Correlation Functions in N=4 SYM from INTEGRABILITY

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Outline

- The AdS/CFT Correspondence
- Integrability in N=4 SYM
- Two-Point Function
- Three-Point Function
- Conclusion and Outlook

The AdS/CFT Correspondence

String Theory on *d+1*-dimensional space-time (AdS)

=

Conformal Field Theory on *d*-dimensional space-time (CFT)

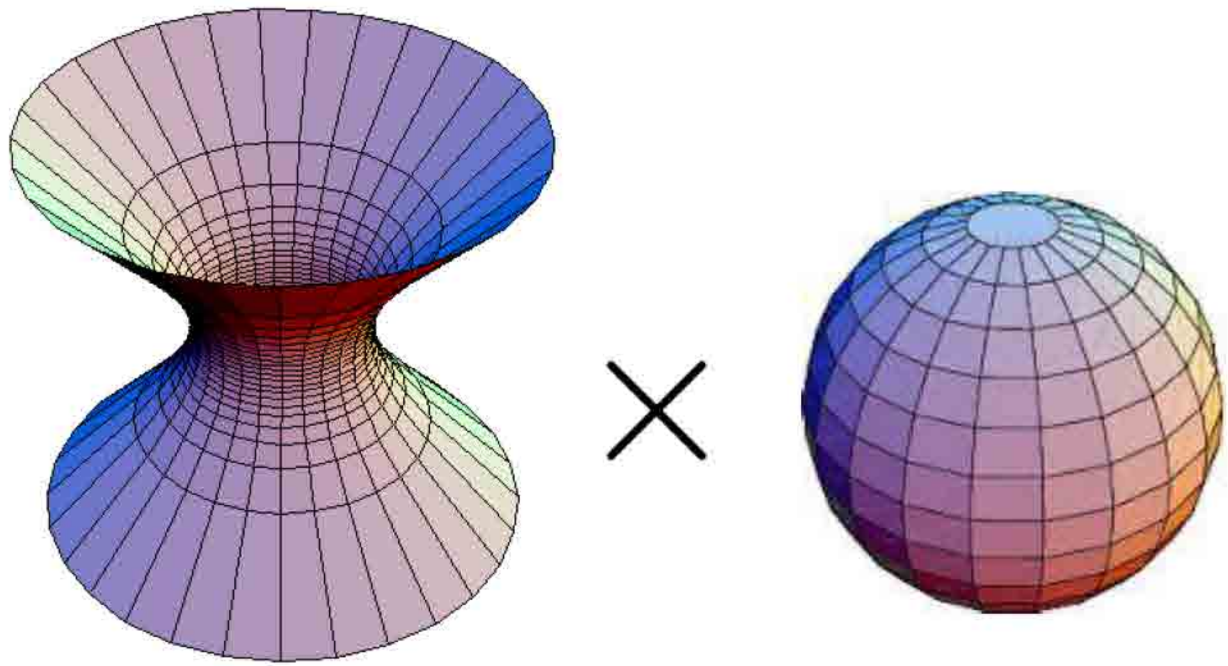
Why it is interesting ?

1. Holographic Principle
2. Strong-weak duality
3. Solve strongly-correlated system
4. Integrability

How does it work ?

Let us consider the first and most well-known example !

Type IIB String Theory



AdS_5 $\times$ S^5

Super Yang-Mills Theory

C.N. Yang and R. Mills

$+$

$a = 1, 2, 3, 4$

Q^a_α

$\bar{Q}_{\dot{\alpha} a}$

Supersymmetry

Symmetry

String Theory

AdS_5 $\qquad\qquad$ S^5

Isometry Group

$$SU(2,2) \times SU(4) \subset SU(2,2|4)$$

Conformal symmetry $\qquad$ *R-symmetry*

Gauge Theory

Parameters

String Theory

String Coupling Constnat

$$4\pi g_s = g_{\mathrm{YM}}^2 \equiv \frac{\lambda}{N}$$

Gauge Coupling Constant

Radius *String tension*

$$\lambda = R^4/\alpha'^2$$

't Hooft Coupling

Gauge Theory

The N=4 SYM

1. Field Contents

$$\Phi_i, \quad i = 1, \cdots, 6$$

$$A_\mu, \quad \mu = 0, 1, 2, 3$$

$$\psi_{a,\alpha}, \quad a = 1, 2, 3, 4 \qquad \alpha = 1, 2$$

$$\bar{\psi}^a_{\dot{\alpha}}, \quad a = 1, 2, 3, 4 \qquad \dot{\alpha} = 1, 2$$

2. Scalar sectors

SO(6) sector

$$\Phi_i, \quad i = 1, \cdots, 6$$

$$Z = \Phi_1 + i\Phi_2 \qquad \bar{Z} = \Phi_1 - i\Phi_2$$

$$X = \Phi_3 + i\Phi_4 \qquad \bar{X} = \Phi_3 - i\Phi_4$$

$$Y = \Phi_5 + i\Phi_6 \qquad \bar{Y} = \Phi_5 - i\Phi_6$$

SO(4)=SU(2) × SU(2) sector

$$\{X, \bar{X}, Z, \bar{Z}\}$$

$$\mathcal{O}_{so(4)} = \mathrm{Tr}\left(XX\bar{X}ZX\bar{Z}XZ\cdots\right)$$

SU(2) sectors

$$\{X, Z\} \qquad \{\bar{X}, Z\}$$

Not Unique !

Integrability

I. **Spin chain for so(6) sector**
(Minahan, Zarembo 2002)

II. **All sectors of psu(2,2|4)**
(Beisert, Staudacher 2003)

III. **Higher loop**
(Beisert, Staudacher, Kristjansen 2003)

IV. **All-loop Asymptotic Bethe Ansatz**
(Beisert Staudacher 2005; Janik 2006; Beisert, Eden, Staudacher 2006)

V. **Finite-size correction (TBA)**
(Janik et al 2005-2012; Aruytunov, Frolov 2007, 2010;
Kazakov, Gromov, Vieira 2008-2013;
Cavaglia, Fioravanti,Tateo 2009, 2010)

Integrability

I. Lax connection of string sigma model
(Bena, Roiban, Polchinski 2002)

II. Exact string solutions
(Frolov, Tseytlin 2002-2004)

III. Spectral curve
(Kazakov, Mashakov, Minahan, Zarembo, 2004;
Beisert, Sakai, Kazakov,Zarembo 2005)

Integrability

I. **Cusp anomalous dimension**

(Basso, Krochemsky, Kostov, Serban, Volin, Frolov, Tseytlin, Polyakov, Kosower, Dixon, Smirnov)

II. **9-loop Konishi**

(Janik, Lukowski, Bajnok Gromov, Kazakov, Vieira, Leurent, Serban, Volin,Heslop, Korchemsky, Eden, Sokachev, Roiban, Tseytlin, Mazzucato, Vallilo, Valatka, Schenderovich)

Two-Point Function

$$\langle \mathcal{O}^A(x)\mathcal{O}^B(y)\rangle \sim \frac{\delta_{AB}}{|x-y|^{2\Delta^A(g)}}$$

$$\Delta(g) = \Delta_0 + \gamma(g) = \Delta_0 + \sum_{n=1}^{\infty} g^{2n}\gamma_n$$

Coupling constant : $g = \frac{\sqrt{\lambda}}{4\pi}$

1. Dilatation Operator

$$\hat{D}(g)\mathcal{O}_A = \Delta_A^B(g)\mathcal{O}_B$$

Operator Mixing

$$\hat{D}(g) = \Delta_0 + \sum_{n=1}^{\infty} g^{2n} H_{2n}$$

2. SO(6) Sector

$$\mathcal{O}^{so(6)}(x) = C^{i_1,\cdots,i_L}\,\mathrm{Tr}\left(\Phi_{i_1}\cdots\Phi_{i_L}\right)$$

$$\hat{D}(g) = \boxed{\Delta_0} + \sum_{n=1}^{\infty} g^{2n} H_{2n}$$

$$\boxed{\Delta_0^{so(6)} = L}$$

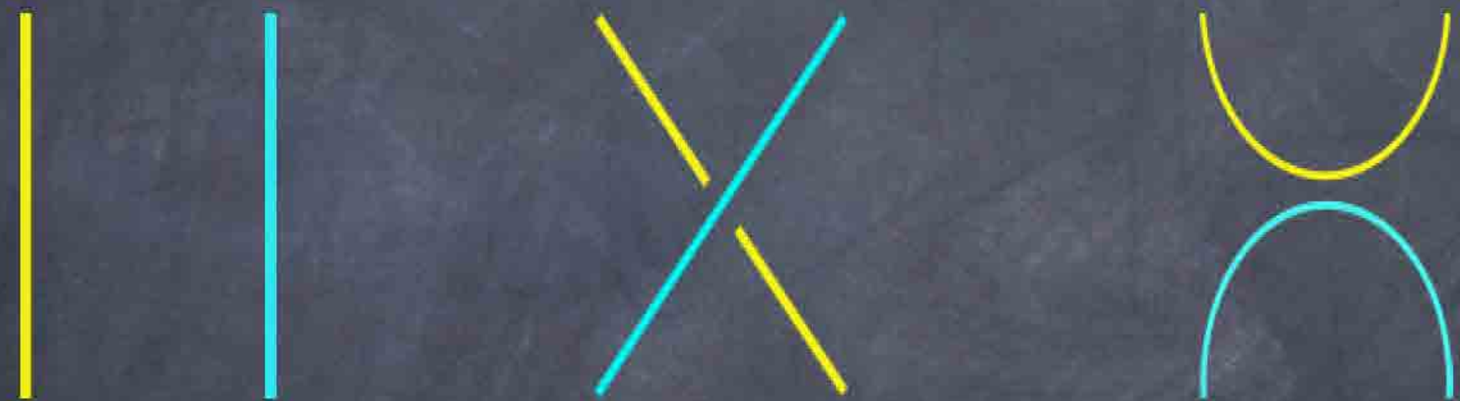
$$\hat{D}(g) = \Delta_0 + \sum_{n=1}^{\infty} g^{2n} H_{2n}$$

$$H_2^{so(6)} = \sum_{l=1}^{L} (\mathrm{K}_{l,l+1} + 2\mathrm{I}_{l,l+1} - 2\mathrm{P}_{l,l+1})$$

SO(6) Integrable Spin Chain !

$$\delta_{i_l}^{j_l}\,\delta_{i_{l+1}}^{j_{l+1}} \qquad \delta_{i_l}^{j_{l+1}}\,\delta_{i_{l+1}}^{j_l} \qquad \delta_{i_l,i_{l+1}}\,\delta^{j_l,j_{l+1}}$$

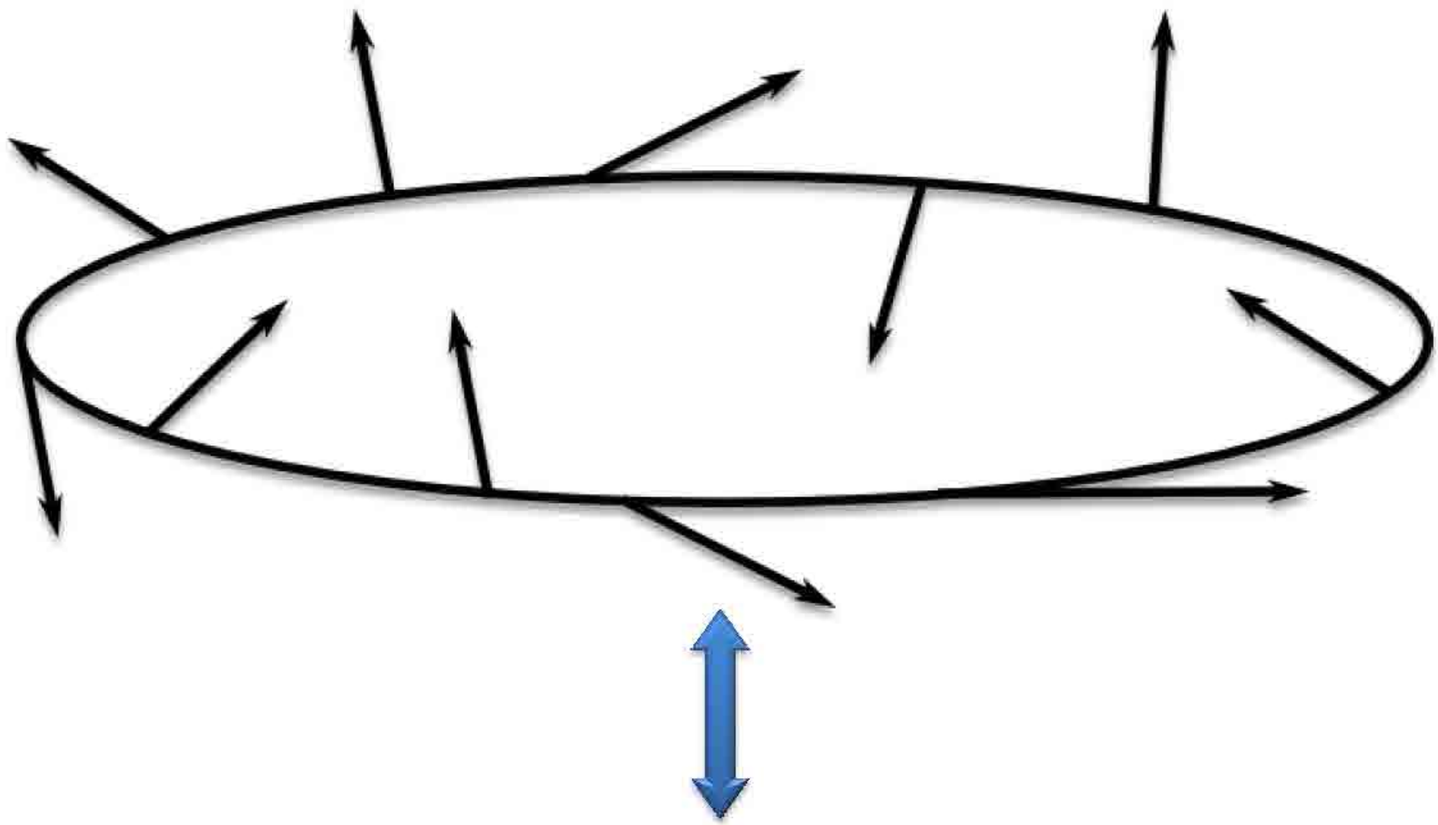
$$j_l \quad j_{l+1} \qquad j_l \quad j_{l+1} \qquad j_l \quad j_{l+1}$$



$$i_l \quad i_{l+1} \qquad i_l \quad i_{l+1} \qquad i_l \quad i_{l+1}$$

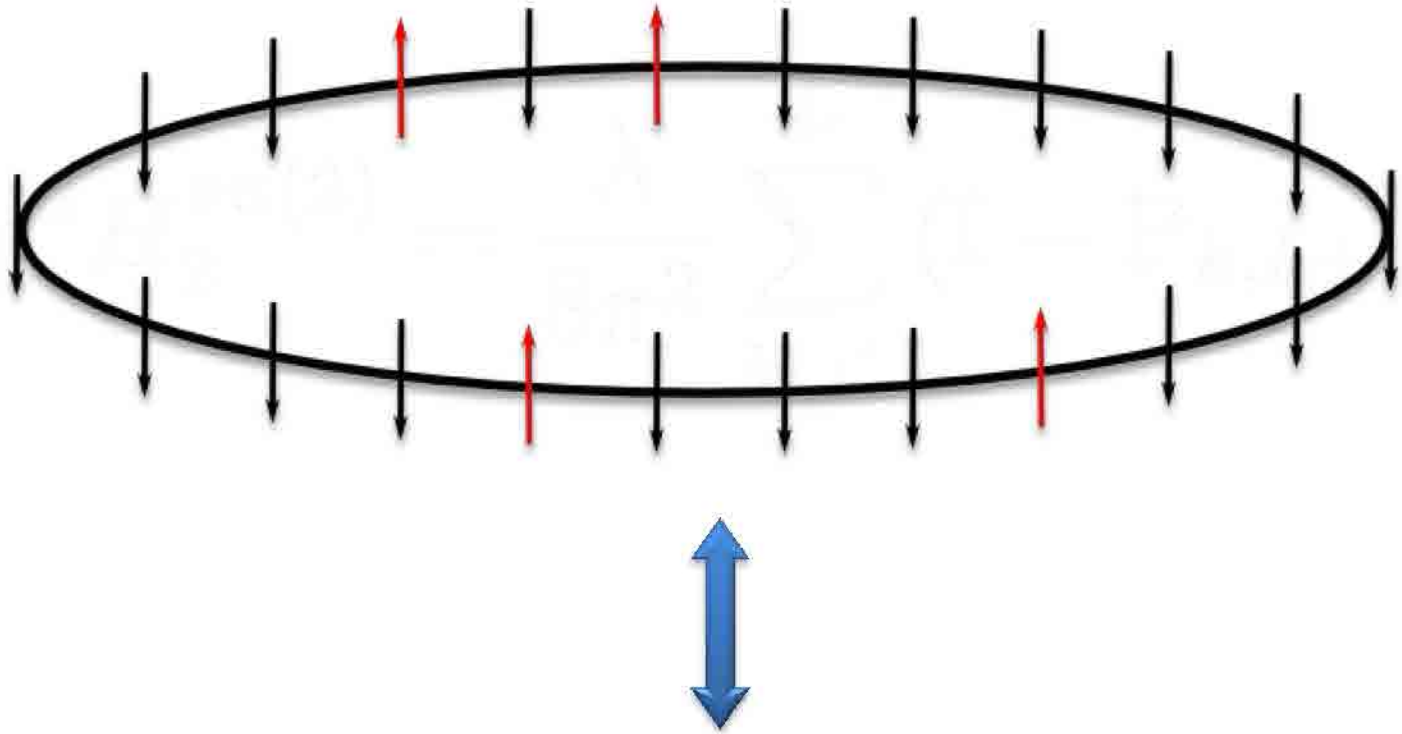
$$\mathrm{I}_{i_l,i_{l+1}}^{j_l,j_{l+1}} \qquad \mathrm{P}_{i_l,i_{l+1}}^{j_l,j_{l+1}} \qquad \mathrm{K}_{i_l,i_{l+1}}^{j_l,j_{l+1}}$$

SO(6) Spin Chain



$$\mathrm{Tr}\left(\Phi_{i_1}\cdots\Phi_{i_L}\right)$$

SU(2) Spin Chain



$$\mathrm{Tr}\,(ZZZXZXZZ\cdots Z)$$

Heisenberg XXX Spin Chain

$$H_2^{su(2)} = 2\sum_{l=1}^{L}\left(\mathrm{I}_{l,l+1} - \mathrm{P}_{l,l+1}\right)$$

$$\mathrm{P}_{k,k+1}\left|\cdots\uparrow\downarrow\cdots\right\rangle = \left|\cdots\downarrow\uparrow\cdots\right\rangle$$

EXACTLY SOLVABLE

How to solve it ?

1. Coordinate Bethe Ansatz
2. Algebraic Bethe Ansatz
3. Separation of Variables
4. Baxter's Q-operator

Bethe Ansatz

"Ansatz" is a German word for a precedure which means "make a guess for the solution, and check whether it works".

— *M. Staudacher*

Bethe Ansatz

Hans Albrecht Bethe
1906-2005

Nobel Prize 1967

Bethe's Guess

- ✓ **Pseudo-vacuum**

$$|\Omega\rangle = |\underbrace{\downarrow \cdots \downarrow}_{L}\rangle$$

- ✓ **One-magnon state**

$$|\Psi_1\rangle = \sum_{n=1}^{L} e^{ipn} \sigma_n^+ |\Omega\rangle$$

- ✓ **Two-magnon state**

$$|\Psi_2\rangle = \sum_{1 \le k_1 < k_2 \le L} \psi(n_1, n_2) \sigma_{n_1}^+ \sigma_{n_2}^+ |\Omega\rangle$$

Scattering Matrix

$$\psi(n_1, n_2) = e^{ip_1 n_1 + ip_2 n_2} + \boxed{S(p_2, p_1)} e^{ip_2 n_1 + ip_1 n_2}$$

Bethe's Guess

- ✓ **Three-magnon state**

$$|\Psi_3\rangle = \sum_{1\le k_1<k_2<k_3\le L} \psi(n_1,n_2,n_3)\sigma^+_{n_1}\sigma^+_{n_2}\sigma^+_{n_3}|\Omega\rangle$$

Factorized Scattering

$$\begin{aligned}\psi(n_1,n_2,n_3) = e^{ip_1n_1+ip_2n_2+ip_3n_3} \\ + e^{ip_2n_1+ip_1n_2+ip_3n_3} S(p_2,p_1) \\ + e^{ip_2n_1+ip_3n_2+ip_1n_3} S(p_2,p_1)S(p_3,p_1) \\ + \text{other possible combinations}\end{aligned}$$

- ✓ ***N*-magnon state**

Bethe Equations

$$e^{ip_k L} = \prod_{j \neq k}^{N} S(p_j, p_k), \quad k = 1, \cdots, N$$

$$S(p_j, p_k) = -\frac{e^{ip_j + ip_k} - 2e^{ip_j} + 1}{e^{ip_j + ip_k} - 2e^{ip_k} + 1}$$

✓ **Change of Variables : Rapidity**

$$u = \frac{1}{2} \cot \frac{p}{2}$$

✓ **Bethe Equations (BE)**

$$\left(\frac{u_k + i/2}{u_k - i/2} \right)^L = \prod_{j \neq k}^{N} \frac{u_k - u_j + i}{u_k - u_j - i}$$

✓ **Number of Equations**

$$k = 1, \cdots, N$$

✓ Solve BAE and calculate energy !

$$E_2^{su(2)} = \frac{\lambda}{8\pi^2} \sum_{k=1}^{L} \frac{1}{u_k^2 + 1/4}$$

JOBS
DONE

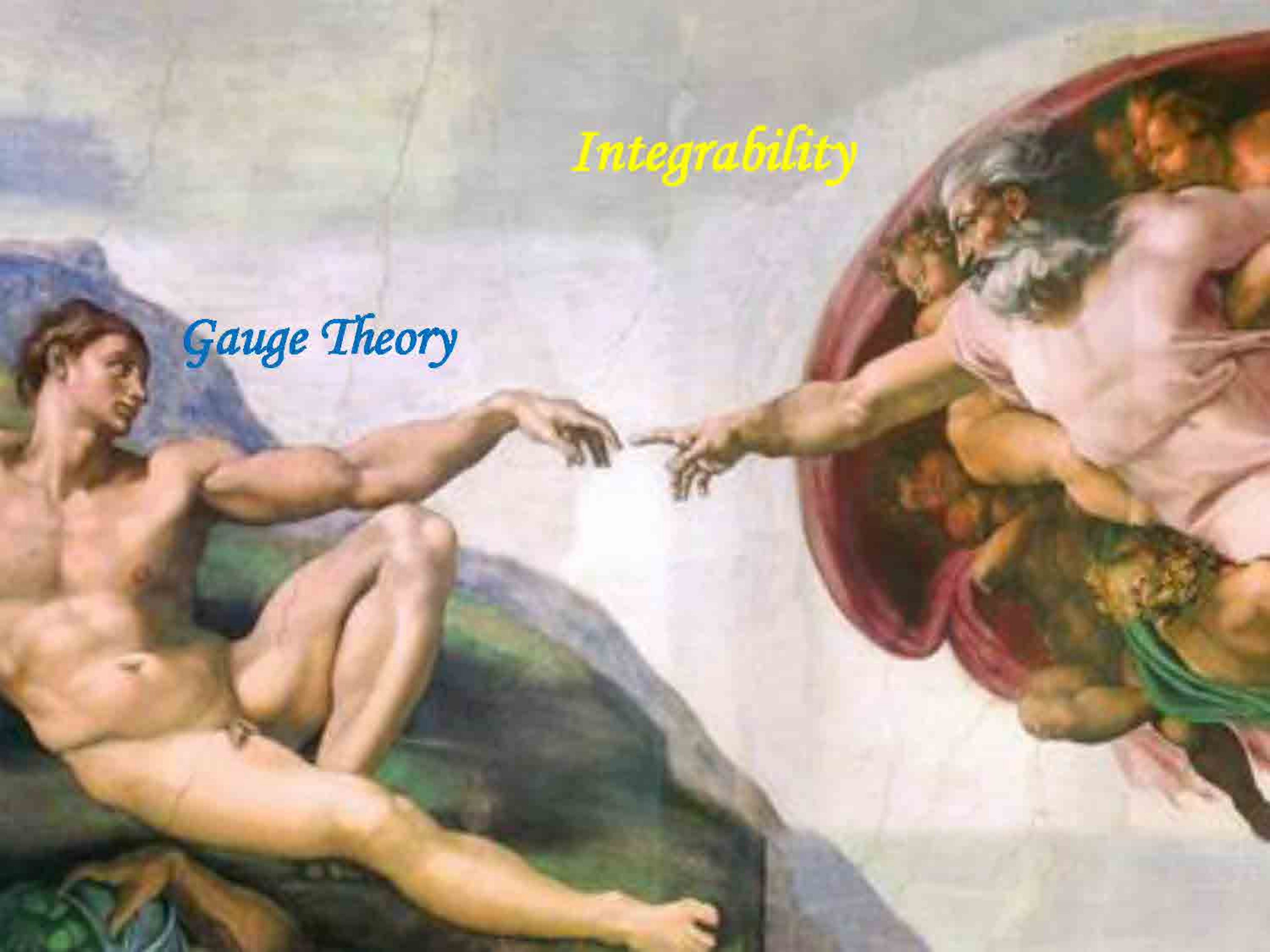
Why it is nice ?

- A problem of dimension:

$$2^L \times 2^L$$

- A system of $N < L/2$ equations !

INTEGRABILITY !


Integrability
Gauge Theory


Algebraic
Functional
Coordinate
Bethe Ansatz ！！！

Three-Point Function at Tree Level and One Loop

Based on the work :

Kostov 2012 arxiv: 1205.4412 ; Kostov Matsuo 2012 arxiv: 1207.2562

Y.J., Kostov, Loebbert, Serban 2013 arxiv: 1310.XXXX

Three-Point Function

1. Formulation of the problem

   (Okuyama, Tseng 2004)

2. Algebraic Bethe ansatz

   (Roiban, Volovich 2004)

3. Scalar products

   (Escobedo, Gromov, Sever, Vieira 2010-2011)

4. Tailoring vs. Freezing

   (Foda 2011; Kostov 2012; Kostov Matsuo 2012)

5. One loop calculation

   (Gromov, Vieira 2012; Serban 2012;)

Three-Point Function

1. Formulation of the problem

   (Janik et al 2010-2012; Buchbinder, Tseytlin 2011)

2. Heavy-Heavy-Light

   (Zarembo 2010; Costa et al 2010)

3. Heavy-Heavy-Heavy

   (Janik el al 2011; Kazama, Komatsu 2011-2013)

4. Comparison

   (Escobedo, Gromov, Sever, Vieira 2011)

Formulation of the problem

1. Operators (EGSV)

- Choice of sectors
- Wave functions

✓ Choice of sectors

Operators	Field Contents	Sectors
$\mathcal{O}_1$	$Z, \quad X$	$SU(2)_L$
$\mathcal{O}_2$	$\bar{Z}, \quad \bar{X}$	$SU(2)_L$
$\mathcal{O}_3$	$Z, \quad \bar{X}$	$SU(2)_R$

✓ Wave functions

Eigenstates of *one-loop* dilatation operator.

2. Structure Constant

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{L_i \delta_{ij}}{|x_{12}|^{2\Delta_i}}$$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{L_1 L_2 L_3 C_{123}(g)}{|x_{12}|^{\Delta_{12}}|x_{23}|^{\Delta_{23}}|x_{13}|^{\Delta_{13}}}$$

$$|x_{ij}| = |x_i - x_j|$$

$$\Delta_{ij} = \Delta_i + \Delta_j - \Delta_k \qquad i, j, k = 1, 2, 3$$

$$N_c C_{123}(g) = C_{123}^{(0)} + g^2 C_{123}^{(1)} + \cdots$$

3. Spin Chain Language

$$\mathcal{O}_1 \to |\mathbf{u}\rangle, \mathcal{O}_2 \to |\mathbf{v}\rangle, \mathcal{O}_3 \to |\mathbf{w}\rangle$$

$$\mathbf{u} = \{u_1, \cdots, u_{N_1}\}$$

$$\mathbf{v} = \{v_1, \cdots, v_{N_2}\}$$

$$\mathbf{w} = \{w_1, \cdots, w_{N_3}\}$$

$$C_{123}^{(0)} = \frac{\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle}{\sqrt{\langle \mathbf{u}|\mathbf{u}\rangle \langle \mathbf{v}|\mathbf{v}\rangle \langle \mathbf{w}|\mathbf{w}\rangle}}$$

The problem becomes

1. Calculate the norm
2. Calculate the cubic vertex
3. Scalar products of Bethe states

Algebraic Bethe Ansatz !

Algebraic Bethe Ansatz

1. R-matrix

2. Monodromy Matrix

3. Transfer Matrix

4. *RTT* relations (Yangian)

Algebraic Bethe Ansatz

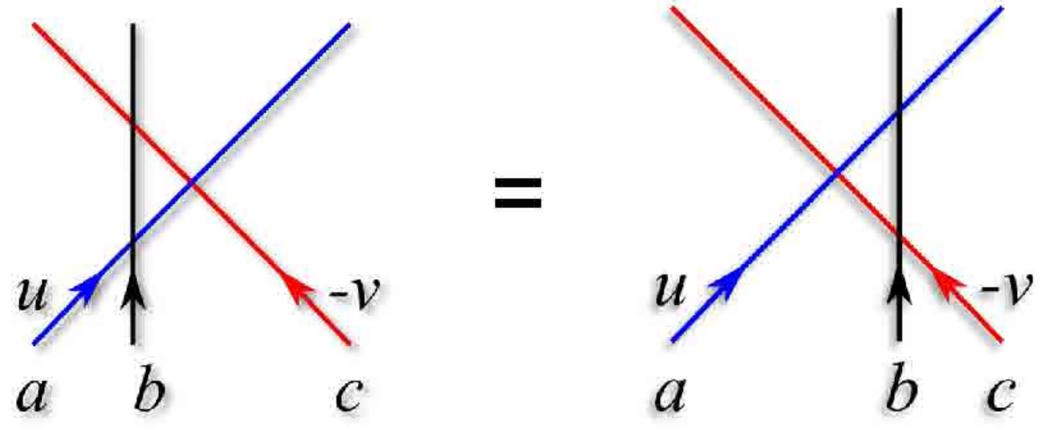
- ✓ **su(2)-Invariant *R*-Matrix**

$$R_{ab}(u) = u\mathrm{I}_{ab} + i\mathrm{P}_{ab}$$

$$V_a \otimes V_b, \quad V_a = V_b = \mathbb{C}^2$$

- ✓ **Yang-Baxter Equation**

$$R_{ab}(u)R_{ac}(u+v)R_{bc}(v) = R_{bc}(v)R_{ac}(u+v)R_{ab}(u)$$



Algebraic Bethe Ansatz

- ✓ **Lax Matrix**

$$L_{an}(u,\theta_n) = \mathrm{I}_{an} + \frac{i}{u - \theta_n - i/2}\mathrm{P}_{an}$$

Inhomogeneous

- ✓ **Monodromy Matrix**

$$\mathcal{T}_a(u) = \prod_{n=1}^{L} L_{an}(u,\theta_n)$$

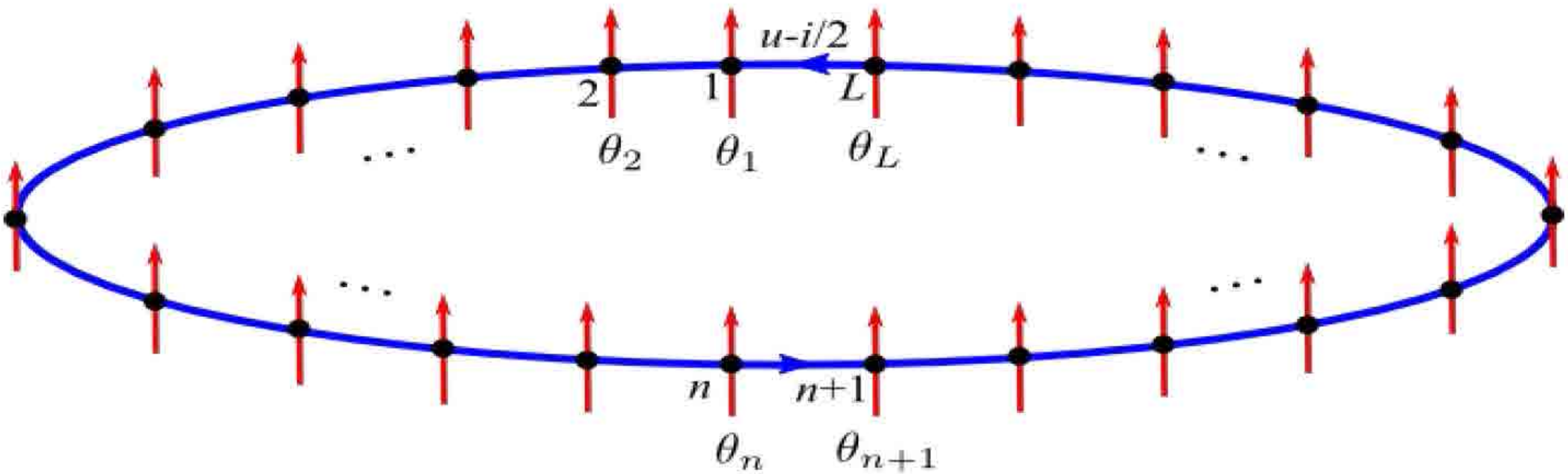
- ✓ **Transfer Matrix**

$$T(u) = \mathrm{Tr}_a\, \mathcal{T}_a(u)$$

Algebraic Bethe Ansatz

✓ **Monodromy Matrix**

$$\mathcal{T}_a(u) = \prod_{n=1}^{L} L_{an}(u, \theta_n)$$



Algebraic Bethe Ansatz

- ✓ **Elements of monodromy matrix**

$$\mathcal{T}_a(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix}_a$$

- ✓ **Pseudo-vacuum**

$$|\Omega\rangle = |\underbrace{\downarrow\downarrow\cdots\downarrow}_{L}\rangle$$

Algebraic Bethe Ansatz

✓ **Action of monodromy matrix**

$$\mathcal{A}(u)|\Omega\rangle = a(u)|\Omega\rangle$$
$$\mathcal{D}(u)|\Omega\rangle = d(u)|\Omega\rangle$$
$$\mathcal{C}(u)|\Omega\rangle = 0$$

✓ **Off-shell Bethe states**

$$|\mathbf{u}_N\rangle = \mathcal{B}(u_1)\cdots\mathcal{B}(u_N)|\Omega\rangle$$

Algebraic Bethe Ansatz

- ✓ **On-shell Bethe states**

$$|\mathbf{u}_N\rangle = \mathcal{B}(u_1)\cdots\mathcal{B}(u_N)|\Omega\rangle$$

$$\frac{a(u_k)}{d(u_k)} = \prod_{j\neq k}^{N} \frac{u_k - u_j + i}{u_k - u_j - i}$$

- ✓ **Eigenstate of transfer matrix**

$$T(u)|\mathbf{u}\rangle = t_{\mathbf{u}}(u)|\mathbf{u}\rangle$$

Algebraic Bethe Ansatz

$$t_{\mathbf{u}}(u) = a(u)\frac{Q_{\mathbf{u}}(u-i)}{Q_{\mathbf{u}}(u)} + d(u)\frac{Q_{\mathbf{u}}(u+i)}{Q_{\mathbf{u}}(u)}$$

$$a(u) = \prod_{n=1}^{L}(u - \theta_n + i/2) \equiv Q_\theta(u + i/2)$$

$$d(u) = \prod_{n=1}^{L}(u - \theta_n - i/2) \equiv Q_\theta(u - i/2)$$

$$Q_{\mathbf{u}}(u) = \prod_{k=1}^{N}(u - u_k)$$

Baxter Polynomial

Algebraic Bethe Ansatz

✓ **Expansion of Transfer Matrix**

$$Q_r = \frac{1}{i(r-1)!} \frac{\mathrm{d}^{r-1}}{\mathrm{d}u^{r-1}} \log T(u) \bigg|_{u=i/2}$$

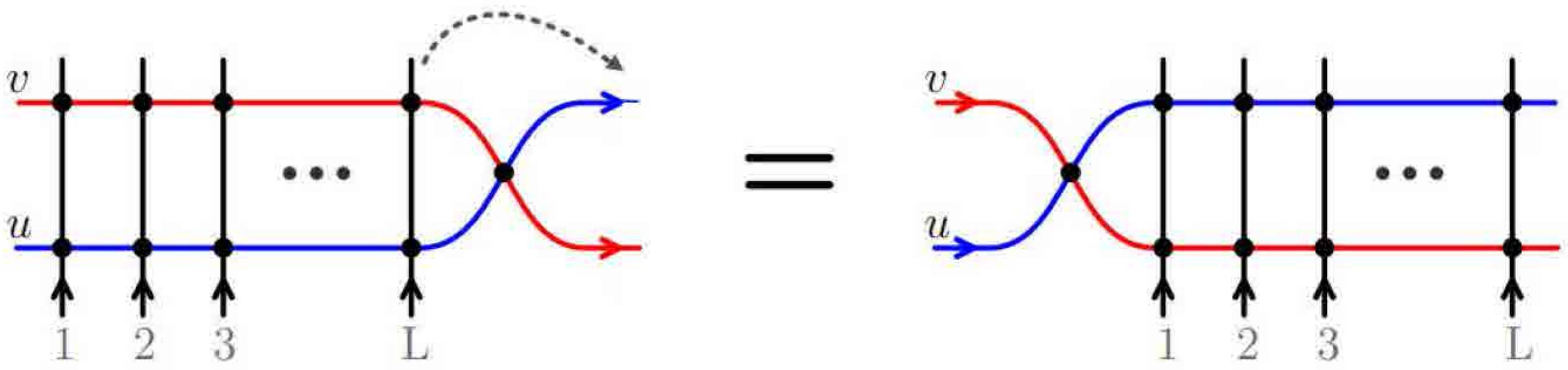
$$Q_2 = H_2 = \sum_{k=1}^{L} (\mathrm{I}_{k,k+1} - \mathrm{P}_{k,k+1})$$

$$Q_3 = \frac{i}{2} \sum_{k=1}^{L} [\mathrm{P}_{k,k-1}, \mathrm{P}_{k,k+1}]$$

Algebraic Bethe Ansatz

✓ **The *RTT* Relations**

$$R_{ab}(u-v)\mathcal{T}_a(u)\mathcal{T}_b(v) = \mathcal{T}_b(v)\mathcal{T}_a(u)R_{ab}(u-v)$$



Algebraic Bethe Ansatz

Algebra between Elements

$$[\mathcal{C}(v), \mathcal{B}(u)] = g(u-v)[\mathcal{A}(v)\mathcal{D}(u) - \mathcal{A}(u)\mathcal{D}(v)]$$

$$\mathcal{A}(v)\mathcal{B}(u) = f(u-v)\mathcal{B}(u)\mathcal{A}(v) + g(v-u)\mathcal{B}(v)\mathcal{A}(u)$$

$$\mathcal{D}(v)\mathcal{B}(u) = f(v-u)\mathcal{B}(u)\mathcal{D}(v) + g(u-v)\mathcal{B}(v)\mathcal{D}(u)$$

$$[\mathcal{B}(u), \mathcal{B}(v)] = [\mathcal{C}(u), \mathcal{C}(v)] = 0$$

$$[\mathcal{A}(u), \mathcal{A}(v)] = [\mathcal{D}(u), \mathcal{D}(v)] = 0$$

$$g(u) \equiv \frac{i}{u}, \quad f(u) = 1 + \frac{i}{u}$$

Scalar Products

1. Definition

2. Evaluation Method

3. Slavnov determinant formulas

Scalar Products

- ✓ **Definition**

$$\mathcal{S}_{\mathbf{v},\mathbf{u}} = \langle \mathbf{v} | \mathbf{u} \rangle = \langle \Omega | \prod_{j=1}^{N} \mathcal{C}(v_j) \prod_{k=1}^{N} \mathcal{B}(u_k) | \Omega \rangle$$

$$\mathcal{B}^\dagger(u) = -\mathcal{C}(u^*)$$

- ✓ **Evaluation Method**

Use the RTT relations repeatedly !


Slavnov
Determinant !

Scalar Products

✓ **Slavnov Determinant Formulas**

When one of the Bethe states is on-shell, the scalar product can be written in a compact form in terms of determinants, which is found by Nikita Slavnov.

$$\langle \mathbf{v} | \mathbf{u} \rangle = \prod_{j=1}^{N} a(v_j) d(u_j) \mathscr{S}_{\mathbf{u},\mathbf{v}}$$

$$\mathscr{S}_{\mathbf{u},\mathbf{v}} = \frac{\det_{jk} \Omega(u_j, v_k)}{\det_{jk} \frac{1}{u_j - v_k + i}}$$

$$\Omega(u, v) = t(u - v) - e^{2ip_{\mathbf{u}}(v)} t(v - u)$$

$$t(u) = \frac{1}{u} - \frac{1}{u + i}$$

$$e^{2ip_{\mathbf{u}}(u)} \equiv \frac{d(u)}{a(u)} \frac{Q_{\mathbf{u}}(u + i)}{Q_{\mathbf{u}}(u - i)}$$

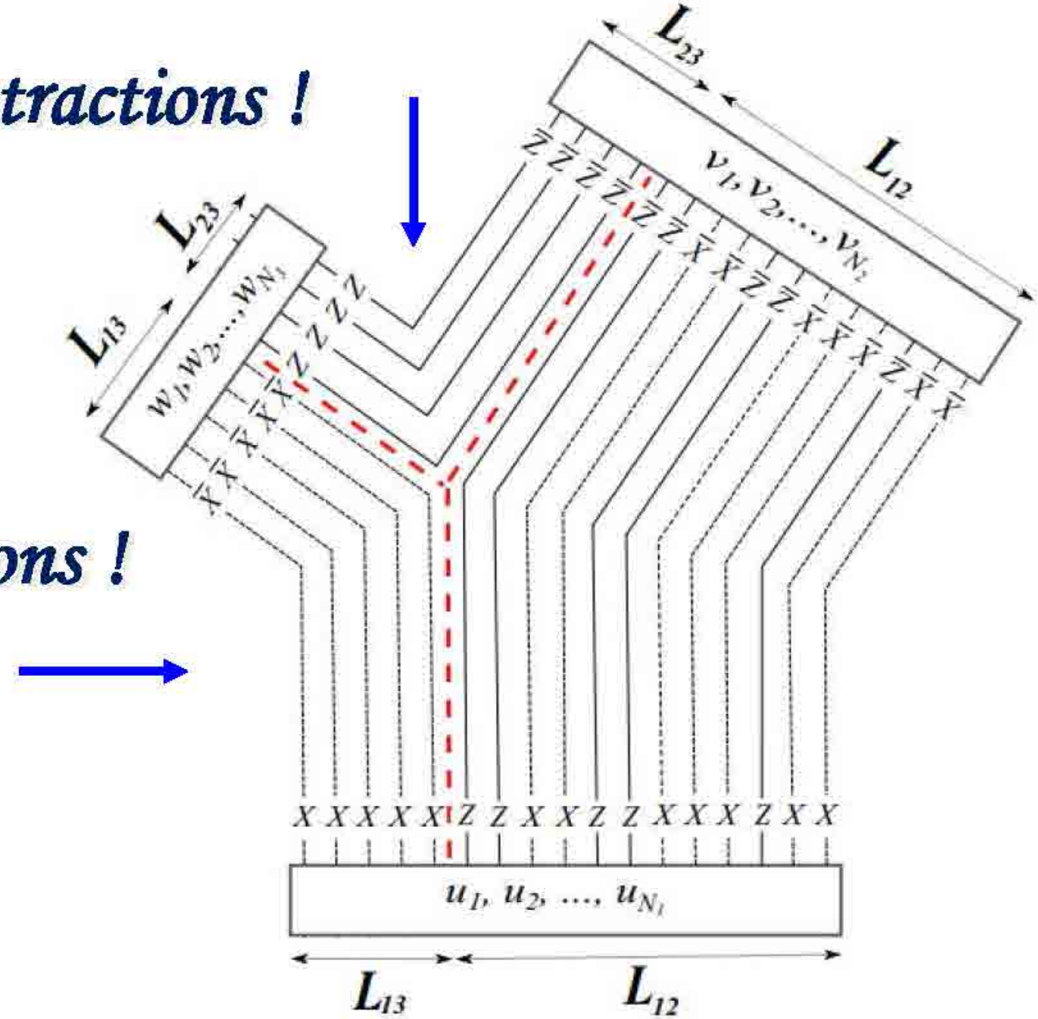
Three-Point Function

1. **Tailoring** Three-Point Function
2. **Freezing** Three-Point Function
3. Sutherland limit
4. **Fixing** Three-Point Function

Planar Diagram

Trivial contractions !

Trivial contractions !



Tailoring

(Escobedo, Gromov, Sever, Vieira 2010-2011)

Cutting

$$|\mathbf{u}\rangle \to \sum_{\mathbf{u}'\cup\mathbf{u}''=\mathbf{u}} |\mathbf{u}'\rangle \otimes |\mathbf{u}''\rangle$$

$$|\mathbf{v}\rangle \to \sum_{\mathbf{v}'\cup\mathbf{v}''=\mathbf{v}} |\mathbf{v}'\rangle \otimes |\mathbf{v}''\rangle$$

$$|\mathbf{w}\rangle \to \sum_{\mathbf{w}'\cup\mathbf{w}''=\mathbf{w}} |\mathbf{w}'\rangle \otimes |\mathbf{w}''\rangle$$

Flipping

$$|\mathbf{u}'\rangle \otimes |\mathbf{u}''\rangle \to |\mathbf{u}'\rangle \otimes \langle \mathbf{u}''^*|$$

$$|\mathbf{v}'\rangle \otimes |\mathbf{v}''\rangle \to |\mathbf{v}'\rangle \otimes \langle \mathbf{v}''^*|$$

$$|\mathbf{w}'\rangle \otimes |\mathbf{w}''\rangle \to |\mathbf{w}'\rangle \otimes \langle \mathbf{w}''^*|$$

Tailoring

Sewing

$$C_{123}^{(0)} \sim \sum_{\text{partitions}} \frac{\langle \mathbf{u}''^* | \mathbf{v}' \rangle \langle \mathbf{v}''^* | \mathbf{w}' \rangle \langle \mathbf{w}''^* | \mathbf{u}' \rangle}{\sqrt{\langle \mathbf{u} | \mathbf{u} \rangle \langle \mathbf{v} | \mathbf{v} \rangle \langle \mathbf{w} | \mathbf{w} \rangle}}$$

- Sum over partitions
- Non-trivial factors at each step
- Scalar products of Off-shell states

Freezing

(Foda 2011)

1. Mapping to a 6-vertex model
2. Cutting out the trivial piece
3. Two independent pieces
4. Calculate the two pieces

Freezing

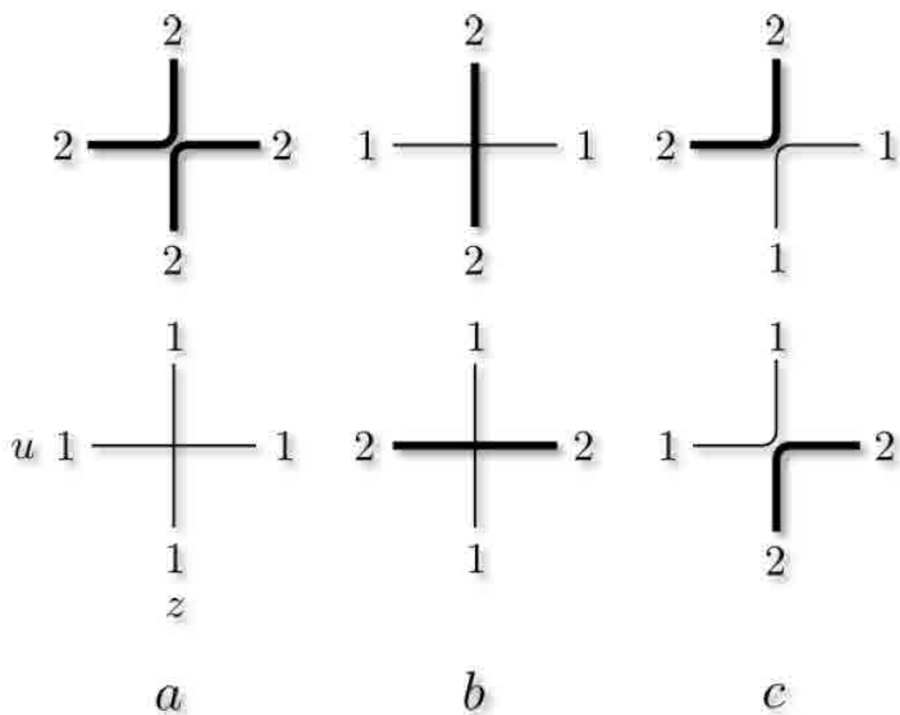
Heisenberg XXX spin chain

$$L(u-z) = \begin{pmatrix} a(u-z) & 0 & 0 & 0 \\ 0 & b(u-z) & c(u-z) & 0 \\ 0 & c(u-z) & b(u-z) & 0 \\ 0 & 0 & 0 & a(u-z) \end{pmatrix}$$

$$a(u-z) = \frac{u-z+i}{u-z}, \quad b(u-z) = 1, \quad c(u-z) = \frac{i}{u-z}$$

Freezing

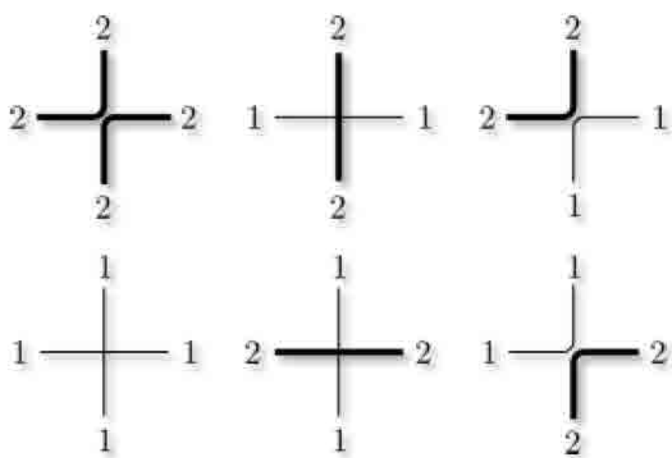
6-Vertex Model



Freezing

Special Value

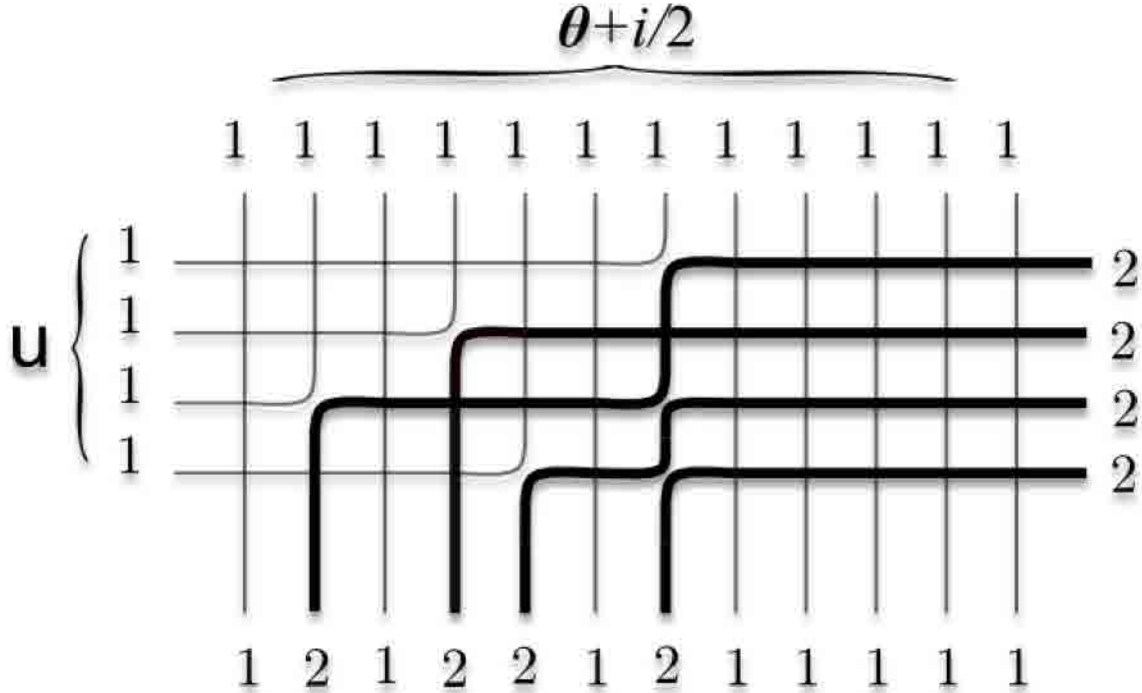
$$u = \theta - i/2, \quad z = \theta + i/2$$



$$a = 0, \quad b = 1, \quad c = -1$$

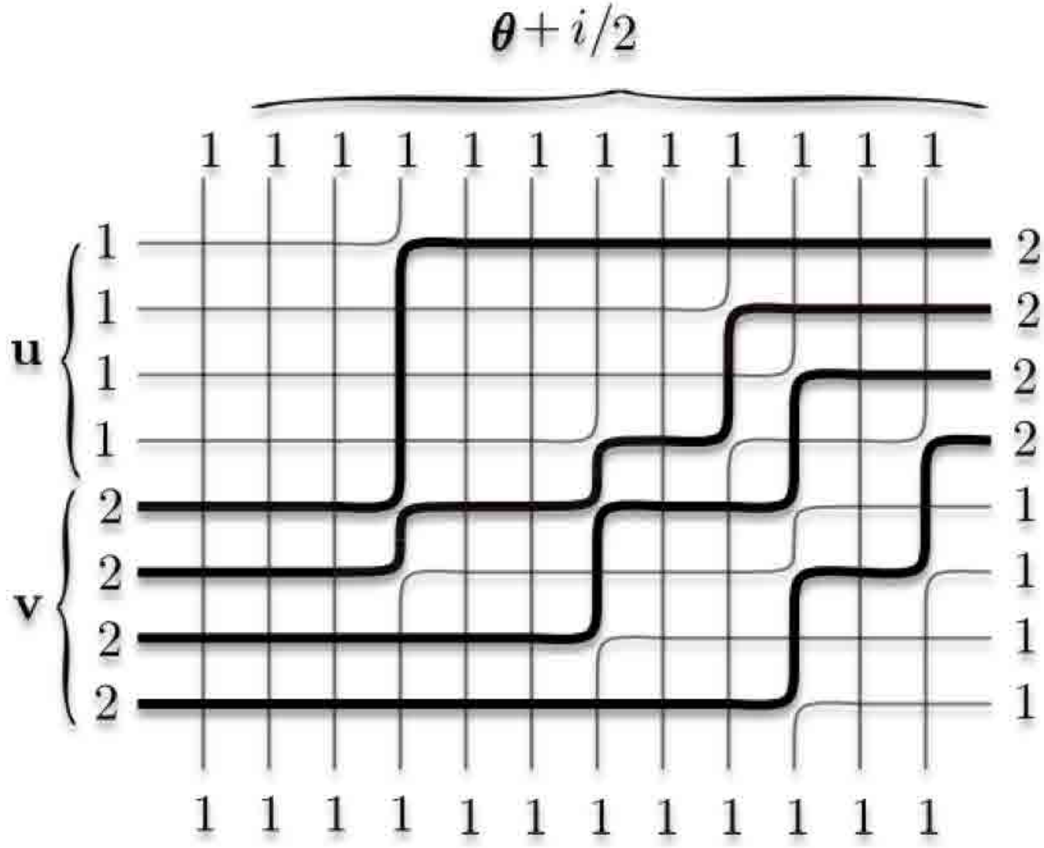
Freezing

Bethe state configuration

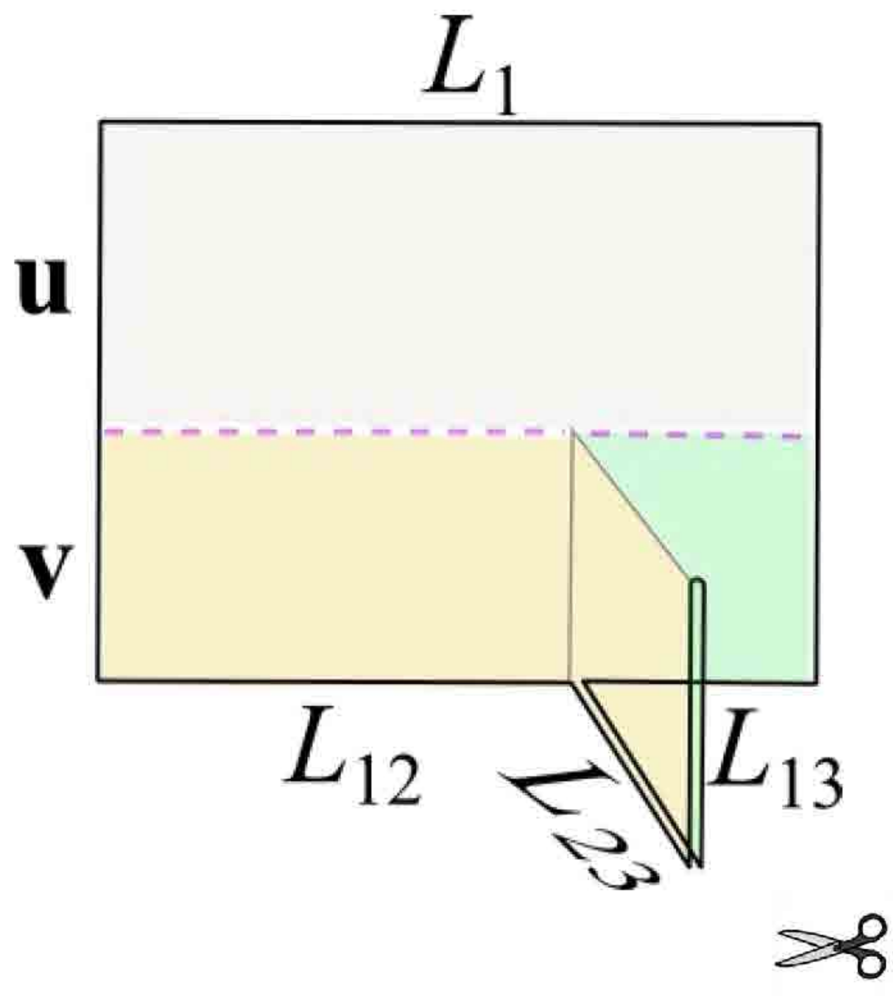


Freezing

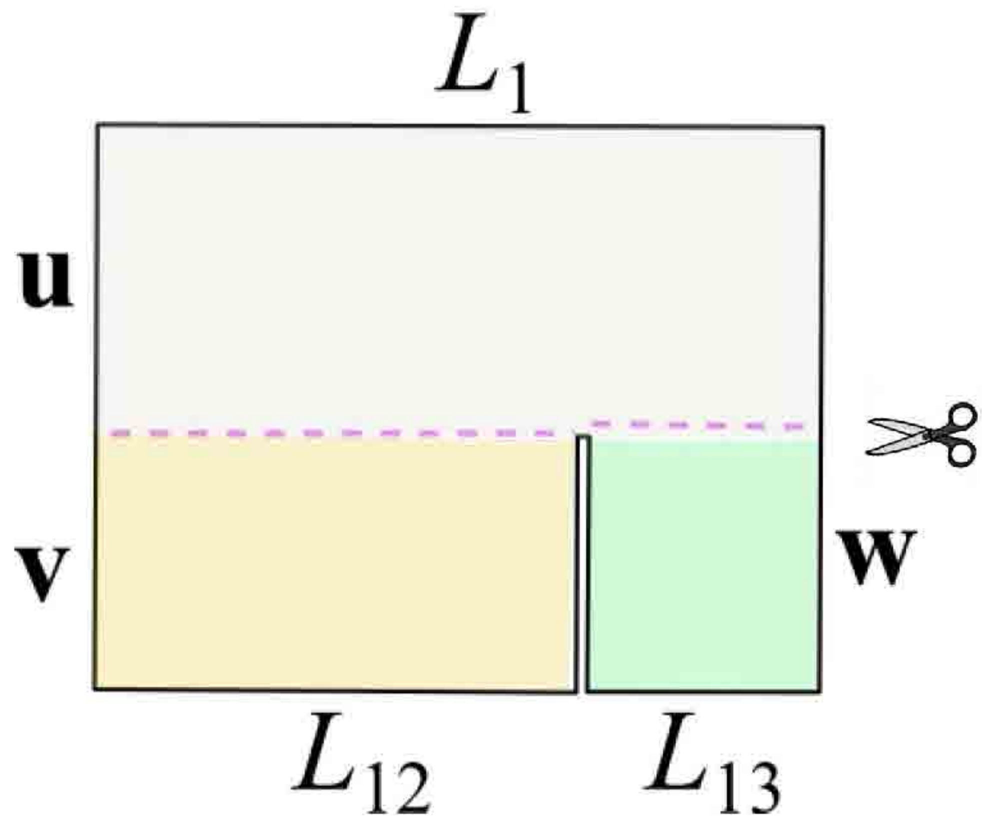
Scalar Product Configuration



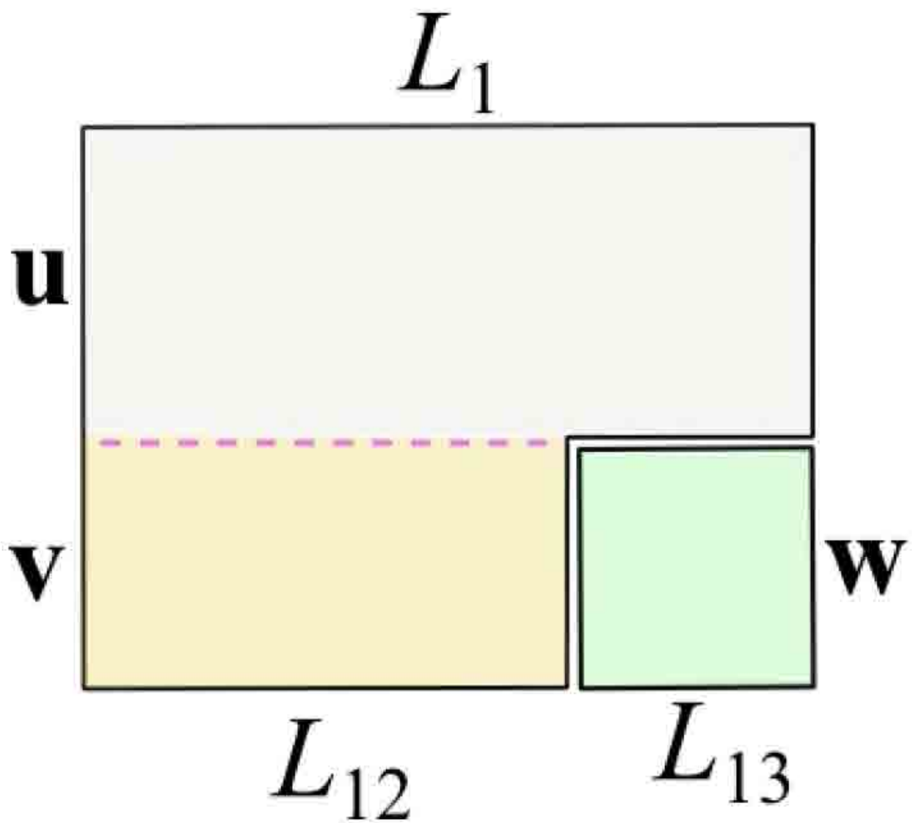
✓ **Cut out trivial piece**

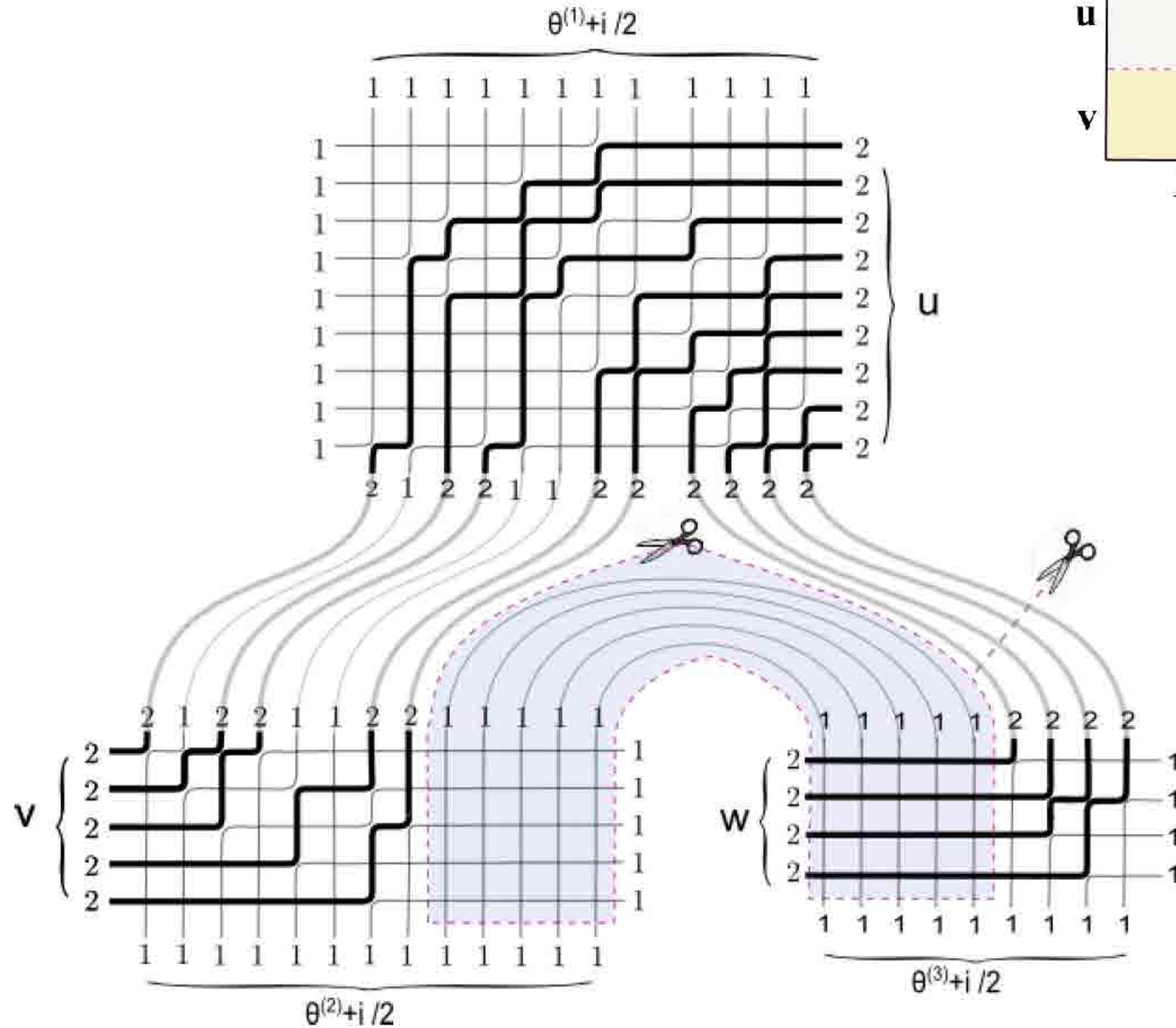


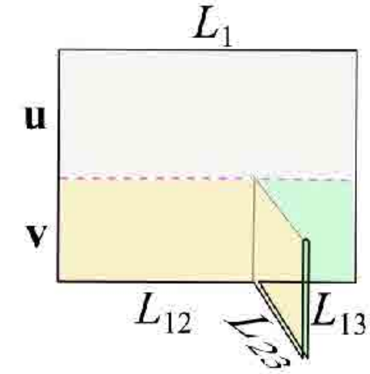
✓ **Cut another piece**

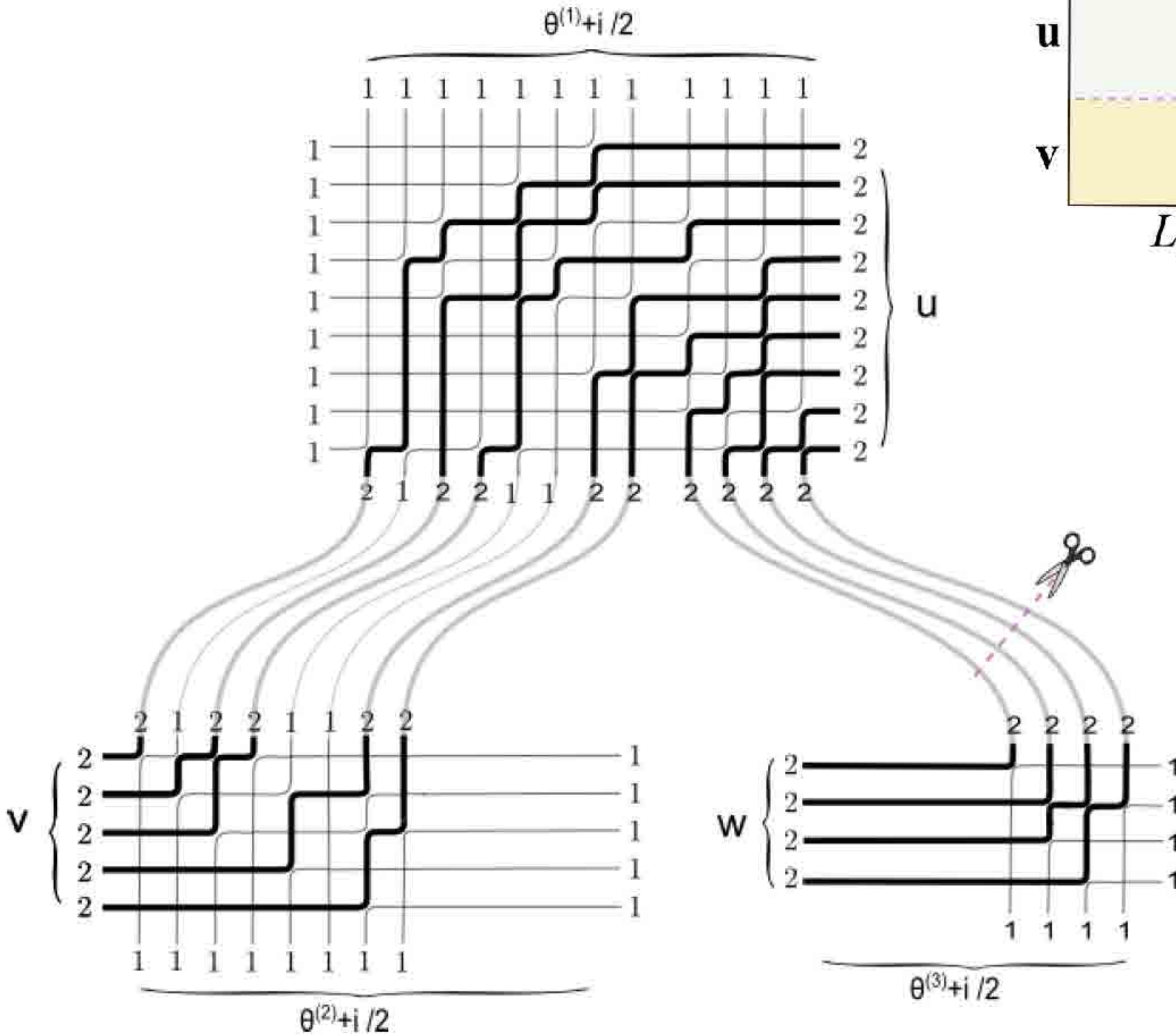


✓ **Two independent pieces**

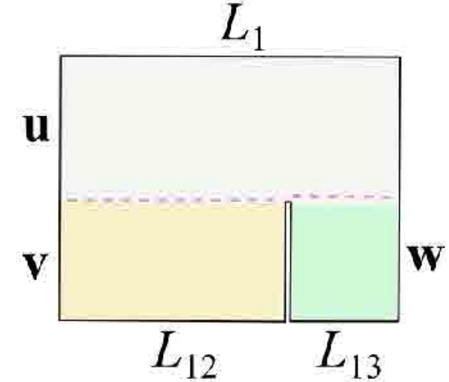



θ(1)+i /2
u
v
w
θ(2)+i /2
θ(3)+i /2

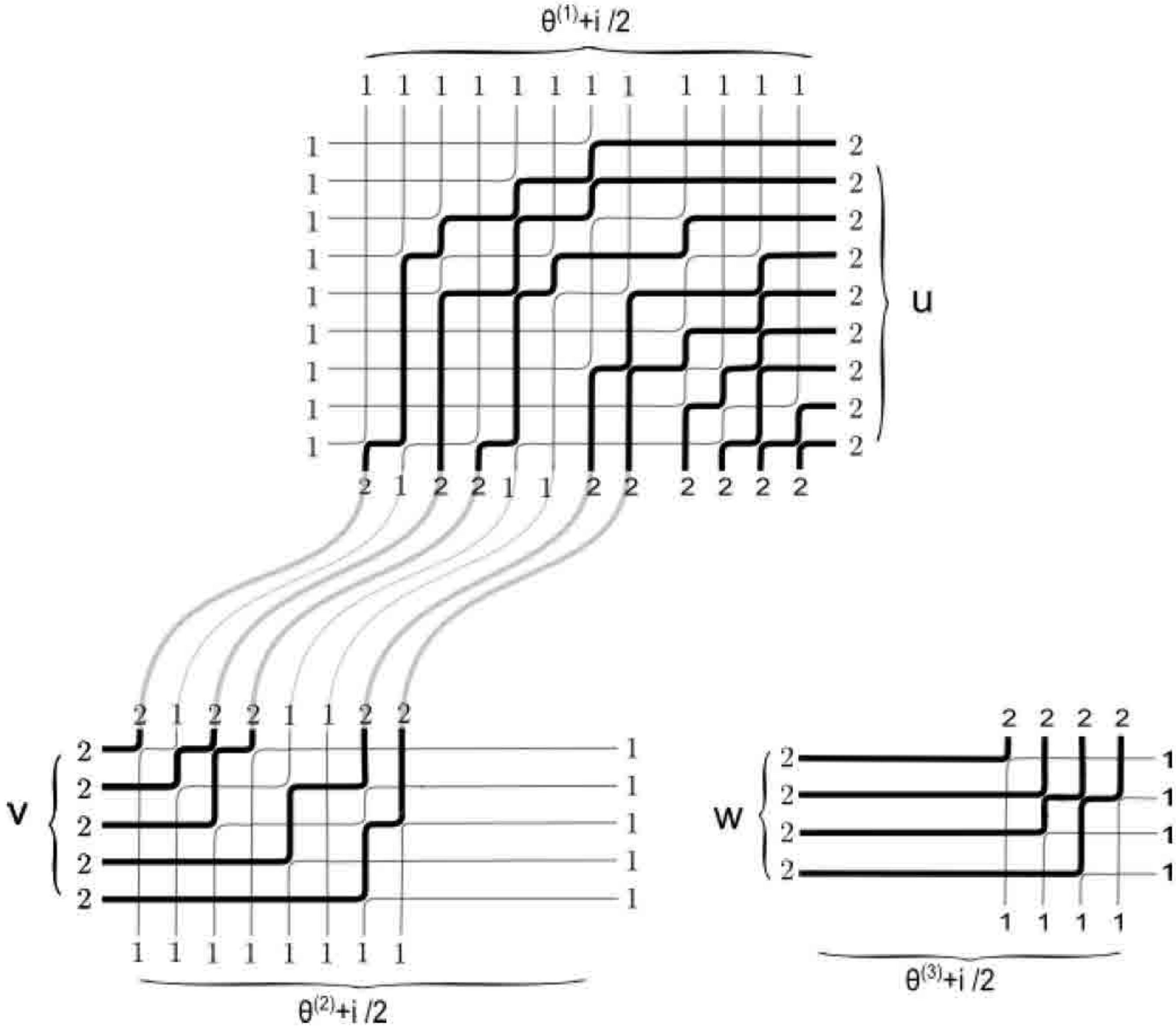

L1
u
v
L12
L23
L13



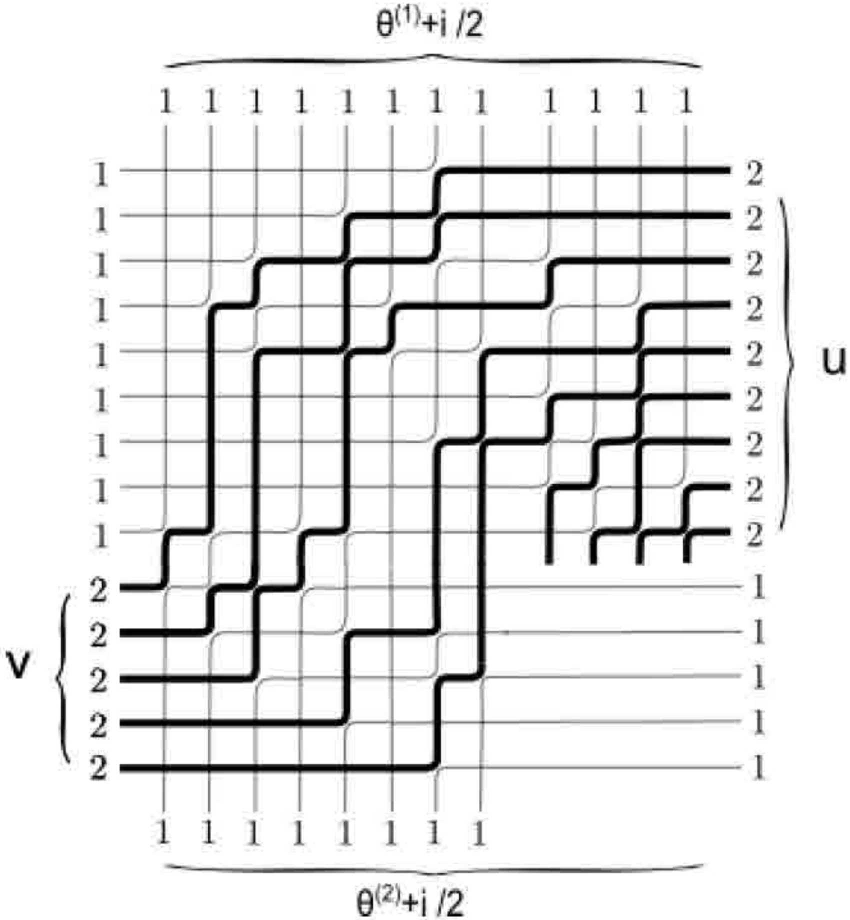
θ(1)+i /2
u
v
w
θ(2)+i /2
θ(3)+i /2

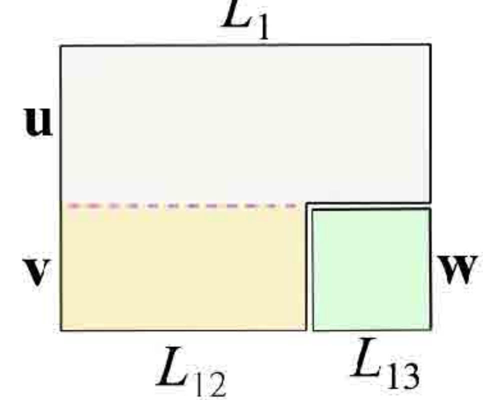


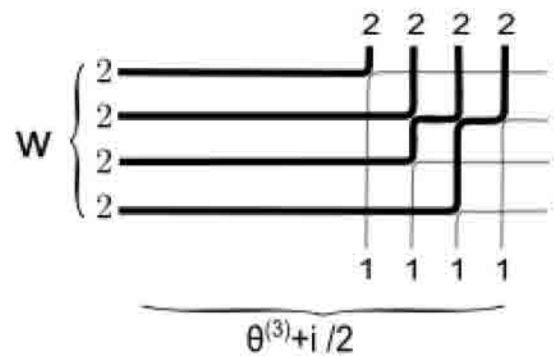
L1
u
v
w
L12
L13

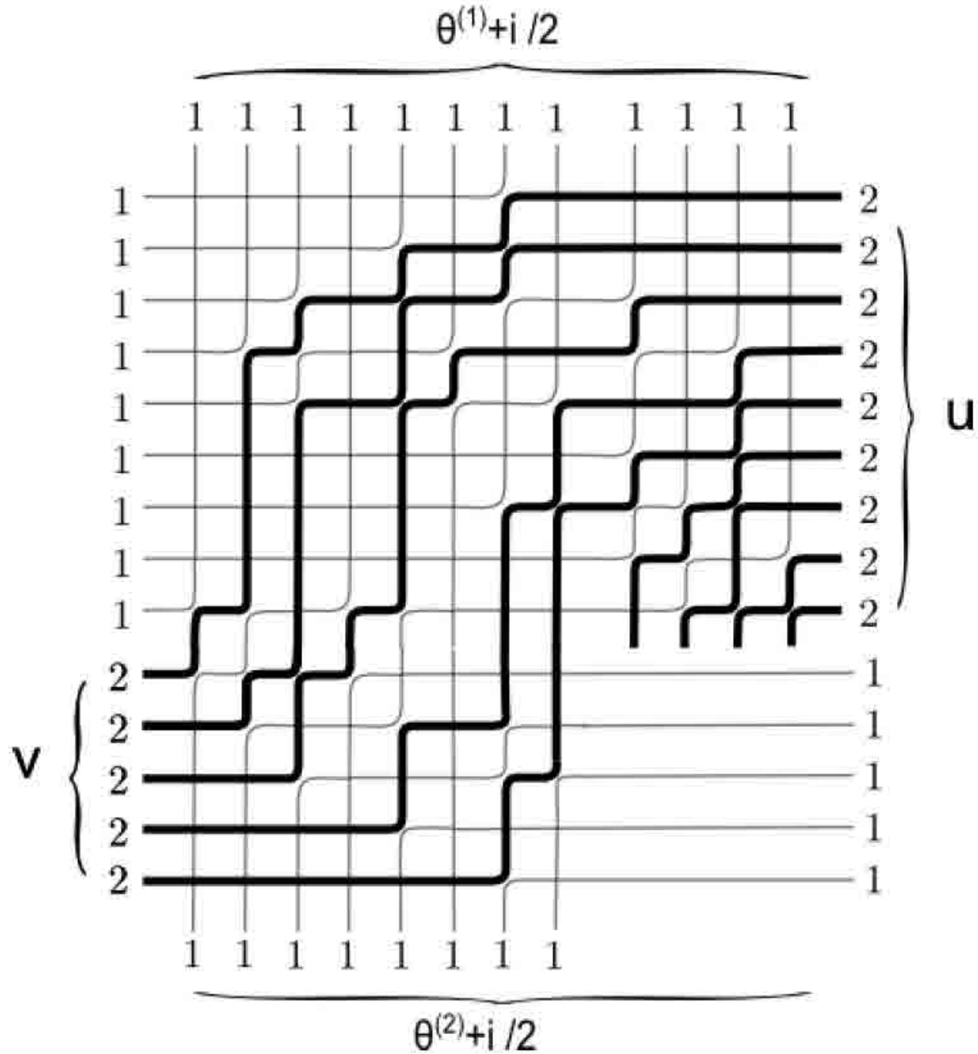


θ(1)+i /2
1 1 1 1 1 1 1 1 1 1 1 1
2 1 2 2 1 1 2 2 2 2 2 2
u
2 1 2 2 1 1 2 2
v
1 1 1 1 1 1 1 1
θ(2)+i /2
2 2 2 2
w
1 1 1 1
θ(3)+i /2


θ(1)+i /2
u
v
θ(2)+i /2


L1
u
v
w
L12
L13


w
θ(3)+i /2



θ^(1)+i /2
1 1 1 1 1 1 1 1 1 1 1 1
1
1
1
1
1
1
1
1
1
2
2
2
2
2
2
2
2
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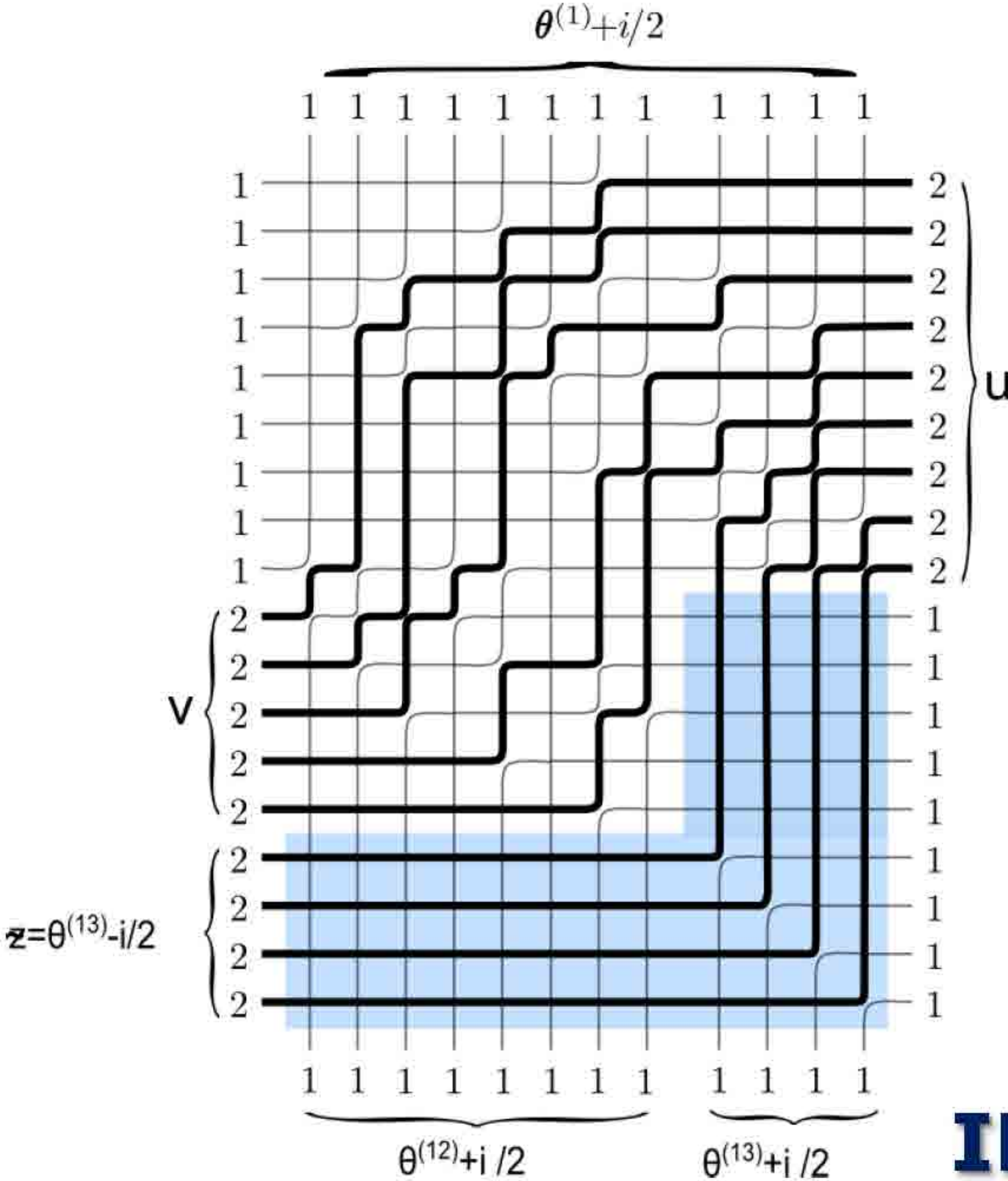
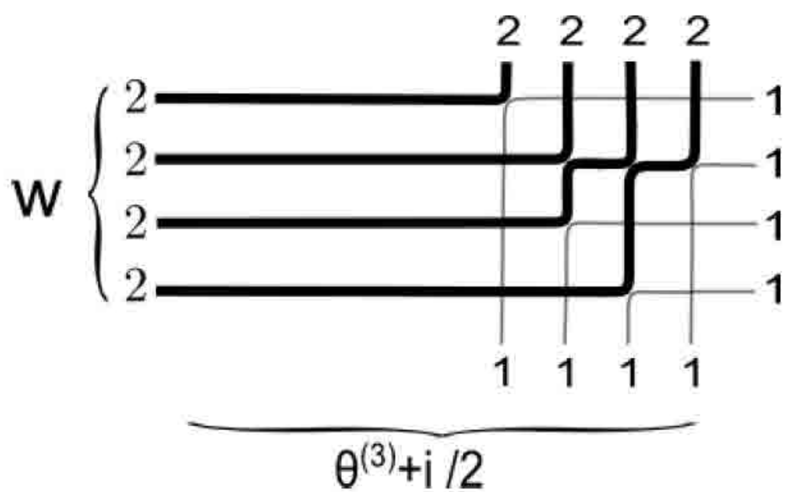
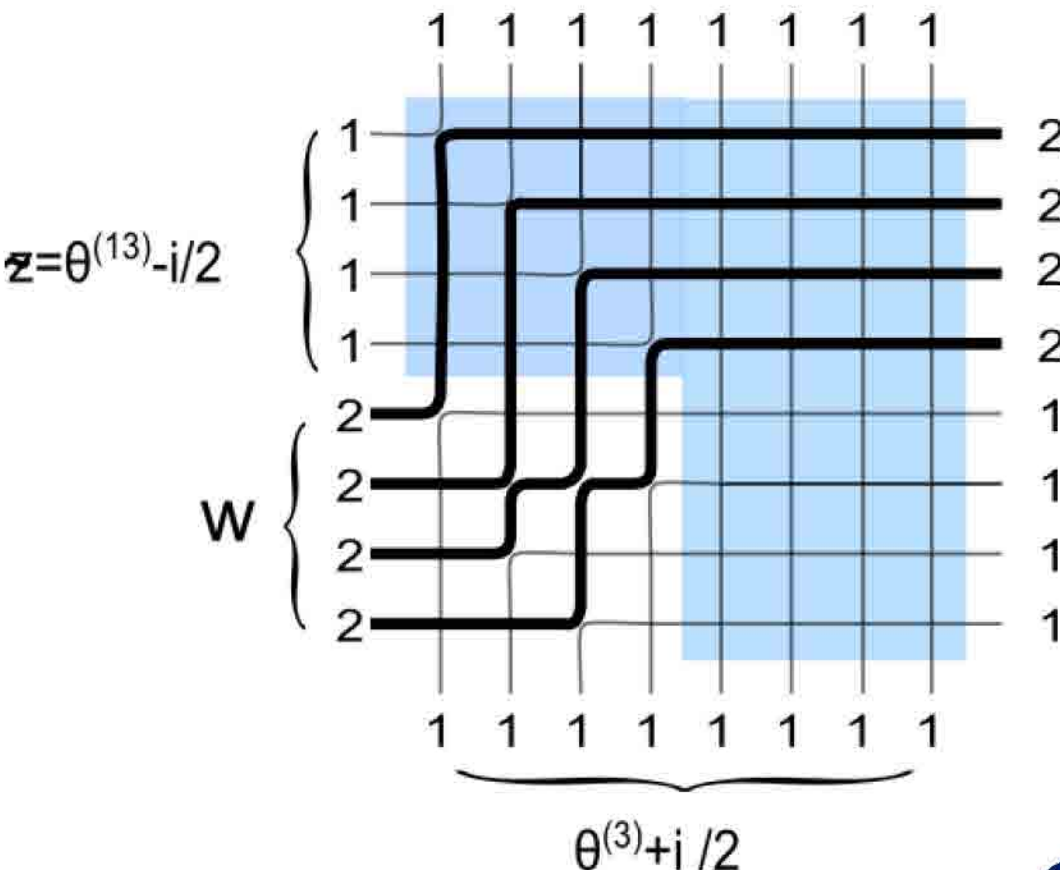
2
1
1
1
1
1
u
v
1 1 1 1 1 1 1 1
θ^(2)+i /2



INVOLVED

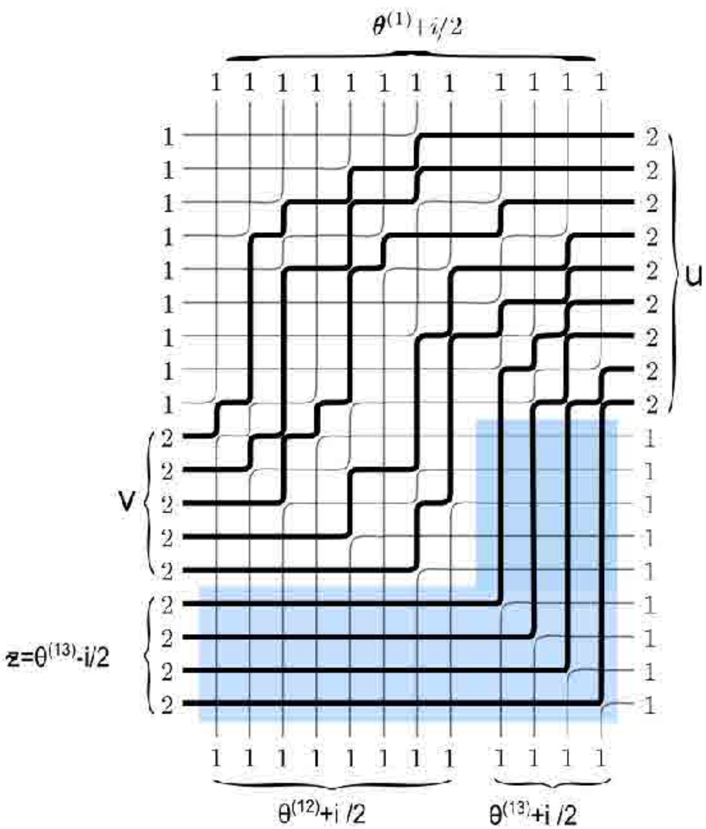

2 2 2 2
2
2
2
2
W
1
1
1
1
1 1 1 1
θ(3)+i /2



SIMPLE

Freezing

Calculation of INVOLVED

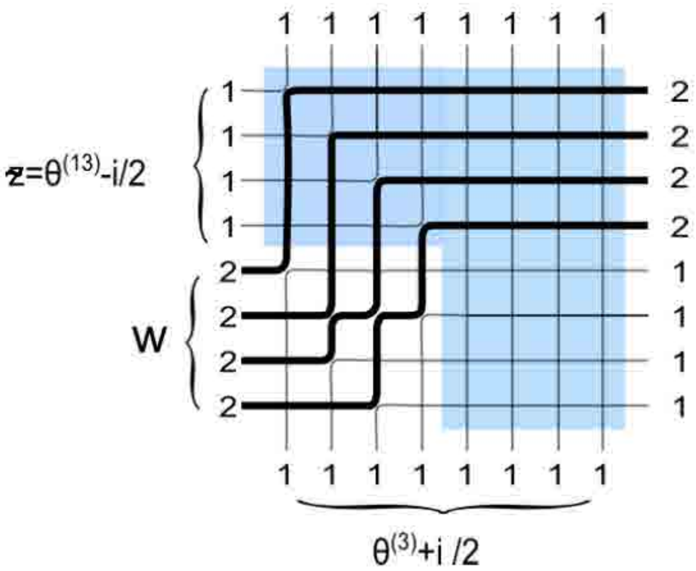


$$= \quad \langle \mathbf{v} \cup \mathbf{z} | \mathbf{u} \rangle_{\boldsymbol{\theta}^{(1)}}$$

$$\mathbf{z} = \boldsymbol{\theta}^{(13)} - i/2$$

Freezing

Calculation of SIMPLE

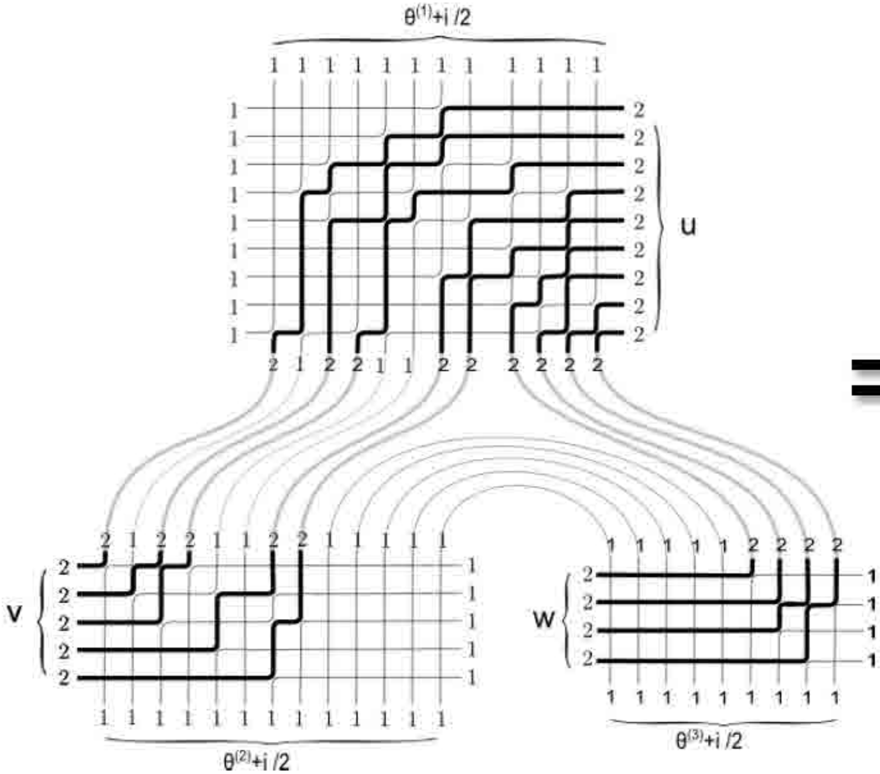


$$= \quad \langle \mathbf{w} | \mathbf{z} \rangle_{\boldsymbol{\theta}^{(3)}}$$

$$\mathbf{z} = \boldsymbol{\theta}^{(13)} - i/2$$

Freezing

Cubic Vertex



$$= \langle \mathbf{v} \cup \mathbf{z} | \mathbf{u} \rangle_{\boldsymbol{\theta}^{(1)}} \langle \mathbf{w} | \mathbf{z} \rangle_{\boldsymbol{\theta}^{(3)}}$$

$$\mathbf{z} = \boldsymbol{\theta}^{(13)} - i/2$$

Freezing

Structure Constant

$$C_{123}^{(0)} = \frac{\langle \mathbf{v} \cup \mathbf{z} | \mathbf{u} \rangle_{\boldsymbol{\theta}^{(1)}} \langle \mathbf{w} | \mathbf{z} \rangle_{\boldsymbol{\theta}^{(3)}}}{\sqrt{\langle \mathbf{u}; \boldsymbol{\theta}^{(1)} | \mathbf{u}; \boldsymbol{\theta}^{(1)} \rangle \langle \mathbf{v}; \boldsymbol{\theta}^{(2)} | \mathbf{v}; \boldsymbol{\theta}^{(2)} \rangle \langle \mathbf{w}; \boldsymbol{\theta}^{(3)} | \mathbf{w}; \boldsymbol{\theta}^{(3)} \rangle}}$$

- For tree level, take the homogeneous limit !
- For higher loop, fix the values of impurities !

Sutherland Limit

(Gromov, Sever, Vieira 2011
Kostov 2012; Kostov Matsuo 2012)

1. The semi-classical limit
2. Factorized Slavnov determinant
3. The $\mathcal{A}$-functional
4. Semi-classical limit of three-point function

The semi-classical limit

1. L, N are large while N/L Finite
2. Compare with string theory
3. Bethe roots condense into cuts
4. Nice analytic results

Factorization

Kostov-Matsuo Formula

(Kostov Matsuo 2012)

$$\mathscr{S}_{\mathbf{u},\mathbf{v}} = (-1)^{|\mathbf{u}|} \mathscr{A}^{+}_{\mathbf{u}\cup\mathbf{v}} \left[\frac{1}{E^{+}_{\mathbf{z}}} \right]$$

$$E^{\pm}_{\mathbf{z}}(u) \equiv \frac{Q_{\mathbf{z}}(u \pm i)}{Q_{\mathbf{z}}(u)}, \quad \mathbf{z} = \boldsymbol{\theta} + i/2$$

$\mathscr{A}$-Functional

Definition of $\mathscr{A}$-functional

$$\mathscr{A}^{\pm}_{\mathbf{u}}[f] \equiv \frac{1}{\Delta_{\mathbf{u}}} \prod_{j=1}^{N} \left(1 - f(u_j) e^{\pm i\partial/\partial u_j}\right) \Delta_{\mathbf{u}}$$

$$\Delta_{\mathbf{u}} \equiv \prod_{j<k}^{N} (u_j - u_k)$$

$$e^{\pm i\partial/\partial u} f(u) = f(u \pm i)$$

$\mathscr{A}$-Functional

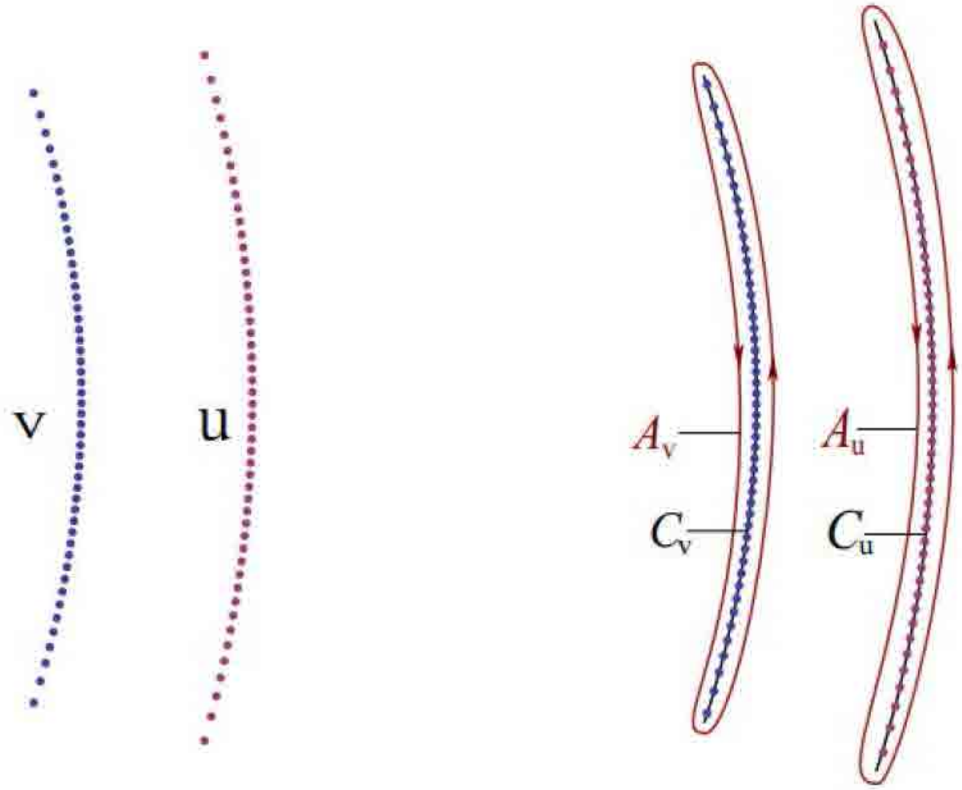
Semi-classical limit of $\mathscr{A}$-functional

$$\mathscr{A}_{\mathbf{u}}^{\pm}[f] = \exp\left[\pm \oint_{A_{\mathbf{u}}} \frac{\mathrm{d}u}{2\pi} \mathrm{Li}_2\left(f(u) e^{\pm i G_{\mathbf{u}}(u)}\right)\right]$$

$$G_{\mathbf{u}}(u) = \sum_{k=1}^{N} \frac{1}{u - u_k} \qquad A_{\mathbf{u}} = \cup_{k=1}^{n} A_{\mathbf{u}}^{k}$$

$\mathcal{A}$-Functional

Cuts and Integral contours



Inhomogeneous Case

$$\log C_{123}^{(0)} = \oint_{A_{\mathbf{u}} \cup A_{\mathbf{v}}} \frac{du}{2\pi} \mathrm{Li}_2\left(e^{ip_{\mathbf{u}}+ip_{\mathbf{v}}+\frac{i}{2}G_{\boldsymbol{\theta}^{(3)}}}\right)$$

$$+ \oint_{A_{\mathbf{w}}} \frac{du}{2\pi} \mathrm{Li}_2\left(e^{ip_{\mathbf{w}}+\frac{i}{2}G_{\boldsymbol{\theta}^{(2)}}-\frac{i}{2}G_{\boldsymbol{\theta}^{(1)}}}\right)$$

$$- \oint_{A_{\mathbf{u}}} \frac{du}{4\pi} \mathrm{Li}_2\left(e^{2ip_{\mathbf{u}}}\right)$$

$$- \oint_{A_{\mathbf{v}}} \frac{du}{4\pi} \mathrm{Li}_2\left(e^{2ip_{\mathbf{v}}}\right)$$

$$- \oint_{A_{\mathbf{w}}} \frac{du}{4\pi} \mathrm{Li}_2\left(e^{2ip_{\mathbf{w}}}\right)$$

Homogeneous Limit

$$\log C_{123}^{(0)} = \oint_{A_\mathbf{u} \cup A_\mathbf{v}} \frac{\mathrm{d}u}{2\pi} \mathrm{Li}_2\left(e^{ip_\mathbf{u}+ip_\mathbf{v}+iL_3/2u}\right)$$
$$+ \oint_{A_\mathbf{w}} \frac{\mathrm{d}u}{2\pi} \mathrm{Li}_2\left(e^{ip_\mathbf{w}+i(L_2-L_1)/2u}\right)$$
$$- \oint_{A_\mathbf{u}} \frac{\mathrm{d}u}{4\pi} \mathrm{Li}_2\left(e^{2ip_\mathbf{u}}\right)$$
$$- \oint_{A_\mathbf{v}} \frac{\mathrm{d}u}{4\pi} \mathrm{Li}_2\left(e^{2ip_\mathbf{v}}\right)$$
$$- \oint_{A_\mathbf{w}} \frac{\mathrm{d}u}{4\pi} \mathrm{Li}_2\left(e^{2ip_\mathbf{w}}\right)$$

One-loop Correction

1. New features at higher loop
2. Unitarity transformation
3. Calculation Method
4. Semi-classical limit

Why it is more difficult

1. Construction of eigenstate
2. Form of the insertions
3. Detail calculation

New Features

BDS Spin Chain

(Beisert, Dipper, Staudacher 2004)

A long-range interacting spin chain that gives the dilatation operator up to three-loop

$$\left(\frac{x(u_k + i/2)}{x(u_k - i/2)}\right)^L = \prod_{j \neq k}^{N} \frac{u_k - u_j + i}{u_k - u_j - i}$$

$$x(u) = \frac{1}{2}\left(u + \sqrt{u^2 - 4g^2}\right)$$

New Features

Dressing Phase

(Beisert, Hernandez, Lopez 2006; Beisert, Eden, Staudacher 2006)

For even higher loops,
a scalar factor need to be included

$$\left(\frac{x(u_k + i/2)}{x(u_k - i/2)}\right)^L = \prod_{j\neq k}^{N} \frac{u_k - u_j + i}{u_k - u_j - i}\,\sigma^2(x_k, x_j)$$

New Features

Two loop Hamiltonian

$$H_4 = 2\sum_{k=1}^{L}\left(4\mathrm{P}_{k,k+1} - 3\mathrm{I}_{k,k+1} - \mathrm{P}_{k,k+2}\right)$$

No systematic way to construct eigenstates !

New Features

Insertions at splitting points

(Okuyama, Tseng 2004)

SIMPLE

$$\langle \uparrow \cdots \uparrow \downarrow \cdots \downarrow | \mathbf{w} \rangle_{\mathrm{XXX}}$$

$$\Downarrow$$

$$\langle \uparrow \cdots \uparrow \downarrow \cdots \downarrow | 1 - g^2 \mathbb{I}_3 \, \mathbf{w} \rangle_{\mathrm{BDS}}$$

New Features

Insertions at splitting points

INVOLVED

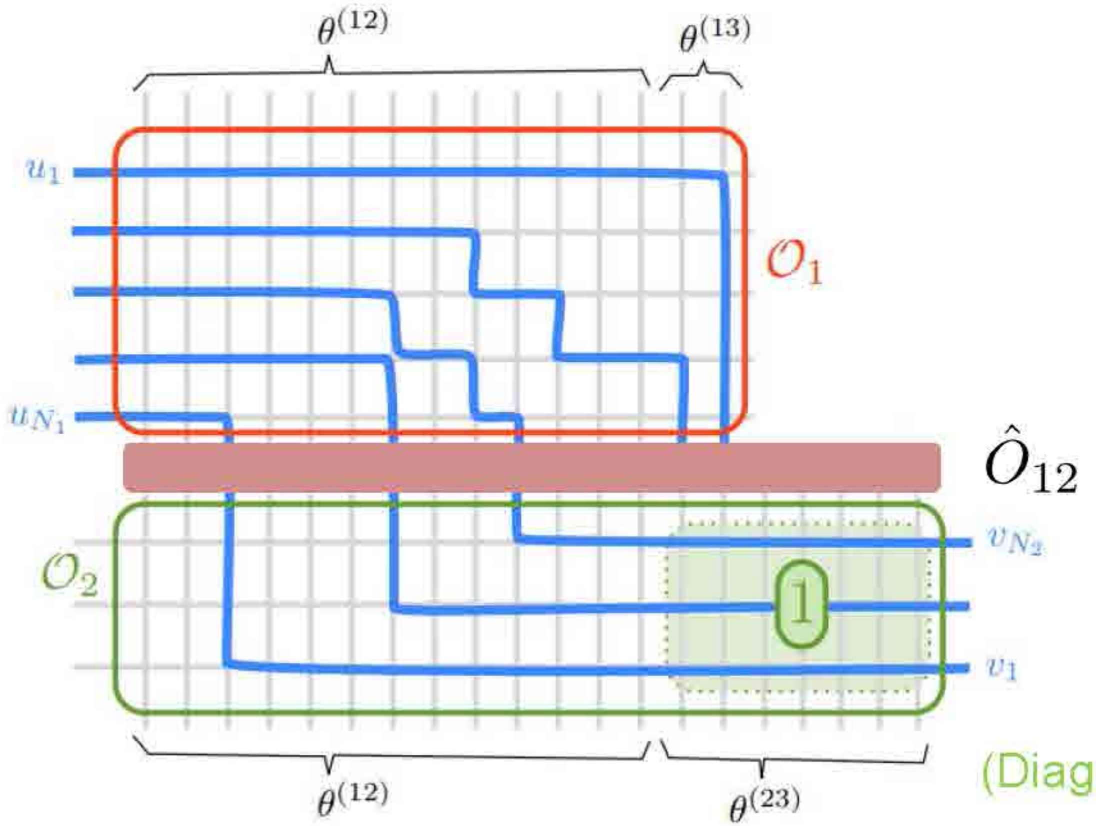
$$ {}_{\mathrm{XXX}}\langle \mathbf{v}|\hat{O}_{12}|\mathbf{u}\rangle_{\mathrm{XXX}} $$

$$\Downarrow$$

$$ {}_{\mathrm{BDS}}\langle \mathbf{v}|1-g^2\mathbb{I}_2|\hat{O}_{12}|1-g^2\mathbb{I}_1|\mathbf{u}\rangle_{\mathrm{BDS}} $$

New Features

$$\hat{O}_{12} = \sum_{i,\cdots,i_{L_{12}}=\uparrow,\downarrow} |i_1 \cdots i_{L_{12}} \underbrace{\downarrow \cdots \downarrow}_{L_{23}}\rangle\langle i_1 \cdots i_{L_{12}} \underbrace{\uparrow \cdots \uparrow}_{L_{13}}|$$



(Diagram from EGSV)

New Features

Insertions at splitting points

$$\mathbb{I}_1 = \mathrm{H}_{L_{12},L_{12}+1} + \mathrm{H}_{L_1,1}$$

$$\mathbb{I}_2 = \mathrm{H}_{L_{12},L_{12}+1} + \mathrm{H}_{L_2,1}$$

$$\mathbb{I}_3 = \mathrm{H}_{L_{13},L_{13}+1} + \mathrm{H}_{L_3,1}$$

$$\mathrm{H}_{k,k+1} = \mathrm{I}_{k,k+1} - \mathrm{P}_{k,k+1}$$

1. *How to find them at higher loop*
2. *How to deal with them*

Unitary Transformation

(Bargeer, Beisert, Loebbert 2010; Y.J, Kostov, Loebbert, Serban 2013)

- ✓ **Spectra are the same:**

BDS Spin Chain

⇕

Inhomogeneous XXX Spin Chain

$$\theta_k^{\mathrm{BDS}} = 2g \sin \frac{2\pi k}{L}$$

Unitary Transformation

✓ **Monodromy matrix**

$$\mathcal{T}_{\mathrm{BDS}}(u) = S\mathcal{T}_{\mathrm{XXX}}(u;\boldsymbol{\theta}^{\mathrm{BDS}})S^{-1}$$

✓ **BDS Bethe state**

$$|\mathbf{u}\rangle_{\mathrm{BDS}} = S|\mathbf{u};\boldsymbol{\theta}^{\mathrm{BDS}}\rangle$$

The S-transformation preserves the RTT relation. This is called morphism property.

Unitary Transformation

✓ **Explicit Form of S-matrix**

$$S = \exp\left(i \sum_{k=1}^{L} \mu_k \mathrm{H}_{k,k+1} - \frac{1}{2} \sum_{k=1}^{L} \rho_k [\mathrm{H}_{k,k-1}, \mathrm{H}_{k,k+1}] + \mathcal{O}(g^4) \right)$$

$$\mu_k = \sum_{j=1}^{k} \theta_j$$

$$\rho_k = 2g^2 k + \theta_k \mu_k - \sum_{j=1}^{k} \theta_j^2$$

Unitary Transformation

✓ **P-D relations** (Gromov, Vieira 2012)

$$\mathrm{H}_{k,k+1}|\mathbf{u}\rangle = -\mathrm{D}_k|\mathbf{u}\rangle \qquad \boxed{\mathrm{D}_k = i(\partial_k - \partial_{k+1})}$$

$$\mathrm{H}_{1,L}|\mathbf{u}\rangle = -\mathrm{D}_L|\mathbf{u}\rangle + E(\mathbf{u})|\mathbf{u}\rangle$$

$$[\mathrm{H}_{k,k-1}, \mathrm{H}_{k,k+1}]|\mathbf{u}\rangle = \left(\frac{1}{2}(\mathrm{D}_{k-1}^2 - \mathrm{D}_k^2) + (\mathrm{D}_{k-1} - \mathrm{D}_k)\right)|\mathbf{u}\rangle$$

$$[\mathrm{H}_{1,L}, \mathrm{H}_{1,2}]|\mathbf{u}\rangle = \left(\frac{1}{2}(\mathrm{D}_L^2 - \mathrm{D}_1^2) + (\mathrm{D}_L - \mathrm{D}_1)\right)|\mathbf{u}\rangle + E(\mathbf{u})\mathrm{H}_{1,L}|\mathbf{u}\rangle$$
$$+ \left(iQ_3(\mathbf{u}) - E(\mathbf{u}) - \frac{1}{2}E^2(\mathbf{u})\right)|\mathbf{u}\rangle$$

Sketch of Calculation

1. Most part of S-operators cancel !
2. Boundary terms treated by P-D relation
3. Insertions treated by P-D relation
4. Cross terms calculated carefully

Final Result

$$C_{123}(g^2) = C_{123}^{\rm BDS}(g^2) + g^2\delta_{123}$$

$$C_{123}^{\rm BDS} = \left.\frac{\langle \mathbf{v}\cup\mathbf{z}|\mathbf{u}\rangle_{\boldsymbol{\theta}^{(1)}}\langle \mathbf{w}|\mathbf{z}\rangle_{\boldsymbol{\theta}^{(3)}}}{\sqrt{\langle \mathbf{u};\boldsymbol{\theta}^{(1)}|\mathbf{u};\boldsymbol{\theta}^{(1)}\rangle\langle \mathbf{v};\boldsymbol{\theta}^{(2)}|\mathbf{v};\boldsymbol{\theta}^{(2)}\rangle\langle \mathbf{w};\boldsymbol{\theta}^{(3)}|\mathbf{w};\boldsymbol{\theta}^{(3)}\rangle}}\right|_{\boldsymbol{\theta}^{\rm BDS}}$$

$$\delta_{123} = -\frac{\left(\partial_1^{(3)}\partial_2^{(3)} - iE(\mathbf{w})\partial_1^{(3)} + \frac{1}{2}E^2(\mathbf{w})\right)\mathscr{A}_{\mathbf{w},\boldsymbol{\theta}^{(13)}}\mathscr{A}_{\mathbf{u}\cup\mathbf{v},\boldsymbol{\theta}^{(12)}}}{\sqrt{\langle\mathbf{u}|\mathbf{u}\rangle\langle\mathbf{v}|\mathbf{v}\rangle\langle\mathbf{w}|\mathbf{w}\rangle}}$$

$$-\frac{\mathscr{A}_{\mathbf{w},\boldsymbol{\theta}^{(13)}}\partial_1^{(1)}\partial_2^{(1)}\mathscr{A}_{\mathbf{u}\cup\mathbf{v},\boldsymbol{\theta}^{(12)}}}{\sqrt{\langle\mathbf{u}|\mathbf{u}\rangle\langle\mathbf{v}|\mathbf{v}\rangle\langle\mathbf{w}|\mathbf{w}\rangle}} \sim \frac{1}{L^2}$$

Classical Limit

(*Conjectured by* **Serban 2012**)

$$\log C_{123}^{\mathrm{BDS}}(g^2) = \oint_{A_{\mathbf{u}} \cup A_{\mathbf{v}}} \frac{\mathrm{d}u}{2\pi} \mathrm{Li}_2 \left(e^{ip_{\mathbf{u}} + ip_{\mathbf{v}} + \frac{iL_3}{2\sqrt{u^2-4g^2}}} \right)$$

$$+ \oint_{A_{\mathbf{w}}} \frac{\mathrm{d}u}{2\pi} \mathrm{Li}_2 \left(e^{ip_{\mathbf{w}} + \frac{i(L_2-L_1)}{2\sqrt{u^2-4g^2}}} \right)$$

$$- \oint_{A_{\mathbf{u}}} \frac{\mathrm{d}u}{2\pi} \mathrm{Li}_2 \left(e^{2ip_{\mathbf{u}}} \right)$$

$$- \oint_{A_{\mathbf{v}}} \frac{\mathrm{d}u}{2\pi} \mathrm{Li}_2 \left(e^{2ip_{\mathbf{v}}} \right)$$

$$- \oint_{A_{\mathbf{w}}} \frac{\mathrm{d}u}{2\pi} \mathrm{Li}_2 \left(e^{2ip_{\mathbf{w}}} \right)$$

Discussions

- The main contribution comes from the expansion of BDS part
- In classical limit, the detailed form of insertions are not important

Conclusion & Outlook

I. We computed structure constant at tree-level and one-loop

II. Compare to string theory

III. Include dressing phase

IV. Global angles

V. Other sectors

Thank you very much !