Derivative interactions in dRGT massive gravity

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+current work
with Daisuke Yamauchi (RESCEU)

Contents of this talk

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Motivation

Can we construct healthy and consistent massive gravity?

“*Linear*” massive gravity

- *Fierz-Pauli massive gravity* (Fierz, Pauli, 1939)

$$S = M_{\rm Pl}^2 \int d^4x \left[\boxed{-\frac{1}{2} h^{\mu\nu} \mathcal{E}_{\mu\nu}^{\alpha\beta} h_{\alpha\beta}} \boxed{- \frac{1}{4} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2)} \right]$$

Linearized
Einstein-Hilbert term

Only allowed mass term
which does not have ghost at linear order

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\mathcal{E}_{\mu\nu}^{\alpha\beta} h_{\alpha\beta} = -\frac{1}{2} (\Box h_{\mu\nu} - \partial_\mu \partial_\alpha h_\nu^\alpha - \partial_\nu \partial_\alpha h_\mu^\alpha + \partial_\mu \partial_\nu h_\alpha^\alpha - \eta_{\mu\nu} \Box h_\alpha^\alpha + \eta_{\mu\nu} \partial_\alpha \partial_\beta h_\beta^\alpha)$$

(1) Linear theory

(2) Lorentz invariant theory, but gauge invariance is broken

(3) No ghost at linear order

(5 DOF=massless tensor+massless vector+massless scalar)

(4) Simple nonlinear extension contains ghost at nonlinear level

(Boulware-Deser ghost, 6th DOF) (Boulware, Deser, 1971)

1st version of nonlinear massive gravity

- *Stuckelberg field* Arkani-Hamed, Georgi, Schwartz (2003)

$$\boxed{f_{\mu\nu}} = \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b$$

ϕ^a are four scalar fields

Covariant tensor

Unitary gauge $\phi^a = x^a \to f_{\mu\nu} = \eta_{\mu\nu}$

Poincare symmetry $\phi^a \to \phi^a + c^a, \quad \phi^a \to \Lambda^a_b\,\phi^b$

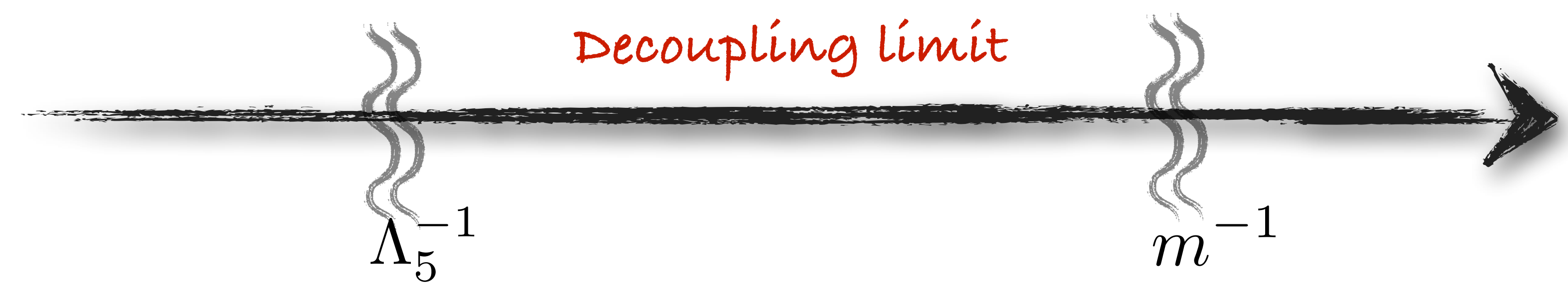
- *Define new covariant fluctuation tensor*

$$H_{\mu\nu} = g_{\mu\nu} - f_{\mu\nu}$$

- *Covariant form of non-linear FP action*

$$S = \frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}\left[R - \frac{1}{4}m^2 g^{\mu\alpha}g^{\nu\beta}(H_{\mu\nu}H_{\alpha\beta} - H_{\mu\alpha}H_{\nu\beta})\right]$$

Decoupling limit



Creminelli et.al. (2005)

- *Expand Stuckelberg field around unitary gauge*

$$\phi^a = (x^\alpha - A^\alpha)\delta^a_\alpha, \quad A^\alpha \to A^\alpha + \partial^\alpha \pi$$

Thanks to Poincare symmetry in field space, we can decompose ϕ into scalar and vector

- *Non-linear leading action within decoupling limit,*

$$m \to 0, \quad M_{\rm Pl} \to \infty, \quad T \to \infty, \quad \Lambda_5 \text{ and } \frac{T}{M_{\rm Pl}} \text{ are fixed}$$

$$S_h = \int d^4x \left[-\frac{1}{2}\hat{h}^{\mu\nu}\mathcal{E}^{\alpha\beta}_{\mu\nu}\hat{h}_{\alpha\beta} + \frac{1}{M_{\rm Pl}}\hat{h}_{\mu\nu}T^{\mu\nu} \right]$$

$$S_A = \int d^4x \left[\boxed{-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}} \right]$$

does not couple with EM tensor

$$S_\pi = \int d^4x \left[-3(\partial\hat{\pi})^2 + \boxed{\frac{1}{\Lambda_5^5}\left\{(\Box\hat{\pi})^3 - (\Box\hat{\pi})(\partial_\mu\partial_\nu\hat{\pi})^2\right\}} + \frac{1}{M_{\rm Pl}}\hat{\pi}T \right]$$

Higher derivative Lagrangian, not galileon

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \hat{\pi}\eta_{\mu\nu}$$

$$\Lambda_5 = (M_{\rm Pl}m^4)^{1/5}$$

6th DOF appears in theory →BD ghost

(Boulware, Deser, 1972)

Adding higher-order potential terms

de Rham, Gabadadze (2010)

- *Action*

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R - \sum_{n=2}^{\infty}\frac{1}{4}m^2 U_n(g,H)\right]$$

$$\begin{aligned}
U_2(g,H) &= b_1[H^2] + b_2[H]^2 \\
U_3(g,H) &= c_1[H^3] + c_2[H][H^2] + c_3[H^3] \\
U_4(g,H) &= d_1[H^4] + d_2[H][H^3] + d_3[H^2][H^2] + d_4[H]^2[H^2] + d_5[H]^4 \\
U_5(g,H) &= f_1[H^5] + f_2[H][H^4] + f_3[H]^2[H^3] + f_4[H^2][H^3] \\
&\quad + f_5[H][H^2]^2 + f_6[H]^3[H^2] + f_7[H]^5 \\
U_6(g,H) &= \cdots
\end{aligned}$$

Fierz-Pauli tuning

$$b_1 = -b_2$$

$$\begin{aligned}
[H] &= g^{\mu\nu}H_{\mu\nu} \\
[H^2] &= g^{\mu\nu}g^{\alpha\beta}H_{\mu\alpha}H_{\nu\beta} \\
[H^3] &= \cdots
\end{aligned}$$

Eliminating 6th DOF

de Rham, Gabadadze (2010)

$$\Pi_{\mu\nu} = \partial_\mu \partial_\nu \hat{\pi}$$
$$\Pi^2_{\mu\nu} = \partial_\mu \partial^\alpha \hat{\pi} \partial_\nu \partial_\alpha \hat{\pi}$$

- *Non-linear Lagrangian*

$$\mathcal{L}_{\Pi^2} = [\Pi]^2 - [\Pi^2]$$
$$\mathcal{L}_{\Pi^3} = -\frac{1}{4\Lambda_5^5}\left((8c_1-4)[\Pi^3] + (8c_2+4)[\Pi][\Pi^2] + 8c_3[\Pi]^3\right)$$
$$\mathcal{L}_{\Pi^4} = \frac{1}{\Lambda_4^8}\left\{\left(3c_1 - 4d_1 - \frac{1}{4}\right)[\Pi^4] + \left(c_2 - 4d_3 + \frac{1}{4}\right)[\Pi^2]^2 + (2c_2 - 4d_2)[\Pi][\Pi^3] + (3c_3 - 4d_4)[\Pi^2][\Pi]^2 - 4d_5[\Pi]^4\right\}$$
$$\mathcal{L}_{\Pi^5} = \cdots$$

This yields the higher order derivative (the origin of BD ghost)

$$\mathcal{L}^{(2)}_{\mathrm{der}} = [\Pi]^2 - [\Pi^2]$$
$$\mathcal{L}^{(3)}_{\mathrm{der}} = 2[\Pi^3] - 3[\Pi][\Pi^2] + [\Pi]^3$$
$$\mathcal{L}^{(4)}_{\mathrm{der}} = -6[\Pi^4] + 3[\Pi^2]^2 + 8[\Pi][\Pi^3] - 6[\Pi^2][\Pi]^2 + [\Pi]^4$$
$$\mathcal{L}^{(5)}_{\mathrm{der}} = \cdots$$

We choose these coefficients so that the Lagrangian becomes total derivative

Eliminating 6th DOF

de Rham, Gabadadze (2010)

- *Choosing the coefficients*

$$c_1 = 2c_3 + \frac{1}{2}, \quad c_2 = -3c_3 - \frac{1}{2},$$

$$d_1 = -6d_5 + \frac{1}{16}(24c_3 + 5), \quad d_2 = 8d_5 - \frac{1}{4}(6c_3 + 1)$$
$$d_3 = 3d_5 - \frac{1}{16}(12c_3 + 1), \quad d_4 = -6d_5 + \frac{3}{4}c_3,$$

$$f_1 = \frac{7}{32} + \frac{9}{8}c_3 - 6d_5 + 24f_7, \quad f_2 = -\frac{5}{32} - \frac{15}{16}c_3 + 6d_5 - 30f_7, \quad f_3 = \frac{3}{8}c_3 - 3d_5 + 20f_7,$$
$$f_4 = -\frac{1}{16} - \frac{3}{4}c_3 + 5d_5 - 20f_7, \quad f_5 = \frac{3}{16}c_3 - 3d_5 + 15f_7, \quad f_6 = d_5 - 10f_7$$

These combinations kill all scalar self-interaction terms !

Action in decoupling limit

de Rham, Gabadadze (2010)

- *The next order interactions*

$$\mathcal{L}_{h\Pi^n} = h^{\mu\nu}X^{(1)}_{\mu\nu} - (6c_3-1)\frac{1}{\Lambda_3^3}h^{\mu\nu}X^{(2)}_{\mu\nu} + (c_3+8d_5)\frac{1}{\Lambda_3^6}h^{\mu\nu}X^{(3)}_{\mu\nu} + \cdots$$

$$X^{(1)}_{\mu\nu} = [\Pi]\eta_{\mu\nu} - \Pi_{\mu\nu}$$

$$X^{(2)}_{\mu\nu} = \Pi^2_{\mu\nu} - [\Pi]\Pi_{\mu\nu} - \frac{1}{2}([\Pi^2]-[\Pi]^2)\eta_{\mu\nu}$$

$$X^{(3)}_{\mu\nu} = 6\Pi^3_{\mu\nu} - 6[\Pi]\Pi^2_{\mu\nu} + 3([\Pi]^2-[\Pi^2])\Pi_{\mu\nu} - ([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3])\eta_{\mu\nu}$$

$$X^{(4)}_{\mu\nu} = \cdots$$

- *The action in the decoupling limit*

$$S = \int d^4x\left[-\frac{1}{2}\hat{h}^{\mu\nu}\mathcal{E}^{\alpha\beta}_{\mu\nu}\hat{h}_{\alpha\beta} + \hat{h}^{\mu\nu}\left(X^{(1)}_{\mu\nu} - \frac{6c_3-1}{\Lambda_3^3}X^{(2)}_{\mu\nu} + \frac{c_3+8d_5}{\Lambda_3^6}X^{(3)}_{\mu\nu}\right) + \frac{1}{M_{\rm Pl}}\hat{h}_{\mu\nu}T^{\mu\nu}\right]$$

(1) The remaining nonlinear interactions are galileons

(EOM is 2nd order differential equations)

(2) The cutoff energy scale is Λ_3

(We cannot trust the theory above Λ_3)

Resummation of nonlinear potential

de Rham, Gabadadze, Tolley (2011)

- *Define the new tensor*

$$\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - \sqrt{\eta_{ab} g^{\mu\alpha}\partial_\alpha\phi^a\partial_\nu\phi^b}$$

- *Property of this tensor in the decoupling limit*

$$\sqrt{T^\mu{}_\alpha}\,\sqrt{T^\alpha{}_\nu} = T^\mu{}_\nu$$

$$\mathcal{K}_{\mu\nu} = g_{\mu\alpha}\mathcal{K}^\alpha_\nu$$

$$\mathcal{K}_{\mu\nu}(g,H)\Big|_{h_{\mu\nu}=0} \equiv \Pi_{\mu\nu}$$

- *dRGT massive gravity*

$$S_{MG} = \frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}\left[R - \frac{m^2}{4}\left(\mathcal{U}_2 + \alpha_3\mathcal{U}_3 + \alpha_4\mathcal{U}_4\right)\right] + S_m[g_{\mu\nu},\psi]$$

$$\mathcal{U}_2 = 2\varepsilon_{\mu\alpha\rho\sigma}\varepsilon^{\nu\beta\rho\sigma}\mathcal{K}^\mu{}_\nu\mathcal{K}^\alpha{}_\beta = 4\left([\mathcal{K}^2] - [\mathcal{K}]^2\right)$$

$$\mathcal{U}_3 = \varepsilon_{\mu\alpha\gamma\rho}\varepsilon^{\nu\beta\delta\rho}\mathcal{K}^\mu{}_\nu\mathcal{K}^\alpha{}_\beta\mathcal{K}^\gamma{}_\delta = -[\mathcal{K}]^3 + 3[\mathcal{K}][\mathcal{K}^2] - 2[\mathcal{K}^3]$$

$$\mathcal{U}_4 = \varepsilon_{\mu\alpha\gamma\rho}\varepsilon^{\nu\beta\delta\sigma}\mathcal{K}^\mu{}_\nu\mathcal{K}^\alpha{}_\beta\mathcal{K}^\gamma{}_\delta\mathcal{K}^\rho{}_\sigma = -[\mathcal{K}]^4 + 6[\mathcal{K}]^2[\mathcal{K}^2] - 3[\mathcal{K}^2]^2 - 8[\mathcal{K}][\mathcal{K}^3] + 6[\mathcal{K}^4]$$

Total derivative in the decoupling limit

No BD ghost in full theory (Hassan, Rosen 2011)

"*Ghost-free*" nonlinear massive gravity

- *de Rham-Gabadadze-Tolley massive gravity* (de Rham, Gabadadze, Tolley, 2011)

$$S_{MG} = \frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}\left[R - \frac{m^2}{4}\left(\mathcal{U}_2 + \alpha_3\mathcal{U}_3 + \alpha_4\mathcal{U}_4\right)\right] + S_m[g_{\mu\nu},\psi]$$

$$\mathcal{U}_2 = \varepsilon_{\mu\alpha\rho\sigma}\varepsilon^{\nu\beta\rho\sigma}\mathcal{K}^\mu{}_\nu\mathcal{K}^\alpha{}_\beta$$
$$\mathcal{U}_3 = \varepsilon_{\mu\alpha\gamma\rho}\varepsilon^{\nu\beta\delta\rho}\mathcal{K}^\mu{}_\nu\mathcal{K}^\alpha{}_\beta\mathcal{K}^\gamma{}_\delta$$
$$\mathcal{U}_4 = \varepsilon_{\mu\alpha\gamma\rho}\varepsilon^{\nu\beta\delta\sigma}\mathcal{K}^\mu{}_\nu\mathcal{K}^\alpha{}_\beta\mathcal{K}^\gamma{}_\delta\mathcal{K}^\rho{}_\sigma$$

$$\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - \sqrt{\delta^\mu{}_\nu - H^\mu{}_\nu}$$
$$= \delta^\mu{}_\nu - \sqrt{\eta_{ab}g^{\mu\alpha}\partial_\alpha\phi^a\partial_\nu\phi^b}$$

ϕ^a is called Stuckelberg field, which restores general covariance

(1) *Nonlinear theory*

(2) *Lorentz invariant theory*

(3) *No ghost at full order (5 DOF, No BD ghost)* (Hassan, Rosen, 2011)

(4) *Unique theory of massive spin-2 field as an extension of general relativity (GR + mass term)*

Decoupling limit

de Rham, Gabadadze (2010)

- *Decoupling limit : Easy to capture high energy behavior within Compton wavelength of massive graviton*

π is the scalar mode of massive graviton

$$H_{\mu\nu} \to \frac{h_{\mu\nu}}{M_{\rm Pl}} + 2\frac{\partial_\mu\partial_\nu\pi}{M_{\rm Pl}m^2} - \frac{\partial_\mu\partial_\alpha\pi\partial_\mu\partial^\alpha\pi}{M_{\rm Pl}^2 m^4}$$

$$M_{\rm Pl} \to \infty, \qquad m \to 0, \qquad \Lambda_3 = (M_{\rm Pl}m^2)^{1/3} = \text{fixed}, \qquad \frac{T_{\mu\nu}}{M_{\rm Pl}} = \text{fixed}$$

- *dRGT Lagrangian in the decoupling limit*

$$\mathcal{L}_{\rm DL} = \boxed{-\frac{1}{4}h^{\mu\nu}\mathcal{E}^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} + \frac{1}{M_{\rm Pl}}h^{\mu\nu}T_{\mu\nu}}$$

Standard gravity part

$$-\boxed{h^{\mu\nu}\left[\frac{1}{4}\varepsilon_\mu{}^{\rho\gamma\alpha}\varepsilon_{\nu\rho\gamma}{}^{\beta}\Pi_{\alpha\beta} + \frac{3\alpha_3+4}{16\Lambda_3^3}\varepsilon_\mu{}^{\gamma\alpha\rho}\varepsilon_{\nu\gamma}{}^{\beta\sigma}\Pi_{\alpha\beta}\Pi_{\rho\sigma} + \frac{\alpha_3+4\alpha_4}{16\Lambda_3^6}\varepsilon_\mu{}^{\alpha\gamma\rho}\varepsilon_\nu{}^{\beta\delta\sigma}\Pi_{\alpha\beta}\Pi_{\gamma\delta}\Pi_{\rho\sigma}\right]}$$

Galileon type interactions

- *2nd order differential EOM (NO BD ghost)*
- *Cutoff energy scale is Λ_3*

$$\varepsilon\varepsilon\Pi \equiv \varepsilon^{\mu\alpha\beta\gamma}\varepsilon^{\nu}{}_{\alpha\beta\gamma}\partial_\mu\partial_\nu\pi$$

$$\varepsilon_\mu\varepsilon_\nu\Pi \equiv \varepsilon_\mu{}^{\alpha\gamma\delta}\varepsilon_\nu{}^{\beta}{}_{\gamma\delta}\partial_\alpha\partial_\beta\pi$$

$$\Pi_{\mu\nu} = \partial_\mu\partial_\nu\pi$$

Cosmologies

- *There is no flat and closed FRW solution (D'Amico et al., 2011)*
- *Open FRW solution (Gumrukcuoglu, Lin, Mukohyama, 2011)*

$$
\begin{aligned}
3\,H^2 + \frac{3\,K}{a^2} &= \Lambda_\pm + \frac{1}{M_{Pl}^2}\rho\,, \\
-\frac{2\dot{H}}{N} + \frac{2\,K}{a^2} &= \frac{1}{M_{Pl}^2}(\rho + P),
\end{aligned}
$$

$$
\Lambda_\pm \equiv -\frac{m_g^2}{(\alpha_3+\alpha_4)^2}\left[(1+\alpha_3)\left(2+\alpha_3+2\,\alpha_3^2-3\,\alpha_4\right) \pm 2\,\left(1+\alpha_3+\alpha_3^2-\alpha_4\right)^{3/2}\right]
$$

- *Linear perturbations are fine, and scalar and vector perturbations are exactly the same as GR because of vanishing the kinetic terms (Gumrukcuoglu, Lin, Mukohyama, 2011)*
- *There is ghost-instability at nonlinear level (Gumrukcuoglu, Lin, Mukohyama, 2012)*
- *Consistent massive gravity : Quasi-dilaton theory (massive graviton + scalar)(de Felice, Mukohyama, 2013), SO(3) massive gravity (Lorentz breaking) (Lin, 2013)*

Derivative interactions in massive gravity

DOF in Fierz-Pauli theory

- *Einstein-Hilbert term*

$$\mathcal{L}_{\rm EH} = \varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma}{}_{\sigma}\,\partial_\mu\partial_\alpha\, h_{\nu\beta}\, h_{\rho\gamma}$$

  - *h_{00} and h_{ij} are Lagrange multipliers*
    (Existence of Hamiltonian and momentum constraints)
  - *DOF of massless graviton = 2*

- *Fierz-Pauli mass term*

$$\mathcal{U}_{\rm FP} = \varepsilon^{\mu\alpha\rho\sigma}\varepsilon^{\nu\beta}{}_{\rho\sigma} h_{\mu\nu} h_{\alpha\beta}$$

  - *h_{00} is Lagrange multiplier*
    (Existence of Hamiltonian constraint)
  - *DOF of massless graviton = 5*

→ *Antisymmetric tensor ensures that h_{00} becomes a Lagrange multiplier*

The Hamiltonian constraint kills BD ghost

Derivative interaction in Fierz-Pauli theory

(Kurt Hinterbichler, 2013)

- *Derivative interaction in Fierz-Pauli theory*

$$\mathcal{L}_{2,3} \sim M_{\rm Pl}^2\, \varepsilon^{\mu\nu\rho\sigma}\, \varepsilon^{\alpha\beta\gamma\delta} \partial_\mu \partial_\alpha\, h_{\nu\beta}\, h_{\rho\gamma}\, h_{\sigma\delta}$$

Levi-Civita structure ensures that the Lagrangian is linear in h_{00}

→ *h_{00} becomes a Lagrange multiplier, which kills BD ghost*

- *In 4 dimension, there is no more derivative interaction, which kills BD ghost. (due to the number of indices in the antisymmetric tensor.)*

- *Fierz-Pauli theory is actually linear theory, but this derivative interactions is nonlinear !! If we want to consider this term, we need to think about Einstein-Hilbert term, instead of linearized Einstein-Hilbert.*

Our work : Is there any consistent nonlinear derivative interactions in de Rham-Gabadadze-Tolley massive gravity??

$$S_{MG} = \frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}\left[R - \frac{m^2}{4}\left(\mathcal{U}_2 + \alpha_3\mathcal{U}_3 + \alpha_4\mathcal{U}_4\right)\right]\boxed{+ S_{int}} + S_m[g_{\mu\nu},\psi],$$

$$\mathcal{U}_2 = \varepsilon_{\mu\alpha\rho\sigma}\varepsilon^{\nu\beta\rho\sigma}\mathcal{K}^\mu{}_\nu\mathcal{K}^\alpha{}_\beta$$

$$\mathcal{U}_3 = \varepsilon_{\mu\alpha\gamma\rho}\varepsilon^{\nu\beta\delta\rho}\mathcal{K}^\mu{}_\nu\mathcal{K}^\alpha{}_\beta\mathcal{K}^\gamma{}_\delta$$

$$\mathcal{U}_4 = \varepsilon_{\mu\alpha\gamma\rho}\varepsilon^{\nu\beta\delta\sigma}\mathcal{K}^\mu{}_\nu\mathcal{K}^\alpha{}_\beta\mathcal{K}^\gamma{}_\delta\mathcal{K}^\rho{}_\sigma$$

$$\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - \sqrt{\delta^\mu{}_\nu - H^\mu{}_\nu} = \delta^\mu{}_\nu - \sqrt{\eta_{ab}g^{\mu\alpha}\partial_\alpha\phi^a\partial_\nu\phi^b}$$

Guidelines for construction of Lagrangian

- *Candidates for derivative interactions using the Riemann tensor*

$$\mathcal{L}_{int} \supset M_{\rm Pl}^2\sqrt{-g}HR,\ M_{\rm Pl}^2\sqrt{-g}H^2R,\ M_{\rm Pl}^2\sqrt{-g}H^3R,\ \cdots$$

$$H_{\mu\nu} = g_{\mu\nu} - \eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b$$

- *Guidelines*

  (1) Linearization of $h_{\mu\nu}$ reproduces Fierz-Pauli theory

  - *Lorentz invariance*
  - *Free of Boulware-Deser ghost at linear level*

  (2) Cut off energy scale is Λ_3

  - *All nonlinear terms below Λ_3 have to be eliminated*

  (3) Free of Boulware-Deser ghost

Energy scales of derivative interactions in the decoupling limit

- *General form of Lagrangian*

$$\mathcal{L}_{int} \supset M_{\rm Pl}^2\sqrt{-g}HR,\ M_{\rm Pl}^2\sqrt{-g}H^2R,\ M_{\rm Pl}^2\sqrt{-g}H^3R,\ \cdots$$

- *The Lagrangian in the decoupling limit can be schematically written as*

$$\mathcal{L}_{int} \sim \Lambda_\lambda^{2-n_h-3n_\pi} h^{n_h-1}\partial^2 h\,(\partial^2\pi)^{n_\pi}$$

	n_h =1	n_h =2
n_π =1	∞	Λ_3
n_π =2	Λ_5	Λ_3
n_π =3	Λ_4	Λ_3
n_π =4	$\Lambda_{11/3}$	Λ_3
...	...	...
n_π =n	$\Lambda_{(3n-1)/(n-1)}$	Λ_3

These has to be eliminated

$$H_{\mu\nu} \to \frac{h_{\mu\nu}}{M_{\rm Pl}} + 2\frac{\partial_\mu\partial_\nu\pi}{M_{\rm Pl}m^2} - \frac{\partial_\mu\partial_\alpha\pi\partial_\mu\partial^\alpha\pi}{M_{\rm Pl}^2 m^4}$$

$$\Lambda_\lambda = (M_p m^{\lambda-1})^{1/\lambda}$$

$$\lambda = \frac{n_h+3n_\pi-2}{n_h+n_\pi-2}$$

HR order term

- *General Lagrangian of HR order*

$$\mathcal{L}_{int,1} = M_{\rm Pl}^2 \sqrt{-g} H_{\mu\nu} (R^{\mu\nu} + d\, R g^{\mu\nu})$$

Linearizing $h_{\mu\nu}$ gives the same order of the linearized Einstein-Hilbert

$$\mathcal{L}^{(2)}_{int,1} \propto M_{\rm Pl}^2 \left[\sqrt{-g} R \right]_{h^2}$$

$$\rightarrow \boxed{d = -1/2}$$

- *In terms of Levi-Civet symbol,*

$$\mathcal{L}_{int,1} = M_{\rm Pl}^2 \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma}{}_{\sigma} R_{\mu\alpha\nu\beta} H_{\rho\gamma}$$

The Lagrangian satisfies requirement (1) : Fierz-Pauli theory at linear theory

HR order term in the decoupling limit

- *The lowest order term in the decoupling limit*

$$\left.\mathcal{L}_{int,1}\right|_{\partial^2 h\,\partial^2\pi} = -\frac{2}{m^2}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma}{}_{\sigma}\partial_\mu\partial_\alpha\, h_{\nu\beta}\,\partial_\rho\partial_\gamma\pi$$

$$= -\frac{2}{m^2}\partial_\gamma\left(\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma}{}_{\sigma}\partial_\mu\partial_\alpha\, h_{\nu\beta}\,\partial_\rho\pi\right)$$

Total derivative

- *The next order term in the decoupling limit*

$$\left.\mathcal{L}_{int,1}\right|_{\partial^2 h\,(\partial^2\pi)^2} = \frac{1}{\Lambda_5^5}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma}{}_{\sigma}\partial_\mu\partial_\alpha\, h_{\nu\beta}\,\partial_\rho\partial_a\pi\partial^a\partial_\gamma\pi$$

This is not zero or total derivative, and EOM contains higher derivatives

→ *The counter part of this term can eliminate this term*

$$\mathcal{L}_{int,1,2} = \frac{1}{4}M_{\rm Pl}^2\sqrt{-g}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma}{}_{\sigma}R_{\mu\alpha\nu\beta}\,H_{\rho a}H^a{}_\gamma$$

Λ_5 term is eliminated !

HR order term in the decoupling limit

- *HR order Lagrangian*

$$\mathcal{L}_{int,1} = M_{\rm Pl}^2 \sqrt{-g}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma}{}_{\sigma} R_{\mu\alpha\nu\beta} H_{\rho\gamma}$$
$$\sim \Lambda_\lambda^{2-n_h-3n_\pi} h^{n_h-1}\partial^2 h\,(\partial^2\pi)^{n_\pi}\Big|_{\rm DL}$$

	n_h =1
n_π =1	∞
n_π =2	Λ_5
n_π =3	Λ_4
n_π =4	$\Lambda_{11/3}$
...	...
n_π =n	$\Lambda_{(3n-1)/(n-1)}$

Automatically total derivative

These terms can be always eliminated by adding higher order terms

HR order term

- *The total Lagrangians including counter terms is given by*

$$\mathcal{L}_{int,1} = M_{\rm Pl}^2\sqrt{-g}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma}{}_{\sigma}R_{\mu\alpha\nu\beta}$$
$$\times\boxed{\left(H_{\rho\gamma}+\frac{1}{4}H_{\rho a}H^{a}{}_{\gamma}+\frac{1}{8}H_{\rho a}H^{a}{}_{b}H^{b}{}_{\gamma}+\frac{5}{64}H_{\rho a}H^{a}{}_{b}H^{b}{}_{c}H^{c}{}_{\gamma}+\cdots\right)}$$
$$= 2\,\mathcal{K}_{\rho\gamma}$$

$$\mathcal{K}^{\mu}{}_{\nu}=\delta^{\mu}{}_{\nu}-\sqrt{\delta^{\mu}{}_{\nu}-H^{\mu}{}_{\nu}}=-\sum_{n=1}^{\infty}\bar{d}_n(H^n)^{\mu}_{\nu},$$

- *The final Lagrangian of HR order term*

$$\mathcal{L}_{int,1}=M_{\rm Pl}^2\sqrt{-g}\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma}{}_{\sigma}R_{\mu\alpha\nu\beta}\,\mathcal{K}_{\rho\gamma}$$

The Lagrangian satisfies requirements (2) : Λ_3 theory in the decoupling limit

H^2R order term

- *Consider the most general combination of this order*

$$\mathcal{L}_{int,2} = M_{\rm Pl}^2 \sqrt{-g}\, R^{\mu\nu\alpha\beta}(c_1 H_{\mu\alpha} H_{\nu\beta} + \cdots)$$

	$n_h = 1$
$n_\pi = 2$	Λ_5
$n_\pi = 3$	Λ_4
$n_\pi = 4$	$\Lambda_{11/3}$
...	...
$n_\pi = n$	$\Lambda_{(3n-1)/(n-1)}$

This order should be a total derivative, otherwise BD ghost appear

$$\sim\ \varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\,\partial_\mu\partial_\alpha h_{\nu\beta}\,\partial_\rho\partial_\gamma\pi\,\partial_\sigma\partial_\delta\pi$$

These terms might be eliminated by adding higher order terms

H^2R order term

- *The Lagrangian is*

$$\mathcal{L}_{int,2} = M_{\rm Pl}^2 \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} R_{\mu\alpha\nu\beta} H_{\rho\gamma} H_{\sigma\delta}$$

This is the only combination that the lowest order Λ_5 term becomes a total derivative

- *With the same method of the previous case, we get the resumed Lagrangian of H^2R order term*

$$\mathcal{L}_{int,2} = M_{\rm Pl}^2 \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} R_{\mu\alpha\nu\beta} \mathcal{K}_{\rho\gamma} \mathcal{K}_{\sigma\delta}$$

- *H^3R, HR^2 or higher order terms?? → No!*

In four dimension, there is no total derivative combination of the lowest order term in the decoupling limit

Riemann derivative interactions

- *In 4 dimension, the general derivative interaction for massive graviton is*

$$\mathcal{L}_{int} = M_{\rm Pl}^2 \sqrt{-g}\, \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} R_{\mu\alpha\nu\beta} \left(\alpha\, g_{\rho\gamma} \mathcal{K}_{\sigma\delta} + \beta\, \mathcal{K}_{\rho\gamma} \mathcal{K}_{\sigma\delta}\right)$$

α and β are parameters

- *We can also construct derivative interactions in arbitrary dimensions D*

$$\mathcal{L}_{int}^{(D,d,m)} = M_{\rm Pl}^{D-2} m^{2-d} \sqrt{-g}\, \varepsilon^{\mu_1\mu_2\cdots\mu_D} \varepsilon^{\nu_1\nu_2\cdots\nu_D} R_{\mu_1\nu_1\mu_2\nu_2} \cdots R_{\mu_{d-1}\nu_{d-1}\mu_d\nu_d}$$
$$\times\, g_{\mu_{d+1}\nu_{d+1}} \cdots g_{\mu_m\nu_m} \mathcal{K}_{\mu_{m+1}\nu_{m+1}} \cdots \mathcal{K}_{\mu_D\nu_D}$$

d is even number

$2 \leq d \leq m \leq D-1$

These Lagrangians satisfy the requirements (1) and (2)

Boulware-Deser ghost??

- *We constructed the Λ_3 nonlinear derivative interactions, but we still need to check the requirement (3) : the existence of BD ghost*

  - *Λ_3 theory in the decoupling limit*

$$\mathcal{L}_{\rm DL} \sim \boxed{\frac{1}{\Lambda_3^3}\pi\left[R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right]_{h^2}} + \boxed{\frac{1}{\Lambda_3^{3n_\pi}}\mathcal{O}[h\partial^2 h\,(\partial^2\pi)^{n_\pi}]}$$

EOM is 2nd order differential equation (coming from $L_{int,2}$)

These terms yield 4th order differential Eq for h and π (coming from $L_{int,1}$ and $L_{int,2}$)

There are extra degrees of freedom, which leads to ghost...

Ghost appears at Λ_3

Other derivative interactions (in progress)

- *In 4 dimension, we found other Λ_3 derivative interactions without the Riemann tensor*

$$\mathcal{L}'_{int,1} = -M_{\rm Pl}^2\sqrt{-g}\,\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma}{}_{\sigma}\,\nabla_\alpha\mathcal{K}_{\nu\beta}\,\nabla_\mu\mathcal{K}_{\rho\gamma}$$
$$\mathcal{L}'_{int,2} = -2M_{\rm Pl}^2\sqrt{-g}\,\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}\,\nabla_\alpha\mathcal{K}_{\nu\beta}\,\nabla_\mu\mathcal{K}_{\rho\gamma}F_{\delta\sigma}(H_{..})$$

- *Λ_3 theory in the decoupling limit*

$$\mathcal{L}_{\rm DL} \sim \boxed{\frac{1}{\Lambda_3^3}\,\pi\left[R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right]_{h^2}} + \boxed{\frac{1}{\Lambda_3^{3n_\pi}}\mathcal{O}[h\partial^2 h\,(\partial^2\pi)^{n_\pi}]}$$

EOM is 2nd order differential equation (coming from $L'_{int,2}$)

These terms yield 4th order differential Eq for h and π (coming from $L'_{int,1}$ and $L'_{int,2}$)

We cannot kill higher derivative terms in EOM even if we combine all four derivative interaction terms…

Appropriate mass scale of derivative interactions?

- *So far, the mass scale of the derivative interactions was M_{pl}*

$$\mathcal{L}_{int} = \boxed{M_{\rm Pl}^2} \sqrt{-g}\, \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} R_{\mu\alpha\nu\beta} \left(\alpha\, g_{\rho\gamma}\, \mathcal{K}_{\sigma\delta} + \beta\, \mathcal{K}_{\rho\gamma}\, \mathcal{K}_{\sigma\delta}\right)$$

If the mass scale M is not M_{Pl} and smaller than M_{Pl},
the ghost scale is roughly above the cutoff energy scale.

- *In addition, if $M < M_{Pl}$,*

$$\mathcal{L}_{int,2} = \overset{M}{\cancel{M_{\rm Pl}^2}} \sqrt{-g} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} R_{\mu\alpha\nu\beta}\, \mathcal{K}_{\rho\gamma}\, \mathcal{K}_{\sigma\delta}$$

$$\mathcal{L}_{\rm DL} \sim \frac{1}{\cancel{\Lambda_3^3}} \pi \left[R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right]_{h^2} + \frac{1}{\cancel{\Lambda_3^{3n_\pi}}} \mathcal{O}[h \partial^2 h\, (\partial^2 \pi)^{n_\pi}]$$

$\Lambda_R > \Lambda_3$ $\Lambda_{ghost} > \Lambda_R > \Lambda_3$

If $\Lambda_{ghost} > \Lambda_{cutoff} > \Lambda_R > \Lambda_3$, the theory might still survive…

Summary

- *We found the most general derivative interactions in dRGT massive gravity*
  - *The energy scales below Λ_3 can be eliminated by adding counter terms*
  - *The Lagrangians can be resumed by using K tensor*
  - *The most general derivative interactions contain four interactions*
  - *Nonlinear terms contribute at Λ_3*
- *Appropriate DOF?*
  - *4th order differential EOM of the scalar and tensor mode in the decoupling limit*
  - *Ghost appears at Λ_3 in dRGT theory + derivative interactions*

The mass scale of the derivative interactions should be $M < M_{pl}$