Derivative interactions in dRGT massive gravity

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+current work
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5. Summary
Motivation

Can we construct healthy and consistent massive gravity?
“Linear” massive gravity

- **Fierz-Pauli massive gravity** (Fierz, Pauli, 1939)

\[
S = M_{Pl}^2 \int d^4 x \left[ -\frac{1}{2} h^{\mu\nu} \mathcal{E}_{\mu\nu}^{\alpha\beta} h_{\alpha\beta} - \frac{1}{4} m^2 (h^{\mu\nu} h_{\mu\nu} - h^2) \right]
\]

Linearized Einstein-Hilbert term

Only allowed mass term which does not have ghost at linear order

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}
\]

\[
\mathcal{E}_{\mu\nu}^{\alpha\beta} h_{\alpha\beta} = -\frac{1}{2} \left( \Box h_{\mu\nu} - \partial_{\mu} \partial_{\alpha} h_{\nu}^{\alpha} - \partial_{\nu} \partial_{\alpha} h_{\mu}^{\alpha} + \partial_{\mu} \partial_{\nu} h_{\alpha}^{\alpha} - \eta_{\mu\nu} \Box h_{\alpha}^{\alpha} + \eta_{\mu\nu} \partial_{\alpha} \partial_{\beta} h_{\beta}^{\alpha} \right)
\]

1. **Linear theory**
2. Lorentz invariant theory, but gauge invariance is broken
3. No ghost at linear order
   (5 DOF=massless tensor+massless vector+massless scalar)
4. **Simple nonlinear extension contains ghost at nonlinear level**
   (Boulware-Deser ghost, 6th DOF) (Boulware, Deser, 1971)
1st version of nonlinear massive gravity

- **Stuckelberg field**  

  \[ f_{\mu \nu} = \eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b \]  
  \[ \phi^a \text{ are four scalar fields} \]

  Covariant tensor

  Unitary gauge  
  \[ \phi^a = x^a \rightarrow f_{\mu \nu} = \eta_{\mu \nu} \]

  Poincare symmetry  
  \[ \phi^a \rightarrow \phi^a + c^a, \quad \phi^a \rightarrow \Lambda^a_b \phi^b \]

- **Define new covariant fluctuation tensor**

  \[ H_{\mu \nu} = g_{\mu \nu} - f_{\mu \nu} \]

- **Covariant form of non-linear FP action**

  \[ S = \frac{M_{Pl}^2}{2} \int d^4 x \sqrt{-g} \left[ R - \frac{1}{4} m^2 g^{\mu \alpha} g_{\nu \beta} (H_{\mu \nu} H_{\alpha \beta} - H_{\mu \alpha} H_{\nu \beta}) \right] \]
Decoupling limit

- Expand Stuckelberg field around unitary gauge

\[ \phi^a = (x^\alpha - A^\alpha) \delta^a_\alpha, \quad A^\alpha \rightarrow A^\alpha + \partial^\alpha \pi \]

- Non-linear leading action within decoupling limit,

\[
S_h = \int d^4x \left[ -\frac{1}{2} \hat{h}^{\mu\nu} \epsilon_{\mu\nu}^{\alpha\beta} \hat{h}_{\alpha\beta} + \frac{1}{M_{Pl}} \hat{h}_{\mu\nu} T^{\mu\nu} \right]
\]

\[
S_A = \int d^4x \left[ -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} \right]
\]

\[
S_\pi = \int d^4x \left[ -3 (\partial \hat{\pi})^2 + \frac{1}{\Lambda_5^5} \left\{ (\Box \hat{\pi})^3 - (\Box \hat{\pi}) (\partial_\mu \partial_\nu \hat{\pi})^2 \right\} + \frac{1}{M_{Pl}} \hat{\pi} T \right]
\]

\[ \Lambda_5 = (M_{Pl} m^4)^{1/5} \]

High derivative Lagrangian, not galileon

**6th DOF appears in theory → BD ghost**

(Boulware, Deser, 1972)
Adding higher-order potential terms

de Rham, Gabadadze (2010)

• Action

\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R - \sum_{n=2}^{\infty} \frac{1}{4} m^2 U_n(g, H) \right] \]

\[ U_2(g, H) = b_1 [H^2] + b_2 [H]^2 \]
\[ U_3(g, H) = c_1 [H^3] + c_2 [H][H^2] + c_3 [H^3] \]
\[ U_4(g, H) = d_1 [H^4] + d_2 [H][H^3] + d_3 [H^2][H^2] + d_4 [H]^2[H^2] + d_5 [H]^4 \]
\[ U_5(g, H) = f_1 [H^5] + f_2 [H][H^4] + f_3 [H]^2[H^3] + f_4 [H^2][H^3] + f_5 [H][H^2]^2 + f_6 [H]^3[H^2] + f_7 [H]^5 \]
\[ U_6(g, H) = \cdots \]

Fierz-Pauli tuning

\[ b_1 = -b_2 \]

\[ [H] = g^{\mu\nu} H_{\mu\nu} \]
\[ [H^2] = g^{\mu\nu} g^{\alpha\beta} H_{\mu\alpha} H_{\nu\beta} \]
\[ [H^3] = \cdots \]
Eliminating 6th DOF

• Non-linear Lagrangian

\[
\mathcal{L}_{\Pi^2} = [\Pi]^2 - [\Pi^2]
\]
\[
\mathcal{L}_{\Pi^3} = -\frac{1}{4\Lambda_5^5} ((8c_1 - 4)[\Pi^3] + (8c_2 + 4)[\Pi][\Pi^2] + 8c_3[\Pi]^3)
\]
\[
\mathcal{L}_{\Pi^4} = \frac{1}{\Lambda_4^8} \left\{ \left( 3c_1 - 4d_1 - \frac{1}{4} \right) [\Pi^4] + \left( c_2 - 4d_3 + \frac{1}{4} \right) [\Pi^2]^2 
\right.
\]
\[
\left. + (2c_2 - 4d_2)[\Pi][\Pi^3] + (3c_3 - 4d_4)[\Pi^2][\Pi] - 4d_5[\Pi]^4 \right\}
\]
\[
\mathcal{L}_{\Pi^5} = \ldots
\]

\[
\mathcal{L}_{\text{der}}^{(2)} = [\Pi]^2 - [\Pi^2]
\]
\[
\mathcal{L}_{\text{der}}^{(3)} = 2[\Pi^3] - 3[\Pi][\Pi^2] + [\Pi]^3
\]
\[
\mathcal{L}_{\text{der}}^{(4)} = -6[\Pi^4] + 3[\Pi^2]^2 + 8[\Pi][\Pi^3] - 6[\Pi^2][\Pi]^2 + [\Pi]^4
\]
\[
\mathcal{L}_{\text{der}}^{(5)} = \ldots
\]

This yields the higher order derivative (the origin of BD ghost)

We choose these coefficients so that the Lagrangian becomes total derivative
Eliminating 6th DOF

de Rham, Gabadadze (2010)

• Choosing the coefficients

\[ c_1 = 2c_3 + \frac{1}{2}, \quad c_2 = -3c_3 - \frac{1}{2}, \]

\[ d_1 = -6d_5 + \frac{1}{16}(24c_3 + 5), \quad d_2 = 8d_5 - \frac{1}{4}(6c_3 + 1) \]
\[ d_3 = 3d_5 - \frac{1}{16}(12c_3 + 1), \quad d_4 = -6d_5 + \frac{3}{4}c_3, \]

\[ f_1 = \frac{7}{32} + \frac{9}{8}c_3 - 6d_5 + 24f_7, \quad f_2 = -\frac{5}{32} - \frac{15}{16}c_3 + 6d_5 - 30f_7, \quad f_3 = \frac{3}{8}c_3 - 3d_5 + 20f_7, \]
\[ f_4 = -\frac{1}{16} - \frac{3}{4}c_3 + 5d_5 - 20f_7, \quad f_5 = \frac{3}{16}c_3 - 3d_5 + 15f_7, \quad f_6 = d_5 - 10f_7 \]

These combinations kill all scalar self-interaction terms!
Action in decoupling limit

- The next order interactions

\[ \mathcal{L}_{h\Pi^n} = h^{\mu\nu} X^{(1)}_{\mu\nu} - (6c_3 - 1) \frac{1}{\Lambda_3^3} h^{\mu\nu} X^{(2)}_{\mu\nu} + (c_3 + 8d_5) \frac{1}{\Lambda_3^6} h^{\mu\nu} X^{(3)}_{\mu\nu} + \cdots \]

\[ X^{(1)}_{\mu\nu} = [\Pi] \eta_{\mu\nu} - \Pi_{\mu\nu} \]

\[ X^{(2)}_{\mu\nu} = \Pi_{\mu\nu}^2 - [\Pi] \Pi_{\mu\nu} - \frac{1}{2} ([\Pi^2] - [\Pi]^2) \eta_{\mu\nu} \]

\[ X^{(3)}_{\mu\nu} = 6\Pi_{\mu\nu}^3 - 6[\Pi] \Pi_{\mu\nu}^2 + 3([\Pi]^2 - [\Pi^2]) \Pi_{\mu\nu} - ([\Pi]^3 - 3[\Pi] [\Pi^2] + 2[\Pi^3]) \eta_{\mu\nu} \]

\[ X^{(4)}_{\mu\nu} = \cdots \]

- The action in the decoupling limit

\[ S = \int d^4 x \left[ -\frac{1}{2} \hat{h}^{\mu\nu} \mathcal{E}_{\mu\nu} \hat{\alpha}\beta \hat{\alpha}\beta + \hat{h}^{\mu\nu} \left( X^{(1)}_{\mu\nu} - \frac{6c_3 - 1}{\Lambda_3^3} X^{(2)}_{\mu\nu} + \frac{c_3 + 8d_5}{\Lambda_3^6} X^{(3)}_{\mu\nu} \right) + \frac{1}{M_{Pl}} \hat{h}_{\mu\nu} T^{\mu\nu} \right] \]

1. **The remaining nonlinear interactions are galileons**
   
   \( \text{(EOM is 2nd order differential equations)} \)

2. **The cutoff energy scale is } \Lambda_3 \)
   
   \( \text{(We cannot trust the theory above } \Lambda_3 \)
Resummation of nonlinear potential

\textit{de Rham, Gabadadze, Tolley (2011)}

- Define the new tensor

\[ K^\mu_\nu = \delta^\mu_\nu - \sqrt{\eta_{ab} g^{\mu \alpha} \partial_\alpha \phi^a \partial_\nu \phi^b} \]

- Property of this tensor in the decoupling limit

\[ K_{\mu \nu}(g, H) \bigg|_{h_{\mu \nu}=0} \equiv \Pi_{\mu \nu} \]

- dRGT massive gravity

\[ S_{MG} = \frac{M_{\text{Pl}}^2}{2} \int d^4 x \sqrt{-g} \left[ R - \frac{m^2}{4} (U_2 + \alpha_3 U_3 + \alpha_4 U_4) \right] + S_m[g_{\mu \nu}, \psi] \]

\[ U_2 = 2 \varepsilon_{\mu \alpha \rho \sigma} \varepsilon^{\nu \beta \rho \sigma} K^\mu_\nu K^\alpha_\beta = 4 ([K^2] - [K]^2) \]

\[ U_3 = \varepsilon_{\mu \alpha \gamma \rho} \varepsilon^{\nu \beta \delta \rho} K^\mu_\nu K^\alpha_\beta K^\gamma_\delta = -[K]^3 + 3[K][K^2] - 2[K^3] \]

\[ U_4 = \varepsilon_{\mu \alpha \gamma \rho} \varepsilon^{\nu \beta \delta \rho} K^\mu_\nu K^\alpha_\beta K^\gamma_\delta K^\rho_\sigma = -[K]^4 + 6[K]^2[K^2] - 3[K^2]^2 - 8[K][K^3] + 6[K]^4 \]

Total derivative in the decoupling limit

No BD ghost in full theory (Hassan, Rosen 2011)
“Ghost-free” nonlinear massive gravity

- **de Rham-Gabadadze-Tolley massive gravity** (de Rham, Gabadadze, Tolley, 2011)

\[
S_{MG} = \frac{M_P^2}{2} \int d^4 x \sqrt{-g} \left[ R - \frac{m^2}{4} (\mathcal{U}_2 + \alpha_3 \mathcal{U}_3 + \alpha_4 \mathcal{U}_4) \right] + S_m[g_{\mu \nu}, \psi]
\]

\[
\mathcal{U}_2 = \varepsilon_{\mu \alpha \rho \sigma} \varepsilon^{\nu \beta \rho \sigma} \mathcal{K}_\nu \mathcal{K}_{\beta} \\
\mathcal{U}_3 = \varepsilon_{\mu \alpha \gamma \rho} \varepsilon^{\nu \beta \delta \rho} \mathcal{K}_\nu \mathcal{K}_{\beta} \mathcal{K}_{\delta} \\
\mathcal{U}_4 = \varepsilon_{\mu \alpha \gamma \rho} \varepsilon^{\nu \beta \delta \sigma} \mathcal{K}_\nu \mathcal{K}_{\beta} \mathcal{K}_{\delta} \mathcal{K}_{\sigma}
\]

\[
\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \sqrt{\delta^\mu_\nu - H^\mu_\nu} \\
= \delta^\mu_\nu - \sqrt{\eta_{ab} g^{\mu \alpha} \partial_\alpha \phi^a \partial_\nu \phi^b}
\]

- **Ghost-free nonlinear massive gravity**

(1) Nonlinear theory
(2) Lorentz invariant theory
(3) No ghost at full order (5 DOF, No BD ghost) (Hassan, Rosen, 2011)
(4) Unique theory of massive spin-2 field as an extension of general relativity
   (GR + mass term)
Decoupling limit

- **Decoupling limit**: Easy to capture high energy behavior within Compton wavelength of massive graviton

\[
H_{\mu\nu} \rightarrow \frac{h_{\mu\nu}}{M_{\text{Pl}}} + 2 \frac{\partial_{\mu} \partial_\nu \pi}{M_{\text{Pl}} m^2} - \frac{\partial_\mu \partial_\alpha \pi \partial_\nu \partial_\alpha \pi}{M_{\text{Pl}}^2 m^4}
\]

\[
M_{\text{Pl}} \rightarrow \infty, \quad m \rightarrow 0, \quad \Lambda_3 = (M_{\text{Pl}} m^2)^{1/3} = \text{fixed}, \quad \frac{T_{\mu\nu}}{M_{\text{Pl}}} = \text{fixed}
\]

- **dRGT Lagrangian in the decoupling limit**

\[
\mathcal{L}_{\text{DL}} = -\frac{1}{4} h^{\mu\nu} \varepsilon_{\mu\nu}^\alpha \varepsilon_{\alpha\beta} h_{\alpha\beta} + \frac{1}{M_{\text{Pl}}} h^{\mu\nu} T_{\mu\nu}
\]

**Standard gravity part**

- **Galileon type interactions**
  - 2nd order differential EOM (NO BD ghost)
  - Cutoff energy scale is \(\Lambda_3\)

\[
\varepsilon \varepsilon \Pi \equiv \varepsilon_{\mu\nu}^\alpha \varepsilon_{\nu\alpha}^\beta \gamma \partial_\mu \partial_\nu \pi
\]

\[
\varepsilon_\mu \varepsilon_\nu \Pi \equiv \varepsilon_{\mu}^\alpha \gamma \varepsilon_{\nu}^\beta \delta \partial_\alpha \partial_\beta \pi
\]

\[
\Pi_{\mu\nu} = \partial_\mu \partial_\nu \pi
\]
Cosmologies

- **There is no flat and closed FRW solution** (D'Amico et al., 2011)
- **Open FRW solution** (Gumrukcuoglu, Lin, Mukohyama, 2011)

\[
3 H^2 + \frac{3 K}{a^2} = \Lambda_\pm + \frac{1}{M_{Pl}^2} \rho, \\
-\frac{2 \dot{H}}{N} + \frac{2 K}{a^2} = \frac{1}{M_{Pl}^2} (\rho + P),
\]

\[
\Lambda_\pm \equiv -\frac{m_g^2}{(\alpha_3 + \alpha_4)^2} \left[ (1 + \alpha_3) (2 + \alpha_3 + 2 \alpha_3^2 - 3 \alpha_4) \pm 2 (1 + \alpha_3 + \alpha_3^2 - \alpha_4)^{3/2} \right]
\]

- **Linear perturbations are fine, and scalar and vector perturbations are exactly the same as GR because of vanishing the kinetic terms** (Gumrukcuoglu, Lin, Mukohyama, 2011)
- **There is ghost-instability at nonlinear level** (Gumrukcuoglu, Lin, Mukohyama, 2012)
- **Consistent massive gravity: Quasi-dilaton theory (massive graviton + scalar)** (de Felice, Mukohyama, 2013), **SO(3) massive gravity (Lorentz breaking)** (Lin, 2013)
Derivative interactions in massive gravity
DOF in Fierz-Pauli theory

- Einstein-Hilbert term

\[ \mathcal{L}_{EH} = \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\alpha\beta\gamma} \partial_\mu \partial_\alpha h_{\nu\beta} h_{\rho\gamma} \]

- \( h_{00} \) and \( h_{ij} \) are Lagrange multipliers
  (Existence of Hamiltonian and momentum constraints)

- DOF of massless graviton = 2

- Fierz-Pauli mass term

\[ \mathcal{U}_{FP} = \varepsilon^{\mu\alpha\rho\sigma} \varepsilon_{\nu\beta\rho\sigma} h_{\mu\nu} h_{\alpha\beta} \]

- \( h_{00} \) is Lagrange multiplier
  (Existence of Hamiltonian constraint)

- DOF of massless graviton = 5

\[ \rightarrow \text{Antisymmetric tensor ensures that } h_{00} \text{ becomes a Lagrange multiplier} \]

The Hamiltonian constraint kills BD ghost
Derivative interaction in Fierz-Pauli theory

• Derivative interaction in Fierz-Pauli theory

\[ L_{2,3} \sim M_{\text{Pl}}^2 \varepsilon^{\mu \nu \rho \sigma} \varepsilon^{\alpha \beta \gamma \delta} \partial_\mu \partial_\alpha \, h_{\nu \beta} \, h_{\rho \gamma} \, h_{\sigma \delta} \]

Levi-Civita structure ensures that the Lagrangian is linear in \( h_{00} \)

\[ \rightarrow \] \( h_{00} \) becomes a Lagrange multiplier, which kills BD ghost

• In 4 dimension, there is no more derivative interaction, which kills BD ghost.
  (due to the number of indices in the antisymmetric tensor.)

• Fierz-Pauli theory is actually linear theory, but this derivative interactions is nonlinear !! If we want to consider this term, we need to think about Einstein-Hilbert term, instead of linearized Einstein-Hilbert.
Our work: Is there any consistent nonlinear derivative interactions in de Rham-Gabadadze-Tolley massive gravity?

\[ S_{MG} = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g} \left[ R - \frac{m^2}{4} (U_2 + \alpha_3 U_3 + \alpha_4 U_4) \right] + S_{int} + S_m[g_{\mu\nu}, \psi], \]

\[ U_2 = \epsilon_{\mu\alpha\rho\sigma} \epsilon^{\nu\beta\rho\sigma} K_{\mu\nu}^{\alpha} K_{\beta}^{\gamma} \]
\[ U_3 = \epsilon_{\mu\alpha\gamma\rho} \epsilon^{\nu\beta\delta\rho} K_{\mu\nu}^{\alpha} K_{\beta}^{\gamma} K_{\delta}^{\rho} \]
\[ U_4 = \epsilon_{\mu\alpha\gamma\rho} \epsilon^{\nu\beta\delta\sigma} K_{\mu\nu}^{\alpha} K_{\beta}^{\gamma} K_{\delta}^{\rho} K_{\sigma}^{\rho} \]

\[ K_{\mu\nu}^{\alpha} = \delta_{\mu\nu} - \sqrt{\delta_{\mu\nu} - H_{\mu}^{\mu}} \]

\[ = \delta_{\mu\nu} - \sqrt{\eta_{ab} \epsilon^{\mu \alpha \beta \sigma} \partial_{\alpha} \phi^a \partial_{\beta} \phi^b} \]
Guidelines for construction of Lagrangian

- Candidates for derivative interactions using the Riemann tensor

\[ \mathcal{L}_{\text{int}} \supset M_{\text{Pl}}^2 \sqrt{-g} H R, \quad M_{\text{Pl}}^2 \sqrt{-g} H^2 R, \quad M_{\text{Pl}}^2 \sqrt{-g} H^3 R, \ldots \]

\[ H_{\mu\nu} = g_{\mu\nu} - \eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b \]

- Guidelines

1. Linearization of \( h_{\mu\nu} \) reproduces Fierz-Pauli theory
   - Lorentz invariance
   - Free of Boulware-Deser ghost at linear level

2. Cut off energy scale is \( \Lambda_3 \)
   - All nonlinear terms below \( \Lambda_3 \) have to be eliminated

3. Free of Boulware-Deser ghost
Energy scales of derivative interactions in the decoupling limit

- **General form of Lagrangian**
  \[
  \mathcal{L}_{int} \supset M_{P1}^2 \sqrt{-g} H R, \quad M_{P1}^2 \sqrt{-g} H^2 R, \quad M_{P1}^2 \sqrt{-g} H^3 R, \ldots
  \]

- **The Lagrangian in the decoupling limit can be schematically written as**
  \[
  \mathcal{L}_{int} \sim \Lambda_\lambda^{2-n_h-3n_\pi} h^{n_h-1} \partial^2 h \left(\partial^2 \pi\right)^{n_\pi}
  \]

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These has to be eliminated

\[
H_{\mu\nu} \rightarrow \frac{h_{\mu\nu}}{M_{P1}} + 2 \frac{\partial_\mu \partial_\nu \pi}{M_{P1} m^2} - \frac{\partial_\mu \partial_\alpha \pi \partial_\mu \partial^\alpha \pi}{M_{P1}^2 m^4}
\]

\[
\Lambda_\lambda = (M_p m^{\lambda-1})^{1/\lambda}
\]

\[
\lambda = \frac{n_h + 3n_\pi - 2}{n_h + n_\pi - 2}
\]
HR order term

- General Lagrangian of HR order

\[ \mathcal{L}_{int,1} = M_{Pl}^2 \sqrt{-g} H_{\mu\nu} (R^{\mu\nu} + d R g^{\mu\nu}) \]

Linearizing \( h_{\mu\nu} \) gives the same order of the linearized Einstein-Hilbert

\[ \mathcal{L}^{(2)}_{int,1} \propto M_{Pl}^2 \left[ \sqrt{-g} R \right]_{\text{h}^2} \]

\[ \rightarrow \quad d = -1/2 \]

- In terms of Levi-Civit\( \alpha \) symbol,

\[ \mathcal{L}_{int,1} = M_{Pl}^2 \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma} R_{\mu\alpha\nu\beta} H_{\rho\gamma} \]

The Lagrangian satisfies requirement (1): Fierz-Pauli theory at linear theory
HR order term in the decoupling limit

• The lowest order term in the decoupling limit

\[
\mathcal{L}_{\text{int,1}} \bigg|_{\partial^2 h \, \partial^2 \pi} = -\frac{2}{m^2} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} \partial_\mu \partial_\alpha h_{\nu\beta} \partial_\rho \partial_\gamma \pi
\]

\[
= -\frac{2}{m^2} \partial_\gamma (\varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} \partial_\mu \partial_\alpha h_{\nu\beta} \partial_\rho \pi)
\]

Total derivative

• The next order term in the decoupling limit

\[
\mathcal{L}_{\text{int,1}} \bigg|_{\partial^2 h \, (\partial^2 \pi)^2} = \frac{1}{\Lambda_5^5} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} \partial_\mu \partial_\alpha h_{\nu\beta} \partial_\rho \partial_\pi \partial_\pi \partial_\gamma \pi
\]

This is not zero or total derivative, and EOM contains higher derivatives

→ The counter part of this term can eliminate this term

\[
\mathcal{L}_{\text{int,1,2}} = \frac{1}{4} M_{\text{Pl}}^2 \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} \partial_\mu \partial_{\alpha} R_{\mu\alpha \nu\beta} H_{\rho a} H^a_{\gamma}
\]

Λ₅ term is eliminated!
HR order term in the decoupling limit

- **HR order Lagrangian**

\[ \mathcal{L}_{\text{int},1} = M_P^2 \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{\alpha\beta\gamma}^{\sigma} R_{\mu\alpha\nu\beta} H_{\rho\gamma} \]

\[ \sim \Lambda_{\lambda}^{2-n_h-3n_\pi} h^{n_h-1} \partial^2 h (\partial^2 \pi)^{n_\pi} \bigg|_{\text{DL}} \]

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Automatically total derivative

These terms can be always eliminated by adding higher order terms
HR order term

- The total Lagrangians including counter terms is given by

\[
\mathcal{L}_{\text{int},1} = M_{\text{Pl}}^2 \sqrt{-g} \varepsilon^{\mu \nu \rho \sigma} \varepsilon_{\alpha \beta}^\gamma \sigma R_{\mu \alpha \nu \beta} \\
\times \left( H_{\rho \gamma} + \frac{1}{4} H_{\rho a} H^{a \gamma} + \frac{1}{8} H_{\rho a} H_{b}{}^{a} H_{b}{}^{\gamma} + \frac{5}{64} H_{\rho a} H_{b}{}^{a} H_{c}{}^{b} H_{c}{}^{\gamma} + \cdots \right) \\
= 2 \mathcal{K}_{\rho \gamma}
\]

\[
\mathcal{K}_{\mu \nu} = \delta_{\mu \nu} - \sqrt{\delta_{\mu \nu} - H_{\mu \nu}^\mu} = - \sum_{n=1}^{\infty} \tilde{d}_n (H^n)^\mu_{\nu},
\]

- The final Lagrangian of HR order term

\[
\mathcal{L}_{\text{int},1} = M_{\text{Pl}}^2 \sqrt{-g} \varepsilon^{\mu \nu \rho \sigma} \varepsilon_{\alpha \beta}^\gamma \sigma R_{\mu \alpha \nu \beta} \mathcal{K}_{\rho \gamma}
\]

The Lagrangian satisfies requirements (2) : $\Lambda_3$ theory in the decoupling limit
**H²R order term**

- Consider the most general combination of this order

\[ \mathcal{L}_{int,2} = M_{Pl}^2 \sqrt{-g} R^\mu_\nu \alpha_\beta (c_1 H_{\mu \alpha} H_{\nu \beta} + \cdots) \]

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<td>( \Lambda_{11/3} )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( n )</td>
<td>( \Lambda_{(3n-1)/(n-1)} )</td>
</tr>
</tbody>
</table>

This order should be a total derivative, otherwise BD ghost appear

\[ \sim \varepsilon^\mu_\nu \rho_\sigma \varepsilon^{\alpha_\beta \gamma \delta} \partial_\mu \partial_\alpha h_{\nu \beta} \partial_\rho \partial_\gamma \pi \partial_\sigma \partial_\delta \pi \]

These terms might be eliminated by adding higher order terms
**H^2R order term**

- The Lagrangian is

\[ \mathcal{L}_{\text{int},2} = M_{\text{Pl}}^2 \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} R_{\mu\alpha\nu\beta} H_{\rho\gamma} H_{\sigma\delta} \]

  This is the only combination that the lowest order \( \Lambda_5 \) term becomes a total derivative

- With the same method of the previous case, we get the resumed Lagrangian of \( H^2R \) order term

\[ \mathcal{L}_{\text{int},2} = M_{\text{Pl}}^2 \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} R_{\mu\alpha\nu\beta} \mathcal{K}_{\rho\gamma} \mathcal{K}_{\sigma\delta} \]

- \( H^3R, HR^2 \) or higher order terms?? → No!

  In four dimension, there is no total derivative combination of the lowest order term in the decoupling limit
Riemann derivative interactions

• In 4 dimension, the general derivative interaction for massive graviton is

\[ \mathcal{L}_{int} = M_{Pl}^2 \sqrt{-g} \varepsilon^{\mu \nu \rho \sigma} \varepsilon^{\alpha \beta \gamma \delta} R_{\mu \alpha \nu \beta} (\alpha g_{\rho \gamma} K_{\sigma \delta} + \beta K_{\rho \gamma} K_{\sigma \delta}) \]

\( \alpha \) and \( \beta \) are parameters

• We can also construct derivative interactions in arbitrary dimensions \( D \)

\[ \mathcal{L}^{(D, d, m)}_{int} = M_{Pl}^{D-2} m^{2-d} \sqrt{-g} \varepsilon^{\mu_1 \mu_2 \cdots \mu_D} \varepsilon^{\nu_1 \nu_2 \cdots \nu_D} R_{\mu_1 \nu_1 \mu_2 \nu_2} \cdots R_{\mu_{d-1} \nu_{d-1} \mu_d \nu_d} \]

\[ \times g_{\mu_{d+1} \nu_{d+1}} \cdots g_{\mu_m \nu_m} K_{\mu_{m+1} \nu_{m+1}} \cdots K_{\mu_D \nu_D} \]

\( d \) is even number

\[ 2 \leq d \leq m \leq D - 1 \]

These Lagrangians satisfy the requirements (1) and (2)
• **We constructed the $\Lambda_3$ nonlinear derivative interactions, but we still need to check the requirement (3): the existence of BD ghost**

• **$\Lambda_3$ theory in the decoupling limit**

\[
\mathcal{L}_{DL} \sim \frac{1}{\Lambda_3^3 \pi} \left[ R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right] h^2 + \frac{1}{\Lambda_3^{3n_\pi}} \mathcal{O}[h \partial^2 h (\partial^2 \pi)^{n_\pi}]
\]

*EOM is 2nd order differential equation (coming from $L_{\text{int},2}$)*

*These terms yield 4th order differential Eq for $h$ and $\pi$ (coming from $L_{\text{int},1}$ and $L_{\text{int},2}$)*

There are extra degrees of freedom, which leads to ghost...

**Ghost appears at $\Lambda_3$**
Other derivative interactions (in progress)

- In 4 dimension, we found other $\Lambda_3$ derivative interactions without the Riemann tensor

\[
\mathcal{L}_{\text{int},1}' = -M_{\text{Pl}}^2 \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma} \nabla_\alpha K_{\nu\beta} \nabla_\mu K_{\rho\gamma}
\]

\[
\mathcal{L}_{\text{int},2}' = -2M_{\text{Pl}}^2 \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} \nabla_\alpha K_{\nu\beta} \nabla_\mu K_{\rho\gamma} F_{\delta\sigma}(H..)
\]

- $\Lambda_3$ theory in the decoupling limit

\[
\mathcal{L}_{\text{DL}} \sim \frac{1}{\Lambda_3^3} \pi \left[ R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right]_{h^2} + \frac{1}{\Lambda_3^{3n_\pi}} O[h\partial^2 h (\partial^2 \pi)^{n_\pi}]
\]

EOM is 2nd order differential equation (coming from $\mathcal{L}_{\text{int},1}'$)

These terms yield 4th order differential Eq for $h$ and $\pi$ (coming from $\mathcal{L}_{\text{int},1}'$ and $\mathcal{L}_{\text{int},2}'$)

We cannot kill higher derivative terms in EOM even if we combine all four derivative interaction terms...
Appropriate mass scale of derivative interactions?

- So far, the mass scale of the derivative interactions was $M_{pl}$

$$\mathcal{L}_{int} = M_{pl}^2 \sqrt{-g} \varepsilon^{\mu \nu \rho \sigma} \varepsilon^{\alpha \beta \gamma \delta} R_{\mu \alpha \nu \beta} (\alpha g_{\rho \gamma} K_{\sigma \delta} + \beta K_{\rho \gamma} K_{\sigma \delta})$$

If the mass scale $M$ is not $M_{pl}$ and smaller than $M_{pl}$, the ghost scale is roughly above the cutoff energy scale.

- In addition, if $M < M_{pl}$,

$$\mathcal{L}_{int,2} = M_{pl}^2 \sqrt{-g} \varepsilon^{\mu \nu \rho \sigma} \varepsilon^{\alpha \beta \gamma \delta} R_{\mu \alpha \nu \beta} K_{\rho \gamma} K_{\sigma \delta}$$

$$\mathcal{L}_{DL} \sim \frac{1}{\Lambda^3} \left[ R^2 - 4R_{\mu \nu} R^{\mu \nu} + R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} \right] h^2 + \frac{1}{\Lambda^3} \mathcal{O}[h \delta^2 h (\delta^2 \pi)^n \pi]$$

$\Lambda_R > \Lambda_3$  \hspace{1cm} $\Lambda_{\text{ghost}} > \Lambda_R > \Lambda_3$

If $\Lambda_{\text{ghost}} > \Lambda_{\text{cutoff}} > \Lambda_R > \Lambda_3$, the theory might still survive…
Summary

• We found the most general derivative interactions in dRGT massive gravity
  • The energy scales below $\Lambda_3$ can be eliminated by adding counter terms
  • The Lagrangians can be resumed by using $K$ tensor
  • The most general derivative interactions contain four interactions
  • Nonlinear terms contribute at $\Lambda_3$

• Appropriate DOF?
  • 4th order differential EOM of the scalar and tensor mode in the decoupling limit
  • Ghost appears at $\Lambda_3$ in dRGT theory + derivative interactions

The mass scale of the derivative interactions should be $M < M_{pl}$