{autons of } \mathcal{L}_{Det} \text{ which lift action of } \mathrm{Aut}(k((t))) \text{ on } \mathrm{Gr}(k((t))) \right\}$$

G - matrix group. We have

$$G(k((t))) \hookrightarrow \mathrm{Aut}(k((t)))$$

$$\widetilde{G}(k((t))) := \widetilde{\mathrm{Aut}}(k((t)))\big|_{G(k((t)))}$$

is the Kac-Moody extension of G.

$\Gamma(\mathcal{L}_{\mathrm{Det}})$ is the basic representation.

Index map gives rise to

Analogous Picture for $k((t_1))\cdots((t_n))$:

Have $Gr(k((t_1))\cdots((t_n)))$

Index map now gives

$$Gr(k((t_1))\cdots((t_n))) \xrightarrow{\text{Index}} B^{n-1}K_k \xrightarrow{B^{n-1}\mathrm{Det}} B^{n-1}\mathrm{Pic}_k^{\mathbb{Z}}$$

$\Big\Downarrow$ This classifies determinantal

$$(n-1)\text{-gerbe} \to \mathcal{L}_{B^{n-1}\mathrm{Det}} \longrightarrow Gr(k((t_1))\cdots((t_n)))$$

$$\widetilde{\mathrm{Aut}}(k((t_1))\cdots((t_n))) = \left\{ \begin{array}{l} \text{automs of } \mathcal{L}_{B^{n-1}\mathrm{Det}} \\ \text{lifting action of} \\ \mathrm{Aut}(k((t_1))\cdots((t_n))) \end{array} \right\}$$

This is an algebraic n-group.

G - matrix group $\rightsquigarrow$ Kac-Moody

$$\tilde{G}(k((t_1))\cdots((t_n))) := \widetilde{\mathrm{Aut}}(-)|_{G(-)}$$

w/ basic representation

$$\Gamma(\mathcal{L}_{B^{n-1}\mathrm{Det}})$$

Much left to understand.

$n=2$: initial results by
Frenkel-Zhu '12
w/o geometric construction.

$n>2$: ?

3) X - curve $/k$.

For $x \in X$, $f, g \in \mathrm{Frac}(\hat{\mathcal{O}}_{X,x})^\times$, the tame <u>symbol</u>

$$\{f, g\}_x := N_x\left((-1)^{(\mathrm{ord}_x f)(\mathrm{ord}_x g)} \frac{f^{\mathrm{ord}_x g}}{g^{\mathrm{ord}_x f}}\right) \in k^\times$$

<u>Thm</u>: (Weil '40)

X - <u>proper</u> curve$/k$.

f, g - nonzero rat$^{\underline{l}}$ fns on X.

Then $\prod_{x \in |X|} \{f, g\}_x = 1$

closed pts.

70s - 80s — Parshin – Kato established analogous reciprocity laws where

proper curve X/k $\rightsquigarrow$ n-dim'l scheme X, almost complete flag $\{$

$$Z_0 \subset \cdots \subset Z_{i-1} \subset - \subset Z_i \subset \cdots \subset Z_{n-1}$$

$(Z_j \subseteq X$ closed, $\dim Z_j = j)$.

1987 - Arbarello, de Concini, Kac deduced Weil reciprocity from properties of

$$\mathcal{L}_{Det} \longrightarrow Gr(k((t))).$$

ACK: Interpret $\prod_{x \in |X|} \{-,-\}_x$ as cocycle specifying a central extension of idèles

$$k^\times \longrightarrow \widetilde{\mathbb{A}(X)}^\times \longrightarrow \mathbb{A}(X)^\times$$

Weil reciprocity $\Longleftrightarrow$ $\widetilde{\mathbb{A}}(X)^\times \big|_{K(X)^\times}$ splits.

ADK now relate $\widetilde{\mathbb{A}}(X)^\times$ to $\mathcal{L}_{Det}$ & use properties of $\mathcal{L}_{Det} \longrightarrow Gr(\mathbb{A}(X))$ to deduce splitting.

Take v.s.

Index map allows us to use these ideas to prove Parshin – Kato reciprocity.

PK for surfaces:

X - 2 dim'l scheme

$x \in X$ closed point

$C \subseteq X$ - curve.

$K_{C,x}$ - 2 dim'l local field at $(x \in C \subseteq X)$

(e.g. $k_x((s))((t))$)

residue field at x.

$\exists$ 2-dim'l tame symbol

$$\{-,-,-\}_{C,x} : K_{C,x}^{\times 3} \longrightarrow k_x^{\times}$$

Thm: (Parshin - Kato)

X - 2 dim'l scheme

$x \in X$ - closed pt.

f, g, h - rational fns. not vanishing at x

$$\prod_{x \in C \subseteq X} \{f, g, h\}_{C,x} = 1 \in k_x^{\times}$$

proof: (Osipov - Zhu '11) interpret

$\prod_{x \in C \subseteq X} \{-,-,-\}_{C,x}$ as a cocycle

specifying a central 2-extension

$$\mathrm{Pic}_k^{\mathbb{Z}} \longrightarrow \widetilde{\mathbb{A}_x^2(X)} \longrightarrow \mathbb{A}_x^2(X)$$

2-group

2 dim'l adèles supported at $x \in X$

PK reciprocity $\iff$ $\widetilde{\mathbb{A}^2_x(X)}\Big|_{K(X)}$ splits.

OZ's proof amounts to deducing the splitting from properties of $\mathcal{L}_{BDet} \to Gr(\mathbb{A}^2_x(X))$ ← 2-Tate v.s.

N.B. OZ's args are categorical.

Index map allows us to present them geometrically.

Geometric presentation allows us to extend args straightforwardly to higher dim'l PK w/o having to pass explicitly through n-categories!

These arguments

N.B. Also have v. natural formulations in alg. K-theory.

This viewpoint reduces all of above to consequences of Thomason-Trobaugh's Localization Thm.

∴ There is a lot of geometry & functional analysis built into the localization Thm!