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Entanglement Renormalization and Black Holes in AdS/CFT

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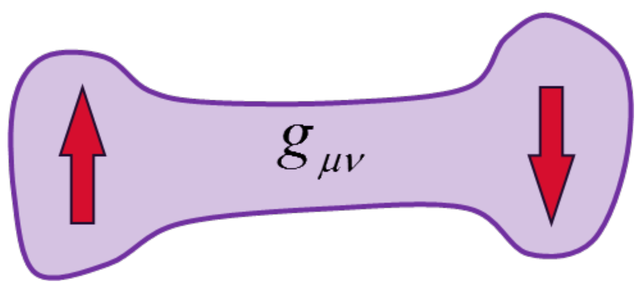


Based on

arXiv:1208.3469 with Masahiro Nozaki (YITP, Kyoto)

and Shinsei Ryu (Illinois, Urbana–Champaign)

arXiv:1311.1643 with Noburo Shiba (YITP, Kyoto)

arXiv:13mm.nnnn with Ali Mollabashi (IPM),

Masahiro Nozaki and Shinsei Ryu

① Introduction

(1-1) What is the quantum entanglement ?

In quantum mechanics, a physical state is described by a vector in Hilbert space.

If we consider **a spin of an electron** (= two dimensional Hilbert space), a state is generally described by **the linear combination:**

$$\left|\Psi\right\rangle = a\left|\uparrow\right\rangle + b\left|\downarrow\right\rangle, \qquad |a|^2 + |b|^2 = 1.$$

Consider **two spin systems**. We can think of the following states:

(i) A direct product state (unentangled state)

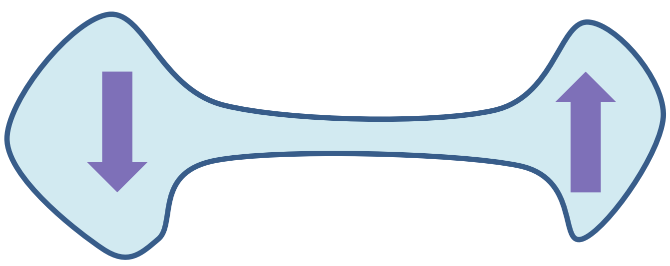
$$|\Psi\rangle = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right]\otimes\left[|\uparrow\rangle_B + |\downarrow\rangle_B\right].$$

Independent

(ii) An entangled state

$$|\Psi\rangle = \left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B\right]/\sqrt{2}.$$

One determines the other !



∃ Non-local correlation

A measure of quantum entanglement is known as the **entanglement entropy** defined as follows.

Divide a quantum system into two subsystems A and B.

$$H_{tot} = H_A \otimes H_B \quad .$$

Define the reduced density matrix ρ_A by

$$\rho_A = \mathrm{Tr}_B \left| \Psi \right\rangle \left\langle \Psi \right| \; .$$

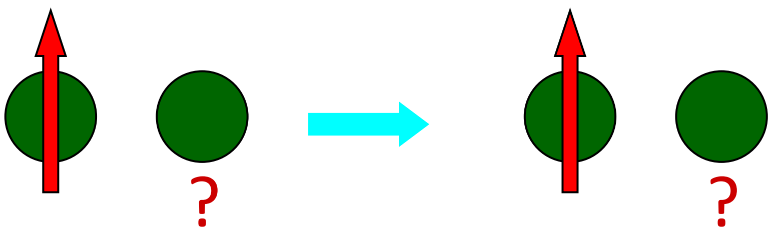
The entanglement entropy S_A is now defined by

$$S_A = -\mathrm{Tr}_A \, \rho_A \log \rho_A \; . \qquad \text{(von-Neumann entropy)}$$

The Simplest Example: two spins (2 qubits)

$$\text{(i)}\quad |\Psi\rangle = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right]\otimes\left[|\uparrow\rangle_B + |\downarrow\rangle_B\right]$$

$$\Rightarrow \rho_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}\left[|\Psi\rangle\langle\Psi|\right] = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right]\cdot\left[\langle\uparrow|_A + \langle\downarrow|_A\right].$$

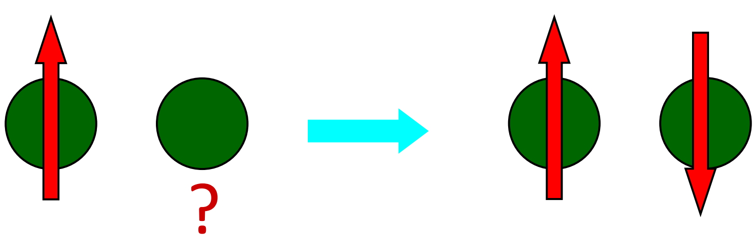


Not Entangled

$$\mathrm{S_A} = 0$$

$$\text{(ii)}\quad |\Psi\rangle = \left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_{\mathrm{B}}\right]/\sqrt{2}$$

$$\Rightarrow \rho_A = \mathrm{Tr}_{\mathrm{B}}\left[|\Psi\rangle\langle\Psi|\right] = \frac{1}{2}\left[|\uparrow\rangle_A\langle\uparrow|_A + |\downarrow\rangle_A\langle\downarrow|_A\right].$$



Entangled

$$\mathrm{S_A} = \log 2$$

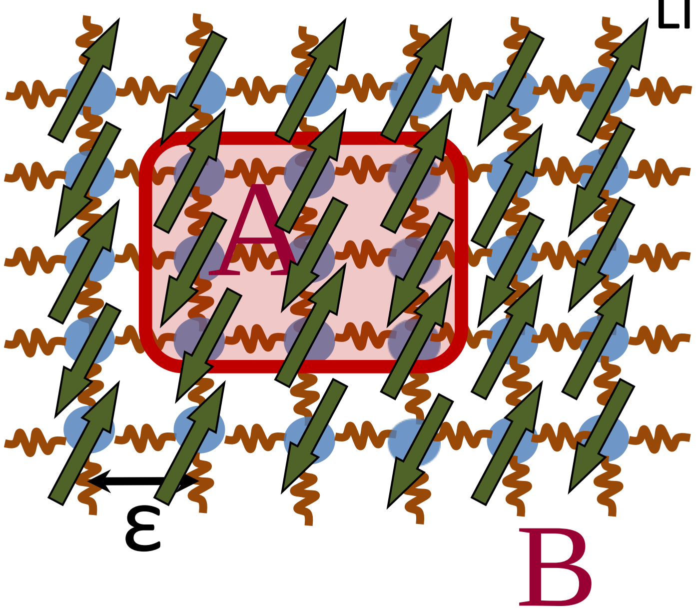
EE in Quantum Many-body Systems and QFTs

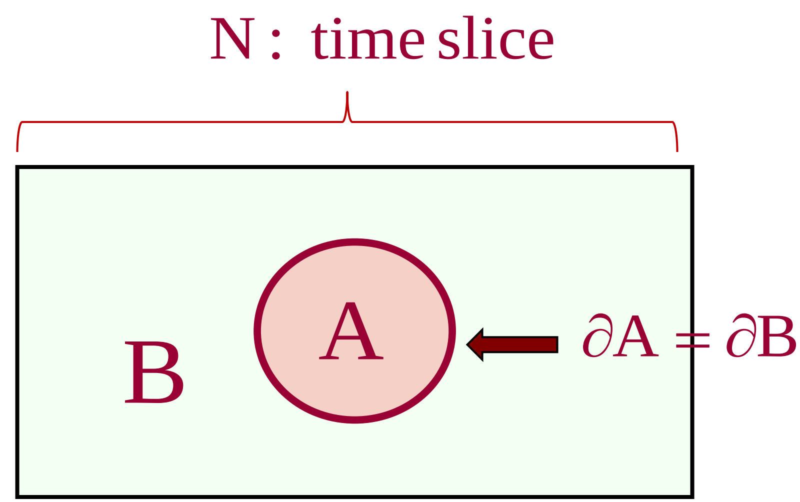
The EE is defined geometrically

(sometime called geometric entropy).

Continuum

Limit ε→0





$$H_{tot} = H_A \otimes H_B \quad .$$

Quantum Many-body Systems

Quantum Field Theories (QFTs)

(1-2) Holographic Entanglement Entropy

HEE formula [Ryu-TT 06, proven by Lewkowycz-Maldacena 13]

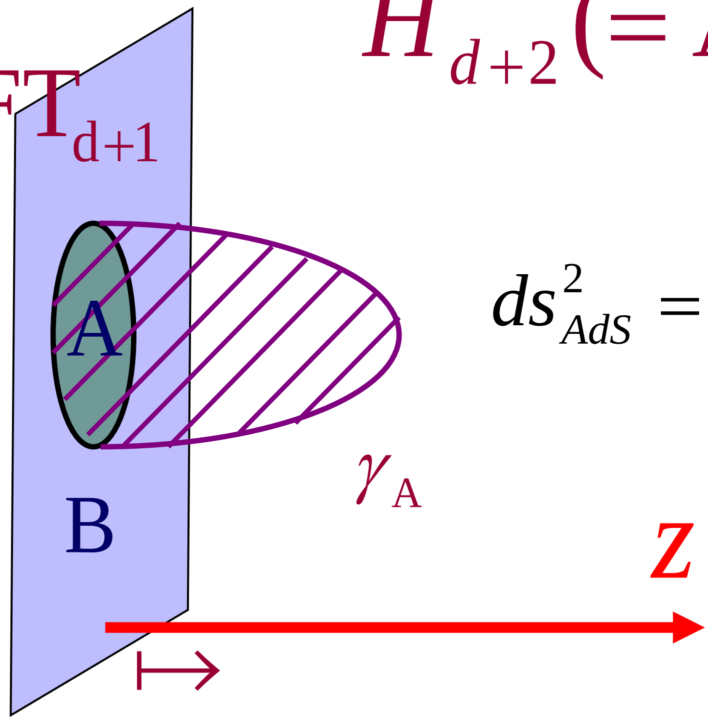
$$\boxed{S_A = \mathrm{Min}\left[\frac{\mathrm{Area}(\gamma_A)}{4G_N}\right].}$$

Entanglement entropy for CFTs (quantum many-body systems at critical points)

Area of minimal surface in hyperbolic space

$$H_{d+2} (= \mathrm{AdS}_{d+2})$$

CFT_{d+1}



$$ds^2_{AdS} = R^2_{AdS} \frac{-dt^2 + \sum_{i=1}^{d} dx_i^2 + dz^2}{z^2}$$

$z > \varepsilon$ (UV cut off)

$\partial A = \partial \gamma_A$ and $A \sim \gamma_A$.

homologous

Note: the HEE formula can be regarded as a generalization of Bekenstein-Hawking formula of black hole entropy:

$$S_A = \frac{\text{Area of BH}}{4G_N}.$$

A Killing horizon (time independent black holes)

⇔ All components of extrinsic curvature are vanishing.

$$\cap$$

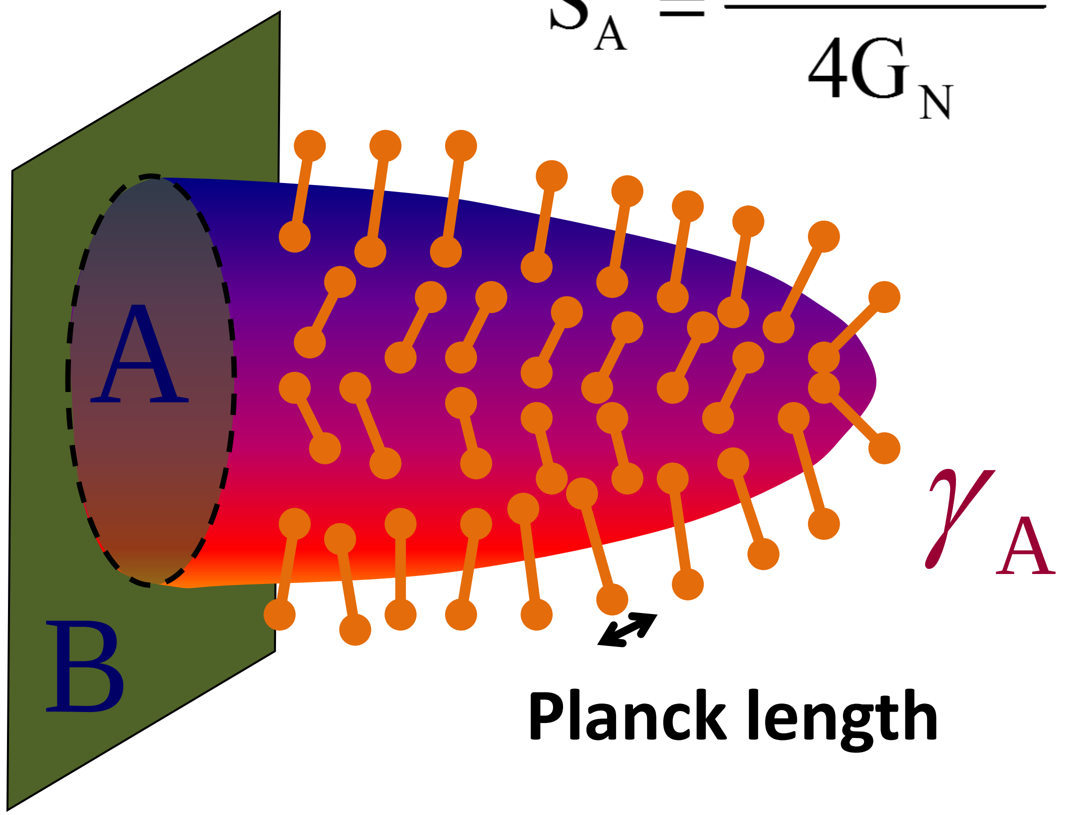
A minimal surface (or extremal surface)

⇔Traces of extrinsic curvature are vanishing.

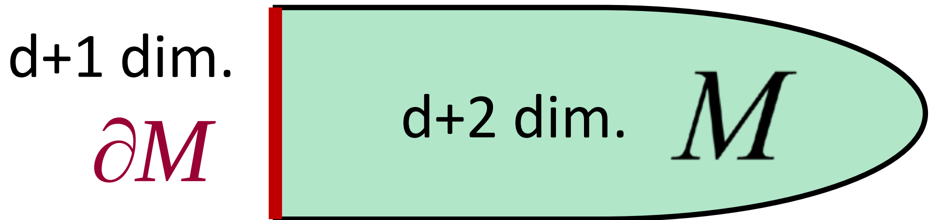
The HEE suggests that

``Spacetime in Gravity
  = Collections of tiny bits of quantum entanglement'' ?

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N} \approx \frac{\mathrm{Area}(\gamma_A)}{l_{pl}^2}.$$



The quantum entanglement can be a key concept to understand the holography.



Boundary **Bulk**

Holography

Quantum many-body system **=** **(Quantum) gravity**

$|\Psi\rangle$ $g_{\mu\nu}$

[Ryu-TT 06]

$$\textbf{Entanglement entropy (EE)} \;=\; \underbrace{\textbf{Area of minimal surface}}_{\textbf{+ quantum corrections}}$$

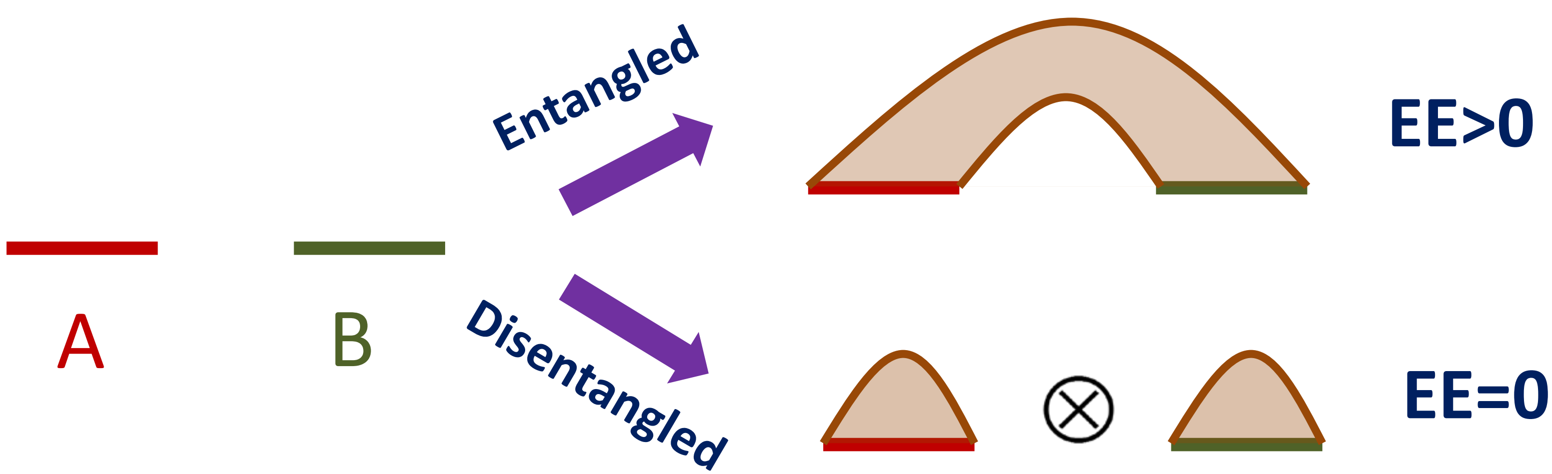
A framework to realize this is the **entanglement renormalization**.

Advantages of EE

- EE is defined for *any quantum many-body systems*. ⇒Universal
  (In cond-mat, EE = a quantum order parameter)

- In the presence of quantum corrections, the metric may not be a good description of the spacetime. But, the EE is robust.

- EE can *capture spacetime topologies*. For example,



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② Entanglement Renormalization and AdS/CFT

(2-1) Our Motivation

In principle, we can obtain a metric from a CFT as follows:

a CFT state ⇒ Information (~EE) = Minimal Areas ⇒ metric

$$|\Psi\rangle \qquad S_A \qquad \mathrm{Area}(\gamma_{\mathrm{A}}) \qquad g_{\mu\nu}$$

One candidate of such frameworks is so called the entanglement renormalization (MERA) [Vidal 05 (for a review see 0912.1651)] as pointed out by [Swingle 09]. [cf. Emergent gravity: Raamsdonk 09, Lee 09]

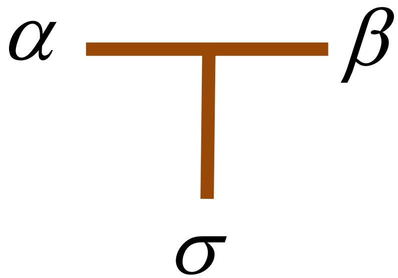
(2-2) Tensor Network (TN)

[See e.g. the review Cirac-Verstraete 09]

Recently, there have been remarkable progresses in numerical algorithms for quantum lattice models, based on so called *tensor product states*.

This leads to various nice variational ansatzs for the ground state wave functions in various quantum many-body systems.

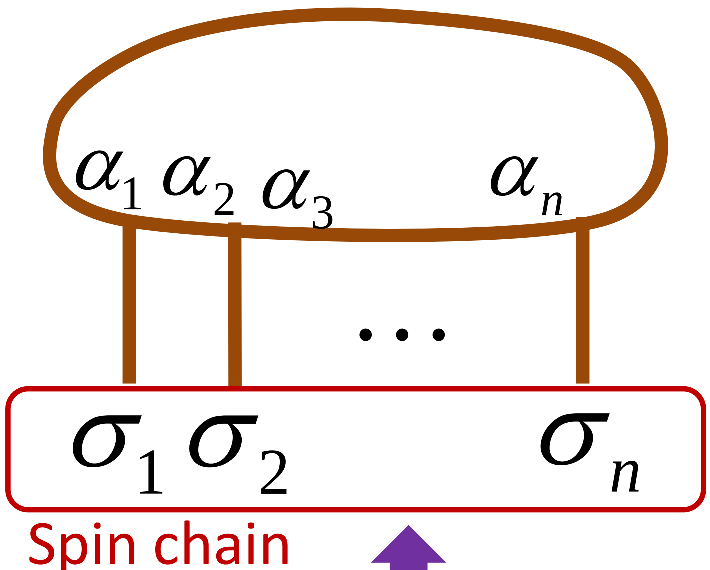
⇒ An ansatz is good if it respects the quantum entanglement of the true ground state.



$$\Leftrightarrow \quad M_{\alpha\beta}(\sigma)$$

Ex. Matrix Product State (MPS)

[DMRG: White 92,…, Rommer-Ostlund 95,..]



$$\alpha_i = 1, 2, \ldots, \chi,$$

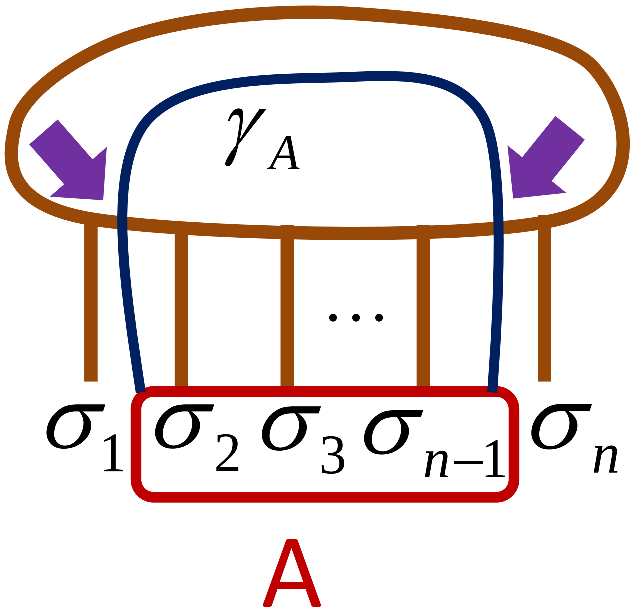
$$\sigma_i = \uparrow \text{ or } \downarrow .$$

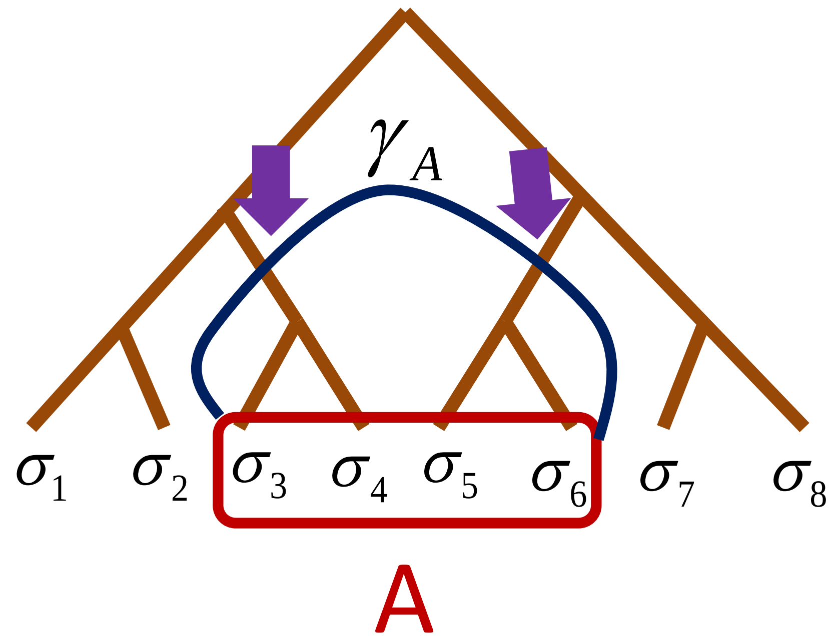
$$\Updownarrow$$

$$|\Psi\rangle = \sum_{\sigma_1, \sigma_2, \cdots, \sigma_n} \mathrm{Tr}[M(\sigma_1) M(\sigma_2) \cdots M(\sigma_n)] \, |\underbrace{\sigma_1, \sigma_2, \cdots, \sigma_n}_{n \ \ Spins}\rangle$$

MPS and TTN are not good near quantum critical points (CFTs) because their entanglement entropies are too small:

$$S_A \leq 2\log\chi \quad (<< \log L \sim S_A^{CFT}).$$





In general,

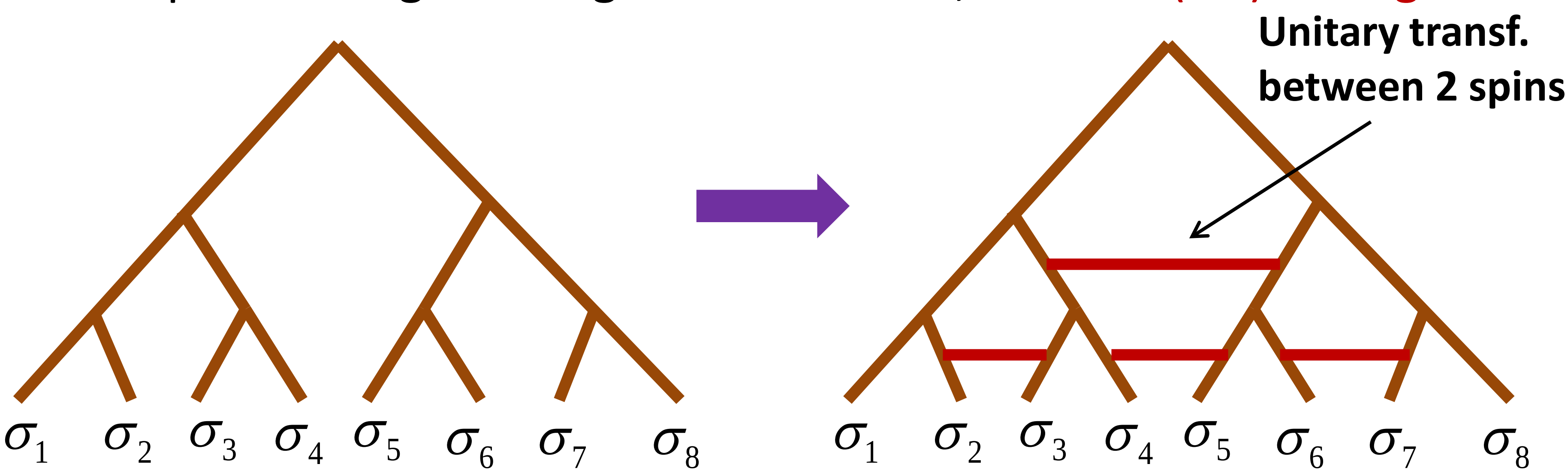
$$\boxed{\begin{aligned} &S_A \sim N_{\text{int}} \cdot \log\chi, \\ &N_{\text{int}} \equiv \min[\#\text{Intersections of } \gamma_A]. \end{aligned}}$$

(2-3) AdS/CFT and (c)MERA

MERA (Multiscale Entanglement Renormalization Ansatz):

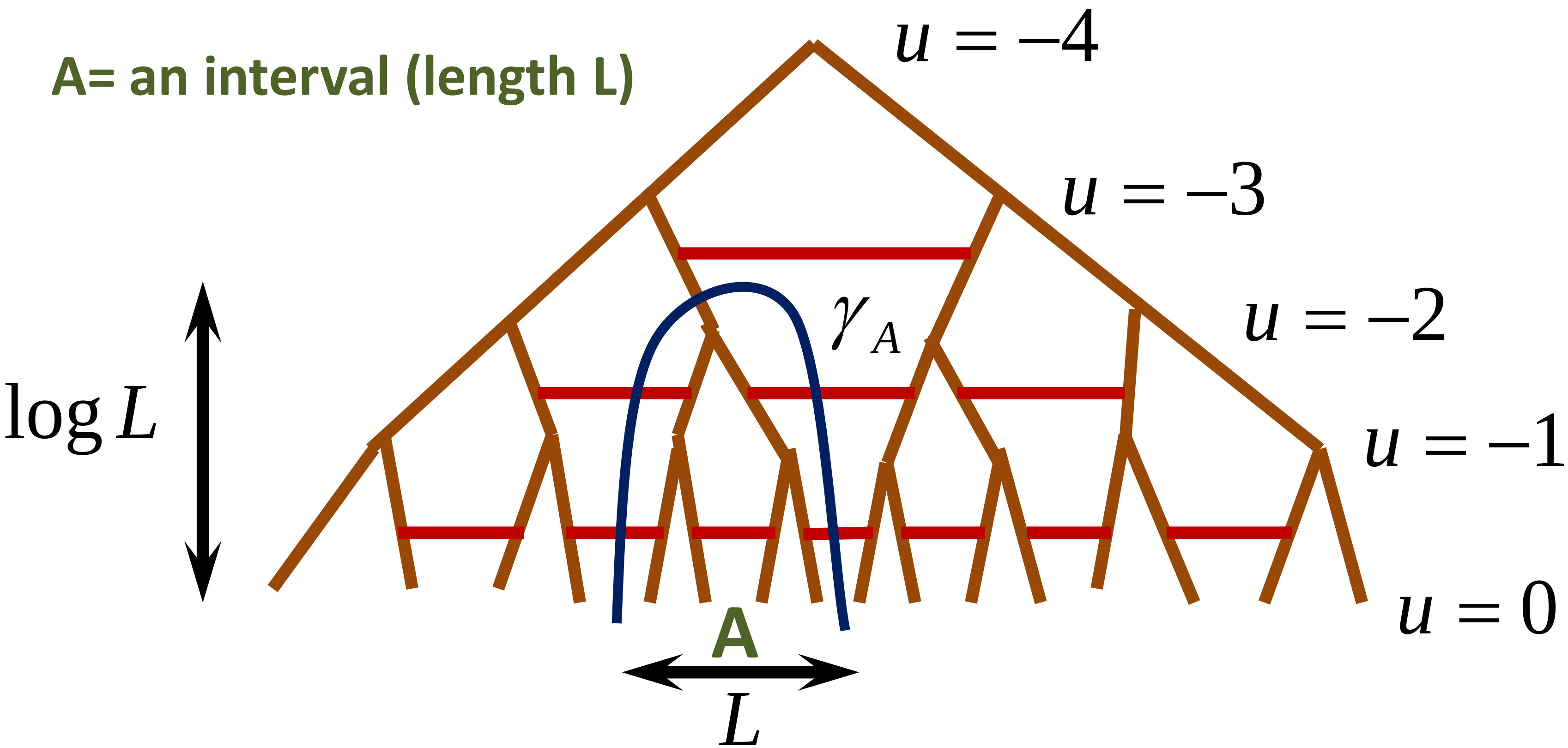
An efficient variational ansatz to find CFT ground states have been developed recently. [Vidal 05 (for a review see 0912.1651)].

To respect its large entanglement in a CFT, we add (dis)entanglers.



Calculations of EE in 1+1 dim. MERA

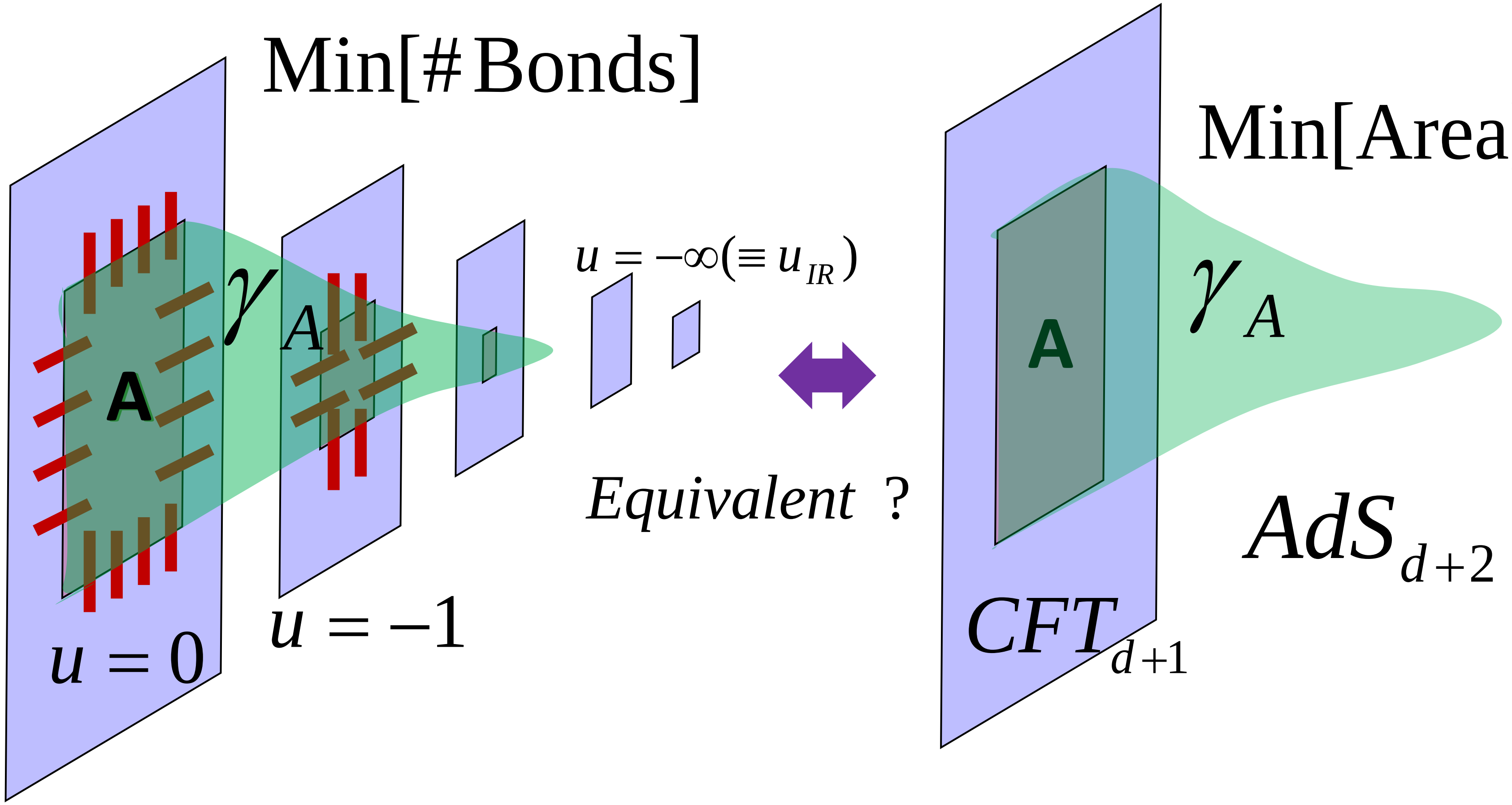
A= an interval (length L)



$$S_A \propto \mathrm{Min}[\#\mathrm{Bonds}] \propto \log L$$

$\Rightarrow$ agrees with 2d CFTs.

A conjectued relation to AdS/CFT [Swingle 09]



$$\text{Metric} = ds^2 + \frac{e^{2u}}{\varepsilon^2}(-dt^2 + d\vec{x}^2) = \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2},$$

$$where \quad z = \varepsilon \cdot e^{-u} \quad .$$

Now, to make the connection to AdS/CFT clearer, we would like to consider the MERA for quantum field theories.

Continuous MERA (cMERA)

[Haegeman-Osborne-Verschelde-Verstraete 11]

$$\underbrace{|\Psi(u)\rangle}_{\substack{\text{True ground state}\\ \text{(highly entangled)}}} = P\cdot\exp\left(-i\int_{u_{IR}}^{u} ds[K(s)+L]\right)\cdot \underbrace{|\Omega\rangle}_{\substack{\text{IR state}\\ \text{(no entanglement)}}} ,$$

$\Rightarrow$ Real space renormalization flow : length scale $\sim \varepsilon\cdot e^{-u}$.

K(u) : disentangler, L: scale transformation

Conjecture

$d+1$ dim. cMERA $=$ gravity on AdS_{d+2} $\quad z=\varepsilon\cdot e^{-u}$.

(2-4) Emergent Metric from cMERA [Nozaki-Ryu-TT 12]

We focus on gravity duals of translational invariant static states, which are not conformal in general.

We conjecture that the metric in the extra direction is given by using the Bures metric (or Fisher information metric):

$$\boxed{g_{uu}du^2 = N \cdot \left(1 - \left|\left\langle \Psi(u) \,|\, e^{iLdu} \,|\, \Psi(u+du) \right\rangle\right|^2\right).}$$

$$N^{-1} \equiv \int dx^d \cdot \int_0^{\Lambda e^u} dk^d =$$ The total volume of phase space at energy scale u.

Bures Metric

The **Bures distance** between two states is defined by

$$D(\psi_1,\psi_2) = 1 - \left|\langle \psi_1 \mid \psi_2 \rangle\right|^2 .$$

More generally, for two mixed states ρ1 and ρ2,

$$D(\rho_1,\rho_2) = 1 - \mathrm{Tr}\sqrt{\sqrt{\rho_1}\,\rho_2\sqrt{\rho_1}} .$$

When the state depends on the parameters {ξi},

the **Bures metric (Fisher information metric)** is defined as

$$D\left[\psi(\xi),\psi(\xi+d\xi)\right] = g_{ij}\,d\xi^i d\xi^j .$$

⇒ Reparameterization invariant (in our case: coordinate u)

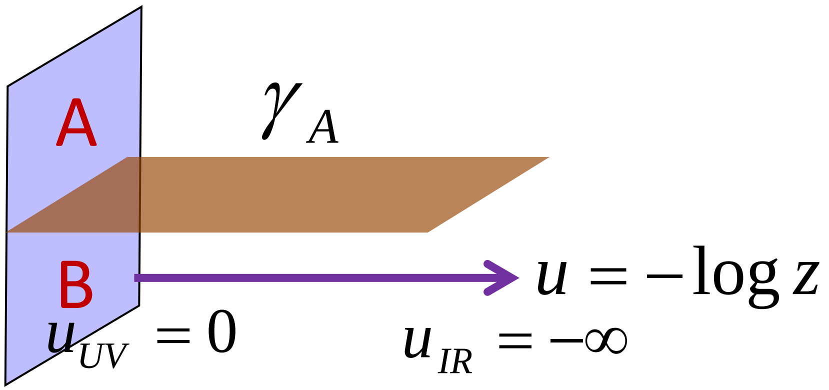
The operation e^{iLdu} removes the coarse-graining procedure to extract the strength of unitary transformations (disentanglers).

⇒ Our metric = the density of disentanglers

= the metric guu in the gravity dual

Understandable from the HEE:

$$S_A \sim \int_{u_{IR}}^{0} du \sqrt{g_{uu}} \cdot e^{(d-1)u}$$



(2-5) Emergent Metric in a (d+1) dim. Free Scalar Theory

Hamiltonian: $H = \frac{1}{2}\int dk^d [\pi(k)\pi(-k) + (k^2 + m^2)\phi(k)\phi(-k)].$

Ground state $|\Psi\rangle$: $a_k|\Psi\rangle = 0.$

Moreover, we introduce the `IR state' $|\Omega\rangle$ which has no real space entanglement.

$$a_x|\Omega\rangle = 0, \qquad a_x = \sqrt{M}\phi(x) + \frac{i}{\sqrt{M}}\pi(x),$$

$$\text{i.e.} \quad |\Omega\rangle = \prod_x |0\rangle_x \qquad a_x^+ = \sqrt{M}\phi(x) - \frac{i}{\sqrt{M}}\pi(x).$$

$$\Rightarrow \quad S_A = 0.$$

For a free scalar theory, the ground state corresponds to

$$\hat{K}(u) = \frac{i}{2}\int dk^d \left[\chi(u)\Gamma\left(ke^{-u}/M\right)a_k^+ a_{-k}^+ + (h.c.)\right],$$

where $\Gamma(\mathrm{x})$ is a cut off function : $\Gamma(\mathrm{x}) = \theta(1 - |\mathrm{x}|)$.

$$\chi(s) = \frac{1}{2}\cdot\frac{e^{2u}}{e^{2u} + m^2/M^2}, \quad (\text{for } m = 0,\ \chi(\mathrm{u}) = 1/2.)$$

For the excited states, $\chi(s)$ becomes time-dependent.

One might be tempting to guess

Density of *bonds*

$$\boxed{ds^2_{Gravity} = g_{uu}du^2 + \frac{e^{2u}}{\varepsilon^2}\cdot d\vec{x}^2 - g_{tt}dt^2} \Rightarrow \sqrt{g_{uu}} \propto |\chi(u)| \ ?$$

Indeed, the previous proposal for guu lead to $g_{uu} = \chi(u)^2$.

Explicit metric

$$\boxed{ds^2_{Gravity} = g_{uu} du^2 + \frac{e^{2u}}{\varepsilon^2} \cdot d\vec{x}^2 - g_{tt} dt^2}$$

(i) Massless scalar (E=k)

$$g_{uu} = \frac{1}{4} \qquad \Rightarrow \text{ the pure } AdS$$

(ii) Lifshitz scalar (E=k$^{\nu}$)

$$g_{uu} = \frac{\nu^2}{4} \qquad \Rightarrow \text{ the Lifshitz geometry}$$

(iii) Massive scalar

$$g_{uu} = \frac{e^{4u}}{4(e^{2u} + m^2/\Lambda^2)^2}.$$

$$\Rightarrow ds^2 = \frac{dz^2}{z^2} + \left(\frac{1}{\Lambda^2 z^2} - \frac{m^2}{\Lambda^2}\right)(d\vec{x}^2 - dt^2).$$

Capped off in the IR z<1/m

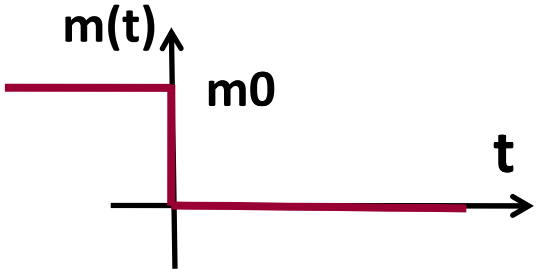
③ Finite Temperature CFT and AdS Black Holes

(3-1) Excited States in MERA

Before we study finite temperature states (mixed state), we would like to examine a class of excited states (pure states), called **quantum quenches**.

Quantum quenches are triggered by sudden large excitations, induced by an instantaneous shift of parameters in the Hamiltonian.

e.g. mass shift:



⇒ an excited state in CFT

Such a excited state is in the class defined as follows:

$$(A_k a_k + B_k a_{-k}^{+})\left|\Psi\right\rangle = 0, \qquad (|A_k|^2 - |B_k|^2 = 1).$$

To realized these states, we need to extend the ansatz such that

$$\hat{K}(u) = \frac{i}{2}\int dk^d \,\Gamma\left(ke^{-u}/M\right)\left[g(u)a_k^{+}a_{-k}^{+} + g^{*}(u)a_k a_{-k}\right],$$

$\Rightarrow$ SU(1,1) Bogoliubov transf. Mk(u)

$$M_k(u) = \begin{pmatrix} p_k(u) & q_k(u) \\ q_k^{*}(u) & p_k^{*}(u) \end{pmatrix} \in SU(1,1), \qquad |p_k(u)|^2 + |q_k(u)|^2 = 1,$$

$$\underbrace{(A_k(u), B_k(u))}_{\text{scale } u} = \underbrace{(\alpha_k, \beta_k)}_{\text{IR limit}} \cdot M_k(u).$$

We can express Mk(u) by introducing 2 × 2 matrix G(u):

$$M_k(u) \equiv \tilde{P}\exp\left(-\int_{-\infty}^{u} G(u)\Gamma(ke^{u}/\Lambda)\right),$$

$$\underbrace{\Rightarrow}_{\text{UV limit}} M_k(0) = \tilde{P}\exp\left(-\int_{\log\frac{k}{\Lambda}}^{0} G(u)\right),$$

$$\Rightarrow G(u) = k\frac{dM_k(0)}{dk}\cdot M_k^{-1}(0). \quad (u = \log\frac{k}{\Lambda})$$

The choice: $\hat{K}(u) = \frac{i}{2}\int dk^d\,\Gamma\left(ke^{-u}/M\right)\left[g(u)a_k^+ a_{-k}^+ + g^*(u)a_k a_{-k}\right],$

corresponds to $G(u) = \begin{pmatrix} 0 & g(u) \\ g^*(u) & 0 \end{pmatrix}.$

For a given UV state $|\Psi\rangle (= |\Psi(0)\rangle)$ or equally $M_k(0)$,
the intermediate state $|\Psi(u)\rangle$ or $M_k(u)$ is determined
up to an ambiguity.

This stems from the phase factor ambiguity of wave function:

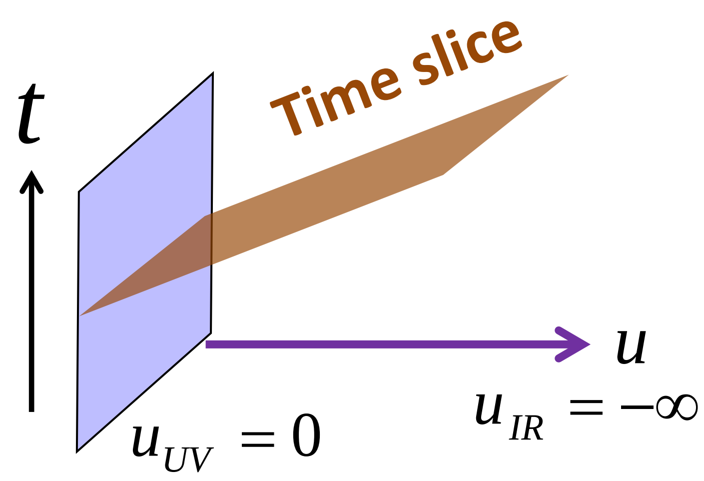
$$(A_k a_k + B_k a_{-k}^{+})|\Psi\rangle = 0 \quad \Rightarrow \quad e^{i\theta_k(t)} \cdot (A_k a_k + B_k a_{-k}^{+})|\Psi\rangle = 0 \ .$$

Our conjecture:

the phase ambiguity $\theta_k(t)$

$\Leftrightarrow$ the choice of the time slice

$$F(t,u) = const.$$



A Description of Quantum Quench by Calabrese-Cardy 05

$$\left|\Psi(t=0)\right\rangle \approx e^{-\frac{\beta H}{4}} \cdot \left|B\right\rangle.$$

Regularization factor because the real excitation energy is finite.

Boundary state

Δm~1/β (~ effective temp.)

Ex. Free Massless Scalar field (Dirichlet b.c.)

$$\left|\Psi(t)\right\rangle = \exp\left(-\frac{1}{2}\int dk e^{-\beta k/2} e^{-2ikt} a_k^{+} a_{-k}^{+}\right)\left|0\right\rangle,$$

$$A_k = \frac{1}{\sqrt{1-e^{-\beta k}}}\cdot e^{ikt+i\theta_k(t)}, \qquad B_k = \frac{e^{-\beta k/2}}{\sqrt{1-e^{-\beta k}}}\cdot e^{-ikt+i\theta_k(t)}.$$

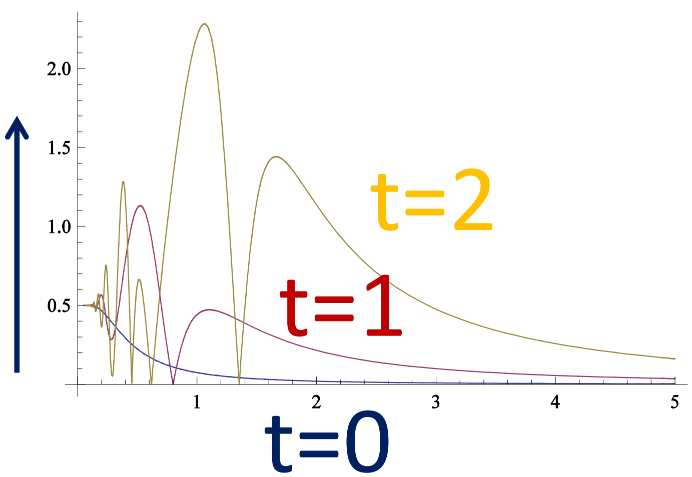
We fix θk(t) such that we have the form: $G(u) = \begin{pmatrix} 0 & g(u) \\ g^*(u) & 0 \end{pmatrix}.$

This leads to $g(u) = \frac{1}{2} + \frac{1}{\sinh(k\beta/2)}\left(kt\sin(2\theta_k(t)) - \frac{k\beta}{4}\cos(2\theta_k(t))\right).$

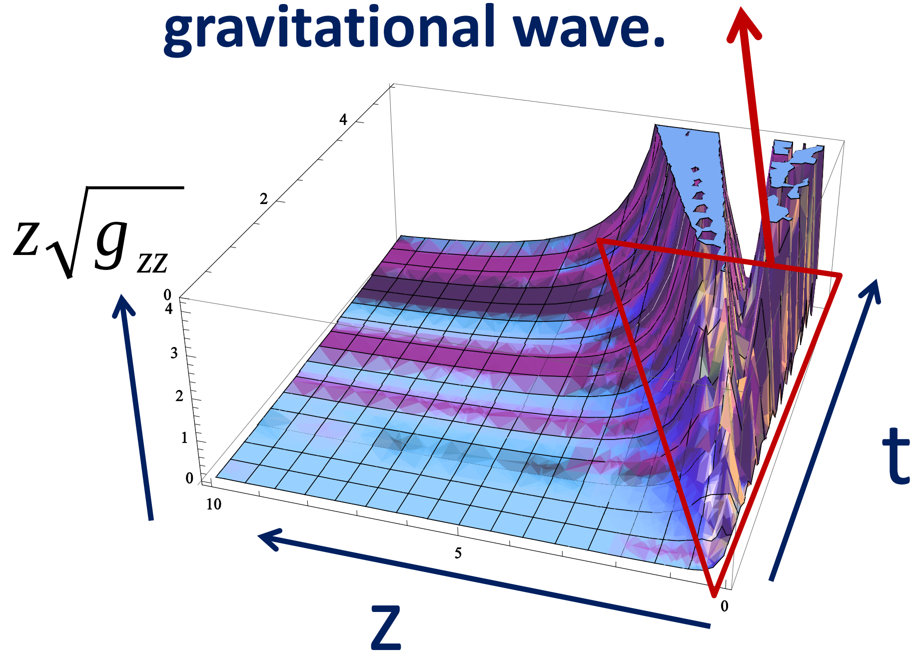
$$\text{Note}: \quad \theta_k(t) \approx -kt \quad \text{ when } k >> \beta.$$

Time dependent metric from the 2d Quantum Quench

$$z\sqrt{g_{zz}} = g(u)$$



Light cone: looks like a gravitational wave.

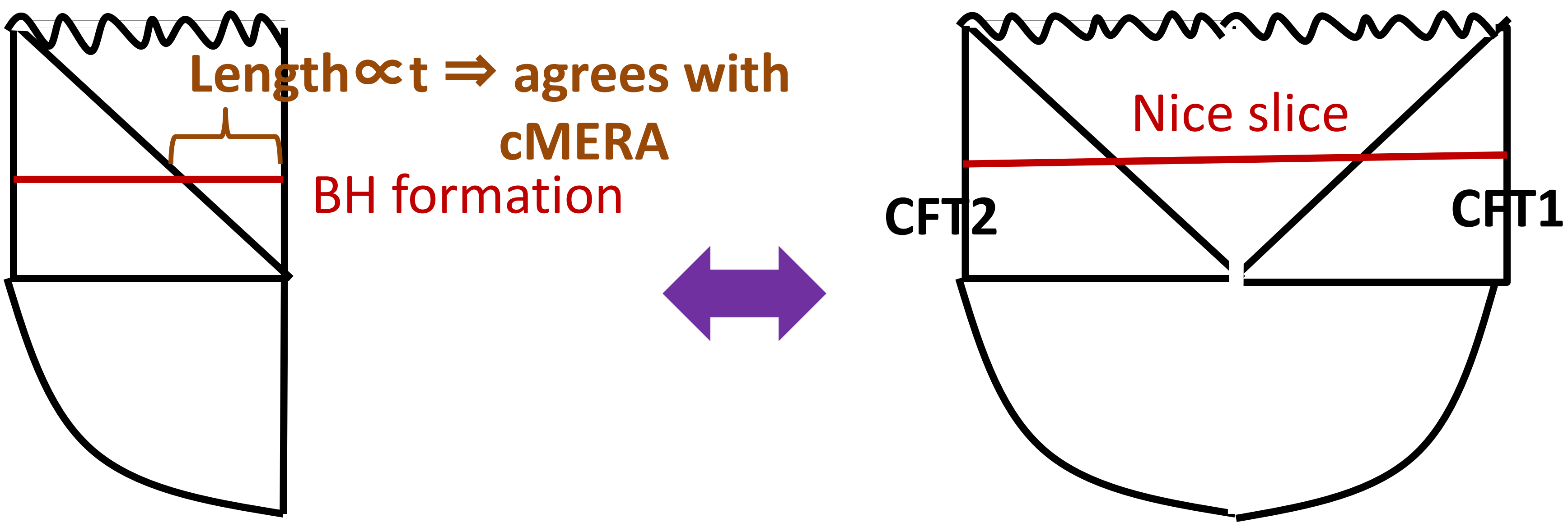


We can also (analytically) confirm the linear growth: SA∝t because g(u)∝t at late time. This is also true in higher dim.

This is consistent with the known CFT (2d) [Calabrese-Cardy 05].
and with the holographic result (any d). [Hartman-Maldacena 13]

Comparison with AdS BH

The holographic dual of a quantum quench was analytically constructed as the half of eternal AdS BH. [Hartman-Maldacena 13]



Gravity dual of quantum quench

Eternal AdS BH
dual to finite temp. CFT

(3-2) Finite Temperature CFT MERA

Indeed, in our free scalar model, we find this relation as follows:

$$\left|\Psi(t)\right\rangle_{QQ} = N \cdot \exp\left(-\frac{1}{2}\int dk e^{-\beta k/2} e^{-2ikt} a_k^+ a_{-k}^+ \right)\left|0\right\rangle.$$

Z_2 projection: CFT1=CFT2

$$\left|\Psi(t)\right\rangle_{\text{Finite}\,T} = N' \cdot e^{-iHt} \cdot \sum_n e^{-\beta E_n/2}\left|n\right\rangle_1 \left|\tilde{n}\right\rangle_2$$

$$= N' \cdot \exp\left(-\int dk e^{-\beta k/2} e^{-2ikt} a_k^+ \tilde{a}_{-k}^+ \right)\left|0\right\rangle_1 \left|\tilde{0}\right\rangle_2.$$

Therefore we can construct the cMERA for the finite temp. CFT.
We can indeed prove that the metric guu, defined in a quantum information theoretically is identical to that of the quantum quench.

④ Possible Gravity Duals of Flat Space and Volume Law

If we consider the almost flat metric (**HEE ∝ Volume**)

$$ds^2 = e^{2wu}du^2 + e^{2u}dx^2 = dy^2 + y^{2/w}dx^2 \quad (y \equiv \varepsilon^{-w}e^{wu}).$$

$$\Leftrightarrow g_{uu} = e^{2wu}.$$

the corresponding dispersion relation reads

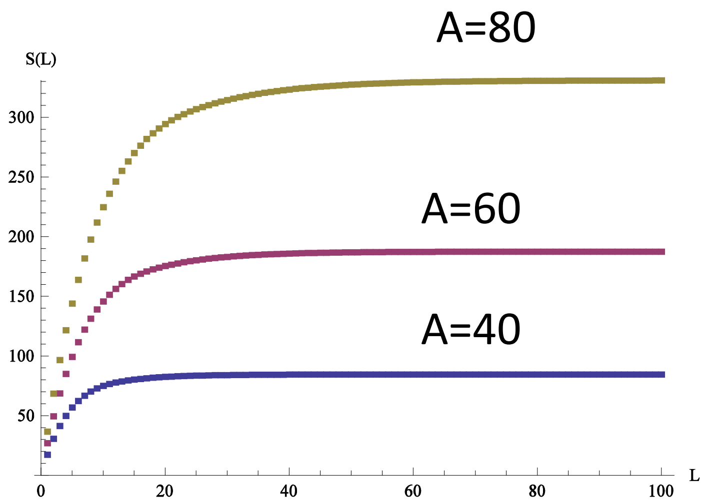
$$\chi(u) = \frac{1}{2}\cdot\left(\frac{k\partial_k E_k}{E_k}\right)\Bigg|_{k=\Lambda e^u} = e^{wu} \qquad \Rightarrow E_k = e^{k^w}.$$

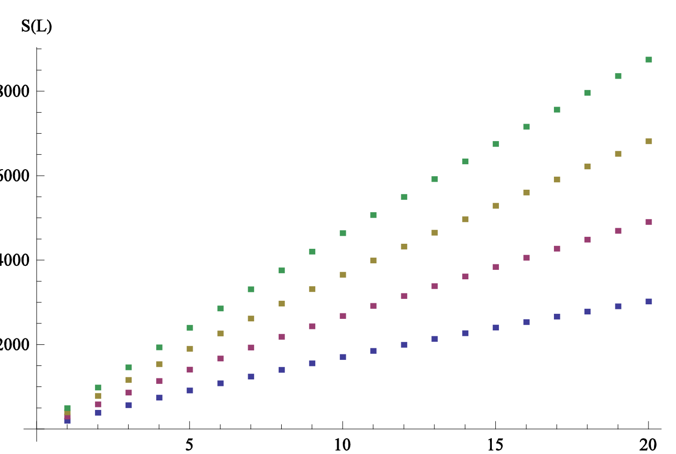
This leads to the highly **non-local** Hamiltonian:

$$H = \int dx^d [\dot{\phi}(x)^2 + \phi(x)e^{A\left(-\partial^2\right)^{\frac{w}{2}}}\phi(x)].$$

Confirmation of Volume law ⇔ Non-local QFTs [Shiba-TT 13]

ex. w=1, d=1





Volume law !

$$S_A \cong \begin{cases} AL/2 & (L << A) \\ cA^2 & (L >> A) \end{cases}.$$

$$\Rightarrow \text{consistent with HEE !}$$

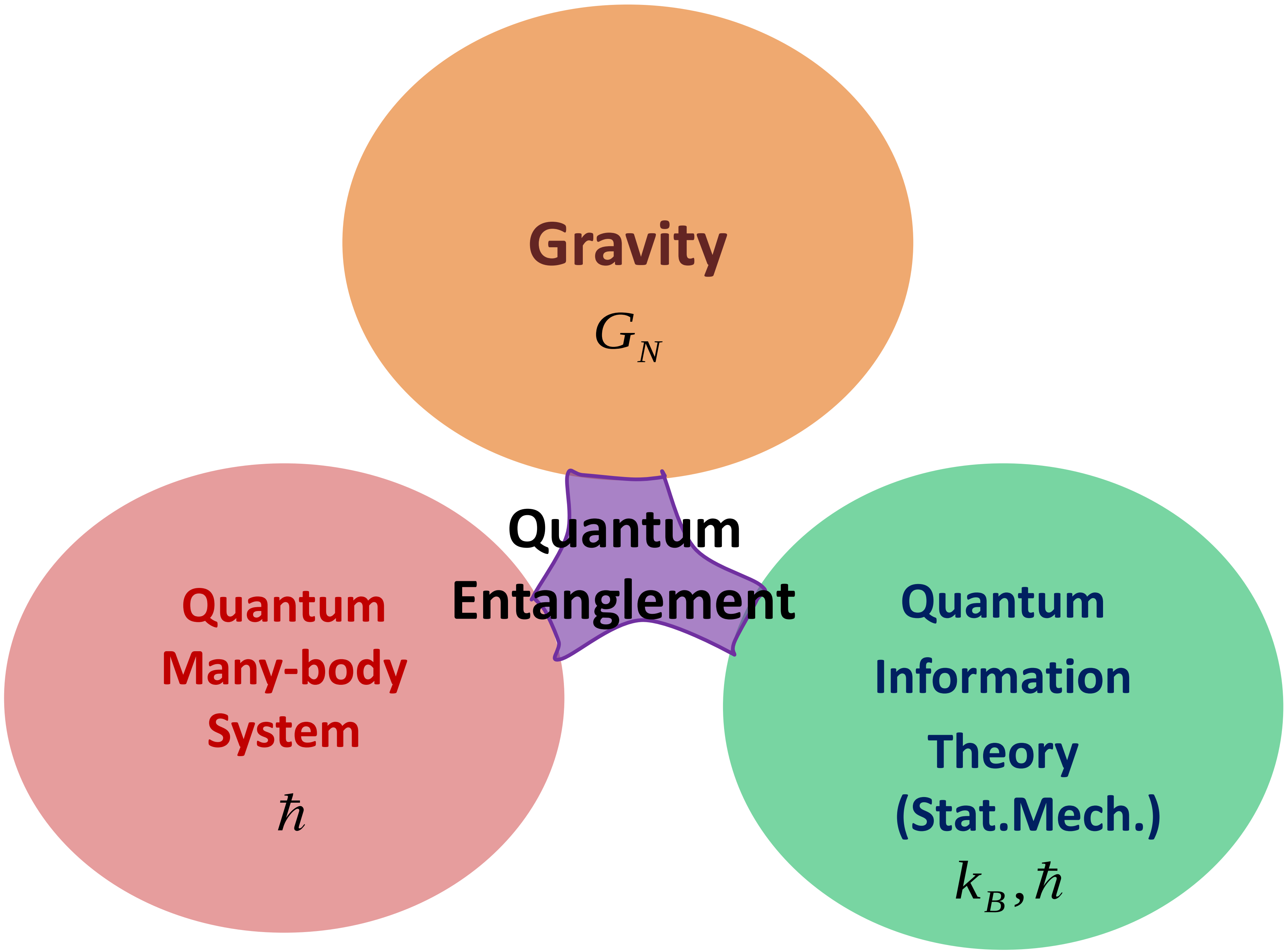
⑤ Conclusions

The idea of the entanglement renormalization can be
a basic mechanism of the AdS/CFT correspondence.

We explored this connection by examining cMERA and proposed a metric in the extra dimension purely in terms of QFT data.

Many future problems:

- How to calculate gtt ? Boosting the subsystem ? Finite temp.?
- The effect of Large N limit in cMERA ?
  (largen N limit ⇔locality⇒saturation of entropy bound ?)
- Time slices and diff. inv. in cMERA ?
- Free field theories ⇒ Higher spin gauge theory?


Gravity
G_N
Quantum Entanglement
Quantum Many-body System
$\hbar$
Quantum Information Theory (Stat.Mech.)
$k_B, \hbar$