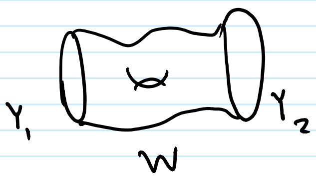
0. Overview

(Seiberg-Witten) monopole Floer homology

$Y^3 \rightsquigarrow HM(Y)$ Abelian groups invariant



$\rightsquigarrow HM(Y_1) \xrightarrow{HM(W)} HM(Y_2)$

- Equivalent to HF, ECH (Kutluhan, Lee, Taubes '10)
- Help prove many deep results in low-dim. topology

Big goal Lift the above picture to stable homotopy category

i.e.

$$\begin{array}{ccl} & \rightsquigarrow & SWF(Y) \quad \text{space / stable object} \\ Y & & \downarrow H_* \\ & \rightsquigarrow & HM(Y) \end{array}$$

Related application

- (Lipshitz + Sarkar '12) Construct a Khovanov homotopy type

$$\begin{array}{ccl} & \rightsquigarrow & \mathcal{X}(L) \\ \text{Link} & & \downarrow H_* \qquad \text{distinguish knots} \\ & \rightsquigarrow & Kh(L) \qquad \text{with the same } Kh \end{array}$$

- (Manolescu '13) Use SWF to develop Pin(2)-Floer homology
  $\Rightarrow$ solve the Triangulation Conjecture

1. Basic tools

1.1 Morse homology M closed, Riem. manifold (fin. dim)
f Morse function

(C_*, ∂) generated by critical points

$a \in C_k$, $\partial a = \sum_{b \in C_{k-1}} n(a,b)\, b$

$a \in C_k$, $\quad \partial a = \sum_{b \in C_{k-1}} n(a,b)\, b$ $\quad \nwarrow$ # flow lines $a \to b$

Then $\quad H_*(C_*, \partial) \simeq H_*(M)$

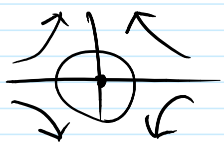
Q: How to retrieve homotopy data

1.2 Conley theory $\quad$ Let φ be a flow on M

$X \subset M$ compact is called an isolating neighborhood if

$$\mathrm{Inv}(X) := \{x \in X : \varphi(x,t) \in X \ \forall t \in \mathbb{R}\} \subset \mathrm{Int}(X)$$

Ex. i)



ii) φ = gradient flow of f

X consists of crit. points and flow lines between them

Fundamental results:

- For an isolating nbhd X, there exists an "index pair" (N, L).
- All such pairs are homotopy equivalent: $N_1/L_1 \simeq N_2/L_2$
- Conley index $\quad \mathcal{I}(X, \varphi) := [N/L]$

1.3 Finite dimensional approximation

Problem in Floer theory: Spaces are infinite-dimensional

Let H be a Hilbert space with a flow φ

$X \subset H$ closed, bounded s.t. $\mathrm{Inv}(X) \subset \mathrm{Int}(X)$

Suppose φ comes from a vector field $F : H \to H$

i.e. $\quad -\frac{\partial}{\partial t}\varphi(x,t) = F(\varphi(x,t))$

Idea: Consider a projection of F on a subspace V

i.e. a flow φ_V induced by $\pi_V F$ on V

If F is nice, then

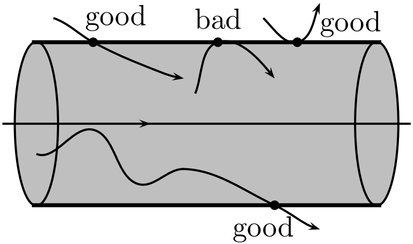
$\mathrm{Inv}(X \cap V) \subset \mathrm{Int}(X \cap V)$ for "large" V

We obtain $\mathcal{I}(X \cap V, \varphi_V)$ Conley indices for a collection of V

They fit together and form a spectrum

Isolating blocks

An ***isolating block*** is a special isolating neighborhood such that every point on its boundary leaves the neighborhood immediately in one or another time direction.



Useful Lemmas

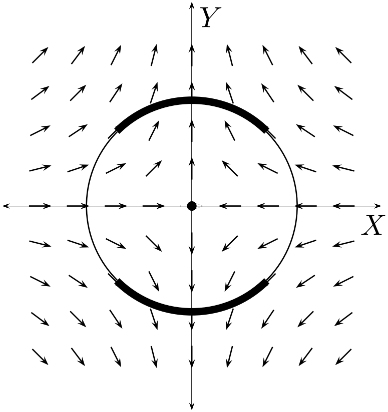
- An isolating block always exists
- For an isolating block N, define its **exit set** n^- to be the set of points on the boundary that leave N immediately.

$$n^- := \{x \in \partial N \mid \exists\, \delta_0 > 0 \text{ such that } x \cdot (0, \delta_0) \cap N = \emptyset\},$$

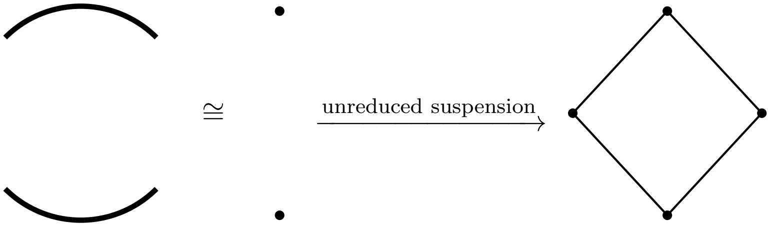
- Then, (N, n^-) is an index pair and $\mathcal{I}(N) = [N/n^-]$
- If Y is contractible, then Y/Z is homotopy equivalent to the unreduced suspension $\mathbf{S}Z$ of Z.

Example

Consider a flow on $\mathbb{R}^2$ given by $(x, y) \cdot t = (2^{-t}x, 2^t y)$. The disk around the origin $(0, 0)$, which is a hyperbolic fixed point, is an isolating block.



Example (Continued)



The Conley index is (homotopy type of) the circle S^1.

Linear flow

Consider a flow given by a bounded, self-adjoint, linear L with no kernel on a Hilbert space.

- Let X be a unit ball in H. Then X is an isolating block.
- Pick a finite-dimensional eigenspace V of L. Similarly, the Conley index $\mathcal{I}(X \cap V)$ is S^{V^-} the sphere with dimension equal to the number of negative eigenvalues in V.

Stable Conley index

It turns out that we can define $E(X)$ as

$$\Sigma^{-V^-}\mathcal{I}(X \cap V) \cong \Sigma^{-W^-}\mathcal{I}(X \cap W) \cong S^0$$

Remark

- Conley theory can be considered as a generalization of Morse theory.
- If a group G acts on M and the flow is G-equivariant, then we can define an equivariant version of Conley indices.
- Seiberg-Witten Floer theory has $U(1)$-symmetry.

2. Seiberg-Witten Theory

Reference

Kronheimer and Mrowka, *Monopoles and three-manifolds*.

Let Y be a closed, oriented, Riemannian 3-manifold equipped with a spin^c structure $\mathfrak{s}$.

- This also means Y is given a unitary rank-two “spinor” bundle $S \to Y$ with Clifford multiplication

$$\rho : TY \to Hom(S, S)$$

Configuration space

Definition

- The ***configuration space*** $\mathcal{C}(Y,\mathfrak{s}) := \mathcal{A}(Y) \oplus \Gamma(S)$ consists of pairs of a spin^c connection and a section of the spinor.
- The ***gauge group*** $\mathcal{G} := \mathrm{Map}(Y, S^1)$ acts on $\mathcal{C}(Y,\mathfrak{s})$ by

$$u \cdot (B, \Psi) = (B - u^{-1}du, u\Psi).$$

- The ***quotient configuration space*** is $\mathcal{B}(Y,\mathfrak{s}) := \mathcal{C}(Y,\mathfrak{s})/\mathcal{G}$

Seiberg-Witten vector field

- The Chern-Simons-Dirac functional is given by

$$CSD(B,\Psi) = -\frac{1}{8}\int_Y (B^t - B_0^t)\wedge(F_{B^t} + F_{B_0^t}) + \frac{1}{2}\int_Y \langle D_B\Psi, \Psi\rangle$$

- The Seiberg-Witten vector field is the gradient of the CSD

$$\nabla CSD(B,\Psi) = (\frac{1}{2} * F_{B^t} + \tau(\Psi), D_B\Psi)$$

- Roughly speaking, monopole Floer homology is a semi-infinite S^1-equivariant Morse homology of $\mathcal{B}(Y,\mathfrak{s})$ using CSD as the Morse function.
- The equation $\nabla CSD = 0$ is the (3D) Seiberg-Witten equation. Critical points are moduli space of the solutions (and they are compact!).

The Coulomb slice

In general, $\mathcal{B}(Y, \mathfrak{s})$ is not a vector space, but a Hilbert bundle over the Picard torus

$$\mathbb{T}^{b_1} = H^1(Y; i\mathbb{R})/(2\pi i H^1(Y; \mathbb{Z})).$$

To apply finite dimensional approximation, we instead consider

Definition

The Coulomb slice $\mathcal{K} := \ker d^* \oplus \Gamma(S) \subset \Omega^1(Y; i\mathbb{R}) \oplus \Gamma(S)$

- This is a universal cover

$$\mathbb{Z}^{b_1} \to \mathcal{K} \to \mathcal{B}(Y, \mathfrak{s}).$$

Outline of the construction of SWF

- $\mathcal{K}$ is a vector space, but we lose compactness.
- We can consider a certain collection $\mathcal{R}_k$ of enlarging closed and bounded subsets of $\mathcal{K}$.
- On each $\mathcal{R}_k$, we can find its stable Conley index $E(\mathcal{R}_k)$.
- Finally, we take limit

$$SWF(Y) := \varinjlim E(\mathcal{R}_k).$$

Previous works

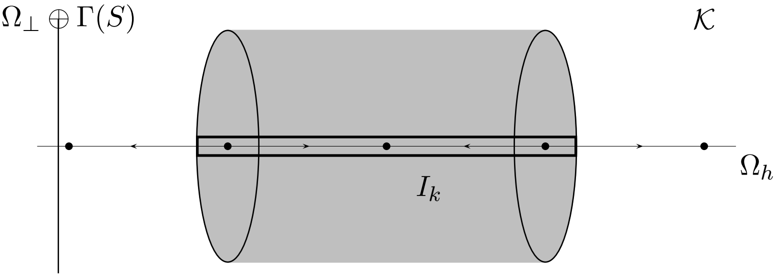
- (1992) Cohen, Jones, and Segal started studying about existence of “Floer homotopy type”.
- (1999) The stable Conley index on Hilbert spaces was developed by Geba, Izydorek, Pruszko
- (2003) In the Seiberg-Witten case, Manolescu successfully constructed Floer homotopy type for 3-manifolds with $b_1 = 0$.
- (2003) Kronheimer and Manolescu extended the construction to 3-manifolds with $b_1 = 1$ with nontorsion spin^c structure.

3. Computation

- We will give an explicit computation for when $Y = T^3$ with a trivial spin^c structure.
- The computation applies to 3-manifolds with nonnegative scalar curvature.
- Using Hodge decomposition, we have $\ker d^* = \Omega_h \oplus \Omega_\perp$, where Ω_h is a space of harmonic 1-forms isomorphic to $H^1(Y; i\mathbb{R}) \cong \mathbb{R}^{b_1}$.

Tubular neighborhoods as isolating neighborhoods

- In this case, all critical points and flow lines lie on Ω_h
- The subset we consider will be a tubular neighborhood $\nu(\epsilon)$ of a subset $I_k \subset \Omega_h \cong \mathbb{R}^3$, i.e. $\mathcal{R}_k = I_k \times B(\Omega_\perp \oplus \Gamma(S), \epsilon)$



Trick: linearizing a vector field

- We can freely scale the radius of tubular neighborhoods. This, in turn, scales the vector field.
- We also use the fact that the (perturbed) vector field can be written as $L + Q$ where L is linear and Q is quadratic.

$$
\begin{array}{llcl}
\text{Isolating neighborhood :} & \nu(\epsilon) & \xrightarrow{\text{Scaling}} & \nu(1) \\
\text{Vector field :} & L(h)v + \pi Q v & \xrightarrow{\text{Scaling}} & L(h)v + \epsilon\,\pi Q v \\
\text{Approximating} & \big\downarrow & & \big\downarrow \\
 & L(h)v + Qv & & L(h)v
\end{array}
$$

The linearized vector field

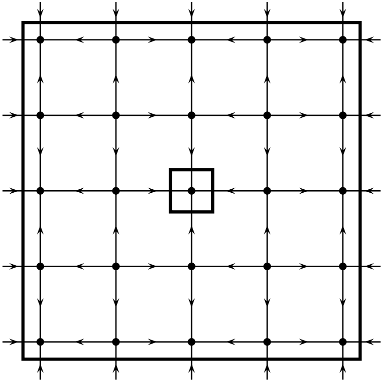
- The linearized vector field on $\Omega_h \times \Gamma(S)$ has a form

$$(h, \psi) \mapsto (\epsilon \nabla f, (D_{B_0+b_h} - \delta)\psi),$$

  where δ, ϵ are sufficiently small positive numbers (the choice of ϵ depends on δ) and f is a Morse function on Ω_h.
- We pick $f(\theta_1, \theta_2, \theta_3) = -\cos\theta_1 - \cos\theta_2 - \cos\theta_3$.

The flow on Ω_h

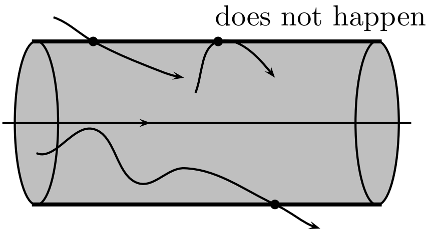
- ∇f induces a flow on Ω_h.
- For subsets on Ω_h, we will consider the cube $I_k = [-(2k+\frac{1}{2})\pi, (2k+\frac{1}{2})\pi]^3$, which is an isolating block with no exit set.



A tubular neighborhood as an isolating block.

Claim

The tubular neighborhood $\nu(I_k)$ is an isolating block.



- One can check directly that the norm $\|v(t)\|^2$ has no local maximum using the second derivative test.
- Consequently, we only need to understand the exit set.

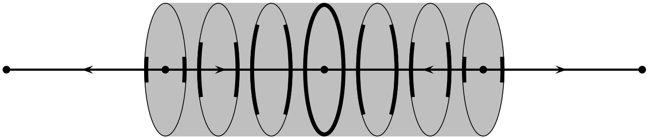
Describing the exit set

From the first derivative of the norm $\|v(t)\|^2$, we can deduce

Exit set of the tubular neighborhood

The exit set of $\nu(I_k)$ is given by the set

$$\{(h,v) \in I_k \times S(1) \,|\, \langle L(h)v, v\rangle \leq 0\}.$$



- We can view this as a union of unit nonpositive cones $\mathrm{Cone}_{\leq}(V, L(h)) = \{v \in S(1) \,|\, \langle L(h)v, v\rangle \leq 0\}$ along fibers.

The spinor bundle

- The spinor bundle is the trivial bundle $T^3 \times \mathbb{C}^2$.
- Using Fourier transform, we have a decomposition

$$\Gamma(S) = \bigoplus_{n_1,n_2,n_3 \in \mathbb{Z}} V_{n_1,n_2,n_3},$$

  each V_{n_1,n_2,n_3} is a 2-dimensional complex vector space spanned by $(e^{i(n_1\theta_1+n_2\theta_2+n_3\theta_3)}, 0)$ and $(0, e^{i(n_1\theta_1+n_2\theta_2+n_3\theta_3)})$.
- Note that the S^1-action comes from multiplication on $\Gamma(S)$ by unit complex numbers.

The Dirac operator

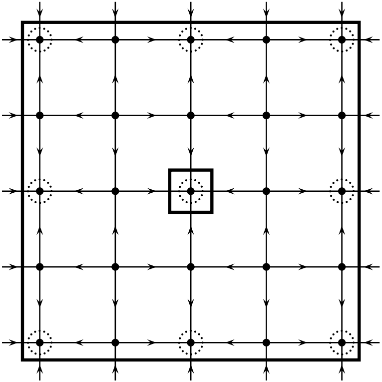
- For $\vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$, the operator $D_{\vec{b}}$ on V_{n_1,n_2,n_3} is given by a matrix

$$\begin{bmatrix} -(n_1 + \frac{b_1}{2\pi}) - \delta & -(n_3 + \frac{b_3}{2\pi}) - (n_2 + \frac{b_2}{2\pi})i \\ -(n_3 + \frac{b_3}{2\pi}) + (n_2 + \frac{b_2}{2\pi})i & n_1 + \frac{b_1}{2\pi} - \delta \end{bmatrix}.$$

- The eigenvalues of the above matrix is $-\delta \pm \sqrt{(n_1 + \frac{b_1}{2\pi})^2 + (n_2 + \frac{b_2}{2\pi})^2 + (n_3 + \frac{b_3}{2\pi})^2}$ with eigenvectors $\begin{bmatrix} -b_1 + R \\ -b_3 + b_2 i \end{bmatrix}$ and $\begin{bmatrix} -b_1 - R \\ -b_3 + b_2 i \end{bmatrix}$ respectively.
- The gauge symmetry can be seen as the matrix of $D_{\vec{b}+2\pi\vec{u}}$ on $V_{\vec{n}}$ is the same as the matrix of $D_{\vec{b}}$ on $V_{\vec{n}+\vec{u}}$.

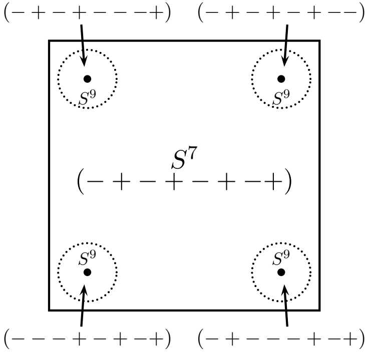
The kernel of $D_{\vec{b}}$

- $D_{\vec{b}}$ has kernel on a small sphere centered at centered at $2\pi(b_1, b_2, b_3)$, which is a critical point of index 0.



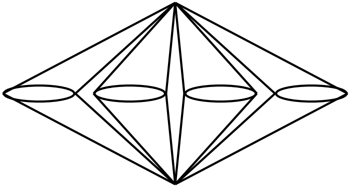
Warm-up (1-dim down)

- Consider a tubular neighborhood of a square $[-\frac{1}{2}, 2\pi + \frac{1}{2}]^2 \times \{0\}$ with respect to a subspace $V = \bigoplus_{i,j=0,1} V_{-i,-j,0}$.
- Look at signature of Dirac operators on V.



Conclusion

- After desuspension, the stable Conley index $E(\nu(I_k))$ is an " S^0-sum" of $(2k+1)^3$ copies of S^2.



- Equivariantly, the 2-sphere is $S^{\mathbb{C}}$

Induced map on Conley indices

- The inclusion $\nu(I_k) \subset \nu(I_{k+1})$ induces a map between Conley indices as well as their associated spectra.

$$E(\nu(I_k)) \to E(\nu(I_{k+1})).$$

- The inclusion $\nu(I_k) \subset \nu(I_{k+1})$ also induces an inclusion of their exit sets

The colimit of spectra

$SWF(T^3)$ is the S^0-sum of $\mathbb{Z}^3$ copies of $S^{\mathbb{C}}$.

4. Homology

- Monopole Floer homology comes with three flavors $\overline{HM}, \widecheck{HM}, \widehat{HM}$ related by a long exact sequence

$$\ldots \to \overline{HM}(Y) \to \widecheck{HM}(Y) \to \widehat{HM}(Y) \to \ldots$$

- There are three S^1-equivariant homology theories: the Borel homology $H_*^{S^1}$, the coBorel homology $\widehat{H}_*^{S^1}$, and the Tate homology $\bar{H}_*^{S^1}$.

Conjecture

These S^1-equivariant homology groups of $SWF(Y)$ agree with the monopole Floer groups of Y.

Example

- As $SWF(S^3) = S^0$, the groups agree.
- But for T^3

	-4	-3	-2	-1	0	1	2	3	4
$\overline{HM}_*(T^3)$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^3$
$\widecheck{HM}_*(T^3)$	0	0	0	0	0	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^3$
$\widehat{HM}_*(T^3)$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	0	0	0
$\bar{H}_*^{S^1}(SWF(T^3))$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$H_*^{S^1}(SWF(T^3))$	0	0	0	0	0	$\mathbb{Z}^{N-1}$	$\mathbb{Z}$	0	$\mathbb{Z}$
$\widehat{H}_*^{S^1}(SWF(T^3))$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}^N$	0	0	0

  where $N = \mathbb{Z}^3$
- This is because there is an action by $\mathbb{Z}^3$ left.

Twisted parametrized spectrum

The construction is based on a concept of twisted parametrized spectra introduced by Douglas.

- Roughly speaking, a twisted parametrized spectrum is a bundle of spectra twisted by automorphisms of the category of spectra, e.g. suspensions.
- One simple way to describe this is to use open cover and transition functions.

Twisted Manolescu-Floer spectrum

- There is a isomorphism between Conley indices induced by the action of a harmonic gauge transformation u

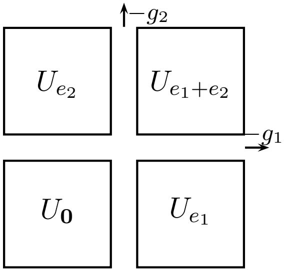
$$\mathcal{I}(\mathcal{R} \cap V) \to \mathcal{I}((u \cdot \mathcal{R}) \cap (uV)).$$

- However, when passing to spectra, we have

$$S^{V^-} \wedge SWF(Y) \to S^{(uV)^-} \wedge SWF(Y),$$

Twisted Manolescu-Floer spectrum

We define a twisted Manolescu-Floer spectrum $\widetilde{SWF}(Y)$ as a twisted parametrized spectrum over the Picard torus.



- We use the standard cover of the torus and the transition functions g_i is the map given by u_i earlier.

The manifold $S^1 \times S^2$

- Consider $S^1 \times S^2$ with torsion spin^c structure.
- We can find that $SWF(S^1 \times S^2, \mathfrak{s}) \simeq S^0$ using the same method as T^3.
- Because the Dirac operator has no kernel, it turns out that $\widetilde{SWF}(S^1 \times S^2, \mathfrak{s}) \simeq S^1 \times SWF(S^1 \times S^2, \mathfrak{s})$.

Homology of $\widetilde{SWF}(T^3, \mathfrak{s})$

There is a spectral sequence for Tate homology.

	0	1	2	3
$\vdots$				
2	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}$
1	0	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}$
-1	0	0	0	0
-2	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}$
$\vdots$				