Deformed supersymmetric gauge theories from string- and M-theory (Part II): Applications

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26 November 2013 – IPMU

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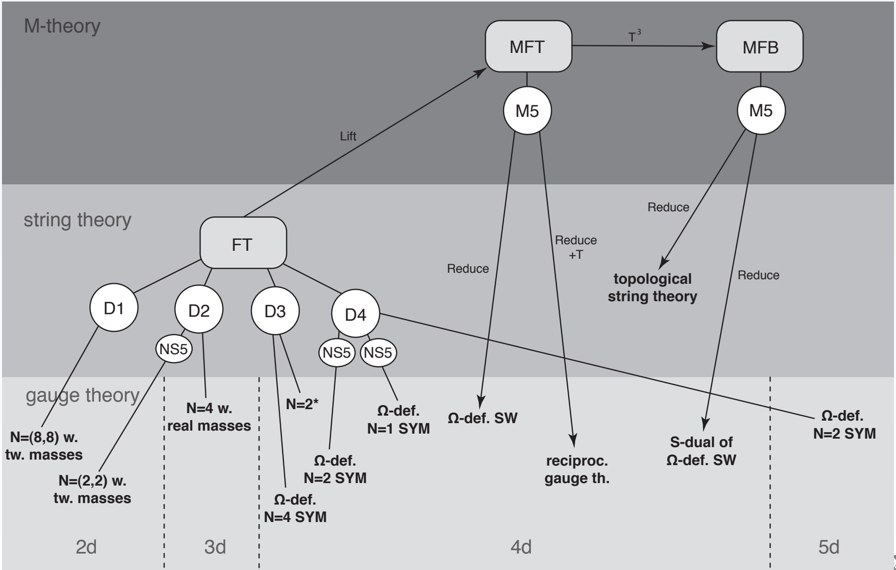
Outline

Outline

Map


M-theory
MFT
T^3
MFB
M5
M5
Lift
string theory
FT
Reduce
Reduce
+T
Reduce
Reduce
topological
string theory
D1
D2
D3
D4
NS5
NS5
NS5
gauge theory
N=(8,8) w.
tw. masses
N=(2,2) w.
tw. masses
N=4 w.
real masses
N=2*
Ω-def.
N=4 SYM
Ω-def.
N=2 SYM
Ω-def.
N=1 SYM
Ω-def. SW
reciproc.
gauge th.
S-dual of
Ω-def. SW
Ω-def.
N=2 SYM
2d
3d
4d
5d

Some D–brane setups

First I want to convince you that there are many applications.

<table>
<tr><th>X</th><th>0</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th></tr>
<tr><td>fluxbrane</td><td colspan="2">ε_1</td><td colspan="2">ε_2</td><td colspan="2">ε_3</td><td></td><td></td><td>$\circ$</td><td>$\circ$</td></tr>
<tr><td>NS5</td><td>$\times$</td><td>$\times$</td><td>$\times$</td><td>$\times$</td><td></td><td></td><td></td><td></td><td>$\times$</td><td>$\times$</td></tr>
<tr><td>D4</td><td>$\times$</td><td>$\times$</td><td>$\times$</td><td>$\times$</td><td></td><td></td><td>$\times$</td><td></td><td></td><td></td></tr>
<tr><td>ξ</td><td>0</td><td>1</td><td>2</td><td>3</td><td></td><td></td><td>4</td><td></td><td></td><td></td></tr>
</table>

The D4 branes describe the Ω-**deformation** of $d = 4, \mathcal{N} = 2$.

Some D–brane setups

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X	0	1	2	3	4	5	6	7	8	9
fluxbrane	ε_1		ε_2		ε_3		ε_4		$\circ$	$\circ$
D3	$\times$	$\times$	$\times$	$\times$						
ξ	0	1	2	3						

The D3 branes describe the **Ω-deformation** of $d = 4, \mathcal{N} = 4$.

Some D–brane setups

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X	0	1	2	3	4	5	6	7	8	9
fluxbrane					ε_1		ε_2		$\circ$	$\circ$
D3	$\times$	$\times$	$\times$	$\times$						
ξ	0	1	2	3						

The D3 branes describe $d = 4, \mathcal{N} = 2^*$ with mass $m = \varepsilon$.

Some D–brane setups

First I want to convince you that there are many applications.

X	0	1	2	3	4	5	6	7	8	9
fluxbrane				ε_1		ε_2		ε_3		$\circ$
D2	$\times$	$\times$	$\times$							
ξ	0	1	2							

The D2 branes describe the **real mass** deformation of $d = 3, \mathcal{N} = 4$ with mass $m_i = \varepsilon_i$

Some D–brane setups

First I want to convince you that there are many applications.

X	0	1	2	3	4	5	6	7	8	9
fluxbrane				$\circ$	ε_1				ε_2	
NS5	$\times$	$\times$	$\times$	$\times$					$\times$	$\times$
D3	$\times$	$\times$	$\times$				$\times$			
ξ	0	1	2				3			

The D3 branes describe the **real mass** deformation of $d = 3, \mathcal{N} = 2$.

Some D–brane setups

First I want to convince you that there are many applications.

X	0	1	2	3	4	5	6	7	8	9
fluxbrane			$\circ$	$\circ$	ε_1				ε_2	
NS5	$\times$	$\times$	$\times$	$\times$					$\times$	$\times$
D2	$\times$	$\times$					$\times$			
ξ	0	1					2			

The D2 branes describe the **twisted mass** deformation of $d = 2, \mathcal{N} = 4$, *i.e.* $d = 2, \mathcal{N} = 2^*$.

Outline

Applications at a glance

D2 branes (spin chains)

D4 branes (Ω–deformation)

Gravity duals

Non-commutativity from geometry

BPS states in the AGT correspondence

Outlook

The Gauge-Bethe Correspondence

- The gauge–Bethe correspondence relates the **Coulomb branch** of certain supersymmetric gauge theories to the **Bethe Ansatz equations** of integrable systems
- More precisely it identifies the twisted effective superpotential with the Yang–Yang function
- The simplest example is the **XXX spin chain**, whose Bethe Ansatz equations read:

$$\left(\frac{\sigma_i - \imath/2}{\sigma_i + \imath/2}\right)^L = \prod_{j\neq i}^{N} \frac{\sigma_i - \sigma_j - \imath}{\sigma_i - \sigma_j + \imath}, \qquad i = 1, \dots, N$$

- These are the same equations that one obtains describing the low energy effective action for a **two dimensional $\mathcal{N} = (2,2)$ theory** with gauge group $U(N)$, L fundamentals, L antifundamentals and an adjoint field with **twisted masses $m_Q = m_{\tilde{Q}} = -\imath/2$, and $m_\Phi = \imath$.**

D2 branes in the fluxtrap

How do we realize this system in String Theory?
Consider the following system of branes

direction	0	1	2	3	4	5	6	7	8	9
fluxtrap					$-\varepsilon$		ε		$\circ$	$\circ$
NS5	$\times$	$\times$					$\times$	$\times$	$\times$	$\times$
D2	$\times$	$\times$	$\times$							
D4	$\times$	$\times$		$\times$	$\times$	$\times$				

From the point of view of the gauge theory on the D2 branes

- $x_8 + \imath x_9 = \sigma$ (twisted chiral)
- $x_6 + \imath x_7 = \Phi$ (chiral adjoint)
- the separation of the NS5 in x_3 is the Fayet-Iliopoulos term
- the separation of the NS5 in x_2 is $1/g^2$

The D2 brane as a BPS object

Consider the **static embedding**

$$F_{\alpha\beta} = 0\,, \quad x_0 = \zeta^0\,, \quad x_1 = \zeta^1\,, \quad x_2 = \zeta^2\,, \quad \phi_1 = \omega\,\zeta^0\,, \quad \phi_2 = -\omega\,\zeta^0\,.$$

The bosonic part of the DBI action reads

$$S = -\mu_2 \int \mathrm{d}^3\zeta \, \sqrt{1 + (x_4^2 + x_5^2 + x_6^2 + x_7^2)\left(|\,\varepsilon\,|^2 - \omega^2\right)}\,.$$

The **equations of motion** can be satisfied in two ways:

- if $x_{3\ldots7} = 0$. This is a **static D2 brane sitting in the trap**.
- if $\omega = \pm|\,\varepsilon\,|$. This is a **rotating D2 brane**. A nice feature is that we are not in the linearized approximation but the frequency is fixed only by the twisted mass $|\,\varepsilon\,|$ and is independent of the amplitude.

Static Embedding

The supersymmetries preserved by the static embedding are those such that

$$\epsilon_L = \Gamma_{D2}\, \epsilon_R\,.$$

- without the NS5 there are 8 preserved supercharges ($\mathcal{N} = 2$ in 3d)

$$\begin{cases} K_L = \left(\mathbb{1} + \Gamma_{11}\right) \Pi_-^{\text{flux}}\, \Gamma_{1208} \exp\left[\frac{1}{2}\left(\phi_1 + \phi_2\right) \Gamma_{67}\right] \eta_1\,, \\ K_R = \left(\mathbb{1} - \Gamma_{11}\right) \hat{\Gamma}_8 \hat{\Gamma}_9 \Pi_-^{\text{flux}} \exp\left[\frac{1}{2}\left(\phi_1 + \phi_2\right) \Gamma_{67}\right] \eta_1\,. \end{cases}$$

- with the NS5 there are 4 preserved supercharges ($\mathcal{N} = (2,2)$ in 2d)

$$\begin{cases} K_L = \left(\mathbb{1} + \Gamma_{11}\right) \Pi_-^{\text{NS5}} \Pi_-^{\text{flux}}\, \Gamma_{1208} \exp\left[\frac{1}{2}\left(\phi_1 + \phi_2\right) \Gamma_{67}\right] \eta_2\,, \\ K_R = \left(\mathbb{1} - \Gamma_{11}\right) \hat{\Gamma}_8 \hat{\Gamma}_9 \Pi_+^{\text{NS5}} \Pi_-^{\text{flux}} \exp\left[\frac{1}{2}\left(\phi_1 + \phi_2\right) \Gamma_{67}\right] \eta_2\,. \end{cases}$$

DBI action for the D2 brane

The last thing that remains to do is to show how the **D2 branes in the fluxtrap background acquire a twisted mass.**
We simply need to write the **Dirac–Born–Infeld action at quadratic order**:

$$S = -\mu_2 \int \mathrm{d}^3\zeta \;\; e^{-\Phi} \sqrt{-\det(g_{\alpha\beta} + B_{\alpha\beta})} \left[1 - \frac{1}{2} \bar{\psi} \left((g+B)^{\alpha\beta} \Gamma_\beta D_\alpha + \Delta^{(1)} \right) \psi \right] ,$$

where

$$D_\alpha = \partial_\alpha X^\mu \left(\nabla_\mu + \frac{1}{8} H_{\mu mn} \Gamma^{mn} \right) , \quad \Delta^{(1)} = \frac{1}{2} \Gamma^m \partial_m \Phi - \frac{1}{24} H_{mnp} \Gamma^{mnp} .$$

Then we expand all the terms at their respective leading order in the fields:

$$g_{\mu\nu}\, \mathrm{d}X^\mu\, \mathrm{d}X^\nu = \mathrm{d}\vec{x}^2_{0\dots 9} + \mathcal{O}(X^4) ,$$

$$H_{\mu\nu\rho}\, \mathrm{d}X^\mu \wedge \mathrm{d}X^\nu \wedge \mathrm{d}X^\rho = 2 |\epsilon| \left(\rho_1\, \mathrm{d}\rho_1 \wedge \mathrm{d}\phi_1 - \rho_2\, \mathrm{d}\rho_2 \wedge \mathrm{d}\phi_2 \right) \wedge \mathrm{d}x_8 + \mathcal{O}(X^5) ,$$

$$e^{-\Phi} = \frac{1}{g_3^2 \sqrt{\alpha'}} \left(1 + \frac{|\epsilon|^2}{2} \left(\rho_1^2 + \rho_2^2 \right) \right) + \mathcal{O}(X^4) .$$

DBI action for the D2 brane

- From the **dilaton** we get a term

$$|\varepsilon|^2 \left(\rho_1^2 + \rho_2^2\right)$$

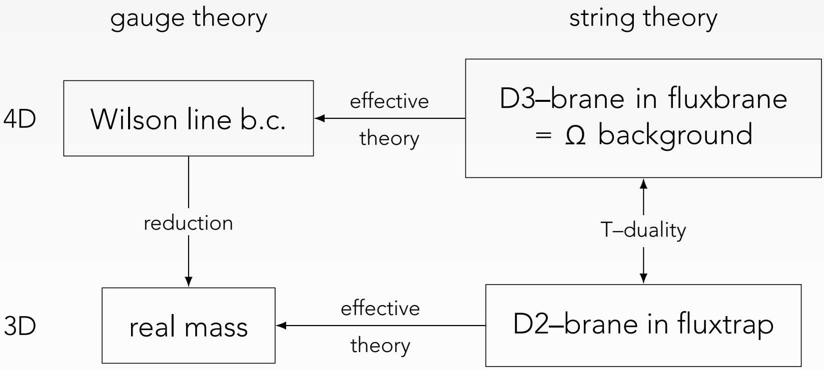
- From the ***B*-field** we get the mass for the fermions:

$$\frac{\varepsilon}{2}\bar{\psi}\left(\Gamma_{45} - \Gamma_{67}\right)\Gamma_{\bar{v}}\psi + \text{c.c.}$$

Putting everything together we reproduce the expected form for the twisted mass term in two dimensions

$$S = \int \mathrm{d}^2\zeta \left[\dot{\Phi}\dot{\bar{\Phi}} - |\varepsilon|^2 \Phi\bar{\Phi} - \bar{\psi}\,\Gamma_0\dot{\psi} - \imath\,\varepsilon\,\bar{\psi}\,\Pi_-^{\Phi}\Gamma_{\bar{v}}\psi + \text{c.c.}\right],$$

A gauge theory perspective



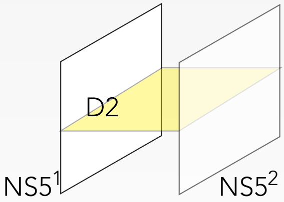
The idea

This is a good moment to stop

- We use a setup with D2 branes stretched between NS5 branes in presence of D4 branes
- Instead of the usual flat space we embed the branes in the fluxtrap
- The ε's are orthogonal to the D2 branes
- We construct BPS states with the right supersymmetries
- The ε deformation turns into a twisted mass term in the effective theory
- This is the mass term required for the two-dimensional Gauge-Bethe correspondence [0901.4744]

Quantum symmetries

N D2 branes stretching between two NS5 branes with L perpendicular D4 branes



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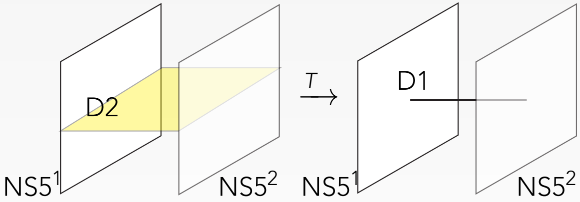

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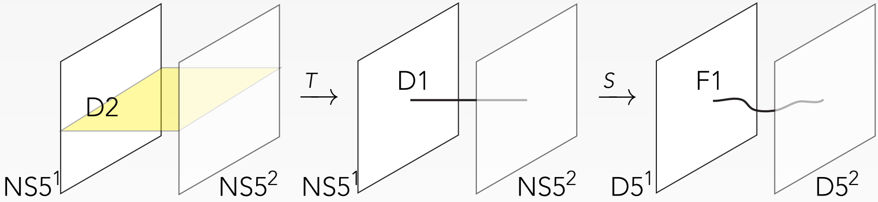

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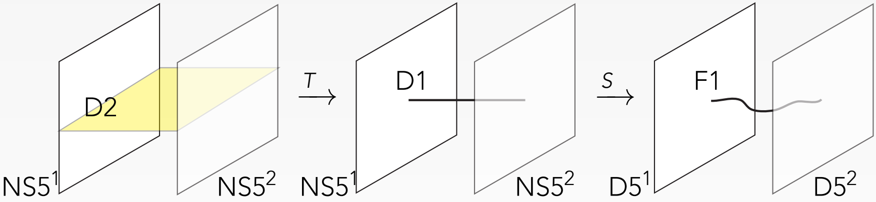

This is the first realization of the enhanced $SU(2)$ symmetry of two coincident NS5 branes in terms of gauge theory.

Outline

The gauge theory

Consider a stack of D4–branes extended **in the directions of the shifts**.

X	0	1	2	3	4	5	6	7	8	9
fluxbrane	ε_1		ε_2		ε_3					$\circ$
NS5	$\times$	$\times$	$\times$	$\times$					$\times$	$\times$
D4	$\times$	$\times$	$\times$	$\times$			$\times$			
ξ	0	1	2	3			4			

The Ω-deformed action

Now we just need to write the DBI action expanded at second order in the fields:

$$\mathscr{L}_{\varepsilon_1,\varepsilon_2} = -\frac{1}{4g_4^2}\left(\|F\|^2 + \frac{1}{2}\|\,\mathrm{d}\varphi + 2\,\mathrm{i}\,\varepsilon\,\imath_{\hat{U}}F\|^2 + \frac{\varepsilon^2}{8}\|\imath_{\hat{U}}\,\mathrm{d}(\varphi + \bar{\varphi})\|^2\right),$$

where $\hat{U}$ is the pullback of the vector field U,

$$\varepsilon\,\hat{U} = \varepsilon\, f^* U = \varepsilon\,\hat{U}^i\partial_{\xi^i} = \varepsilon_1\left(\xi^0\partial_1 - \xi^1\partial_0\right) + \varepsilon_2\left(\xi^2\partial_3 - \xi^3\partial_2\right).$$

Lagrangian of the Ω–deformation of $\mathcal{N} = 2$ SYM. [Nekrasov-Okounkov]
The advantage is that now **we can understand it as coming from string theory** and we have an algorithmic way to generalize it.

The interpretation

$$\mathscr{L}_{\varepsilon_1,\varepsilon_2} = -\frac{1}{4g_4^2}\left(1 + \|F\|^2 + \frac{1}{2}\|\,\mathrm{d}\varphi + 2\mathrm{i}\,\varepsilon\,\imath_{\hat{U}}F\|^2 + \frac{\varepsilon^2}{8}\|\imath_{\hat{U}}\,\mathrm{d}(\varphi+\bar{\varphi})\|^2\right),$$

- the terms in ε are odd under charge conjugation $A_\mu \to -A_\mu$. This is because they come from the B field. This is the leading deformation of the background
- the terms in ε^2 come from metric and dilaton. They control classical gauge configurations and hence directly to the instanton moduli space
- A single **D–instanton is a $D(-1)$ brane**. Its action is

$$\mathscr{L}_{\text{inst}} = e^{-\Phi} = \sqrt{1 + \varepsilon^2\|U\|^2}$$

a critical point for the action is a **critical point for the dilaton profile: $U = 0$**. This is the **string theoretical version of localization**.

Generalizations

Let us now remove the NS5–branes. The **Ω–deformed Lagrangian of $\mathcal{N} = 4$ super Yang–Mills**:

$$\mathscr{L} = \frac{1}{4g^2}\left[F_{ij}F_{ij} + \frac{1}{2}\left|\partial_i \varphi + \varepsilon\, U^k F_{ki}\right|^2 - \frac{1}{8}\left(\varepsilon\, U^i\, \partial_i \bar{\varphi} - \text{c.c.} + 2\,|\,\varepsilon\,|^2\, U^k \bar{U}^l F_{kl}\right)^2 \right.$$
$$+ \frac{1}{4}\left(\delta^{\,ij} + |\,\varepsilon\,|^2\, U^i \bar{U}^j\right)\left(\partial_i z\, \partial_j \bar{z} + \partial_i w\, \partial_j \bar{w} + \text{c.c.}\right) +$$
$$\left. + \frac{1}{2\,\mathrm{i}}\left(\bar{\varepsilon}_{\,3}\, \varepsilon\, U^i + \text{c.c.}\right)\left(\bar{w}\, \partial_i w - \text{c.c.}\right) + \frac{1}{2}\,|\,\varepsilon_{\,3}|^2\, w\bar{w}\right],$$

The fields w and z describe the oscillations of the D3–brane respectively in $x^4 + \mathrm{i}\, x^5$ and $x^6 + \mathrm{i}\, x^7$.
The effect of the deformation on these two fields consists in a **modification of the kinetic term**. Moreover, the field w acquires a mass term and a one-derivative term, which is allowed by the broken Poincaré invariance.

Generalizations

Now put the brane completely orthogonal to the fluxtrap (i.e. set $\varepsilon_1 = \varepsilon_2 = 0$ and $\varepsilon_3 = \varepsilon_4 = m$).
The Lagrangian is the same as $\mathcal{N} = 4$ but with mass terms for two of the complex bosons:

$$\mathscr{L} = \frac{1}{8g^2}\left[2F_{ij}F_{ij} + \left|\partial_i \varphi\right|^2 + \left|\partial_i z\right|^2 + \left|\partial_i w\right|^2 + \left|m\right|^2 \left|w\right|^2 + \left|m\right|^2 \left|z\right|^2\right].$$

There is only one deformation parameter, so we have eight supercharges.
This is an alternative **realization of $\mathcal{N} = 2^*$**, obtained as dimensional reduction of $\mathcal{N} = 2$ in five dimensions on a Wilson line (analogous to the real mass term in two dimensions coming from three dimensions).
The description is completely local.

Generalizations

The **Ω–deformed Lagrangian of $\mathcal{N} = 1$ super Yang–Mills**. Just by looking at the D-brane diagram we find that this is only possible in the $\varepsilon_2 = 0$ limit and if the (dual) Melvin direction is parallel to the brane.

x	0	1	2	3	4	5	6	7	8	9
fluxbrane	ε_1		$\circ$		$-\varepsilon_1$					
D4	$\times$	$\times$	$\times$	$\times$			$\times$			
NS5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$				
NS5	$\times$	$\times$	$\times$	$\times$					$\times$	$\times$
ξ	0	1	2	3						

The effective action has two supercharges and a one-derivative term

$$\mathscr{L} = \frac{1}{4g^2}\left[F_{ij}F_{ij} + 2\,\varepsilon\,U^i F_{ij}\mathbf{e}_2^j\right]$$

Outline

A gravity dual for the four-dimensional deformed theories

We have realized Ω–deformed $\mathcal{N} = 4$ gauge theories in terms of D3–branes in the fluxtrap background. The most general example is $\mathcal{N} = 2^*$ with $\varepsilon_1 \neq \varepsilon_2$.

Up to now the D3–brane has been a probe object. We have disregarded its backreaction on the bulk fields.

Adding the backreaction will provide us with the gravity dual to the deformed gauge theories.

Literature

This type of deformation has already been studied in the literature. There are two famous examples:

1. The β–deformation [Maldacena–Lunin] preserves the conformal symmetry of $\mathcal{N} = 4$ via a combination of shifts and T–dualities.
2. Confining Four-Dimensional Gauge Theory [Polchinski–Strassler] deformation of $\mathcal{N} = 4$ leading to $\mathcal{N} = 1^*$.

Even though there are obvious similarities with 1., since we break conformal invariance we are forced to follow 2.

Polchinski–Strassler

We have seen that for $\epsilon_1 = \epsilon_2 = 0$, the brane in the fluxtrap realizes $\mathcal{N} = 2^*$. This is a special case of the deformation studied by Polchinski and Strassler.

In their work they give masses to all the three scalars of $\mathcal{N} = 4$ to realize $\mathcal{N} = 1^*$.

- The [PS] solution is found at **first order in the deformation parameter and is valid around the horizon**.
- The fluxtrap is the opposite limit (far away from the D3–brane) and the solution is known for any finite value of the deformation.

The two descriptions agree in some defining properties. In both cases **the lowest order effect** is in the three-form and metric and dilaton are corrected starting from second order.

Interpolation

We look for a solution that

- is a deformation of $\mathrm{AdS}_5 \times S^5$ that breaks conformal invariance,
- interpolates between [PS] at the horizon and the fluxtrap at infinity,
- at first order only changes the three-form fields of the standard D3 solution
- has a non-trivial dilaton at second order

The good news is that we have a natural parameter (ε) that we can use to decouple the equations of motion of supergravity.

Equations of motion

The equations for the three-form decouple from the others and the solution at first order can be fed into the dilaton equation and the result in turn can be used for the curvature equation.

$$
\begin{aligned}
\nabla^2 \Phi &= e^{2\Phi} \partial_M C \partial^M C - \tfrac{1}{12} e^{-\Phi} H^2 + \tfrac{1}{12} e^{\Phi} \tilde{F}_3^2 \\
\nabla^M (e^{2\Phi} \partial_M C) &= -\tfrac{1}{6} e^{-\Phi} H \cdot \tilde{F}_3 \\
d * (e^{\Phi} \tilde{F}_3) &= F_5 \wedge H \\
d * (e^{-\Phi} H - C e^{\Phi} \tilde{F}_3) &= -F_5 \wedge F_3 \\
d * \tilde{F}_5 &= - F_3 \wedge H \\
Ric_{MN} &= \frac{1}{2} \partial_M \Phi \, \partial_N \Phi + \frac{e^{2\Phi}}{2} \partial_M C \partial_N C + \frac{1}{96} (\tilde{F}_5^2)_{MN} \\
&\quad + \frac{1}{4} \left(e^{-\Phi} (H^2)_{MN} + e^{\Phi} (\tilde{F}_3^2)_{MN} \right) - \frac{1}{48} G_{MN} \left(e^{-\Phi} H^2 + e^{\Phi} \tilde{F}_3^2 \right)
\end{aligned}
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&\quad + \frac{1}{4} \left(e^{-\Phi} (H^2)_{MN} + e^{\Phi} (\tilde{F}_3^2)_{MN} \right) - \frac{1}{48} G_{MN} \left(e^{-\Phi} H^2 + e^{\Phi} \tilde{F}_3^2 \right)
\end{aligned}
$$

The solution for $\mathcal{N} = 2^*$

In the $\mathcal{N} = 2^*$ case, the lowest order deformation appears in the two-form fields:

$$
\begin{aligned}
B &= aV \wedge dx^8 + \frac{Q}{r^4}\left(V \wedge dx^8 + x^8 \omega\right), \\
C_2 &= -\frac{Q}{r^4}\left(V \wedge dx^9 + x^9 \omega\right),
\end{aligned}
$$

where $2\omega = \mathrm{d}V$.

- far away from the brane ($Q = 0$), we recover the fluxtrap;
- at the horizon ($a = 0$) this is the Polchinski-Strassler solution.

The breaking of conformal invariance is shown by the presence of a dilaton depending on r

$$
\begin{cases}
\Phi = -\frac{a\varepsilon^2 \rho^2}{2} - \frac{Q\varepsilon^2}{2}\frac{x_9^2 - x_8^2}{r^4} \\
C_0 = Q\varepsilon^2 \frac{x^8 x^9}{r^4}
\end{cases}
$$

The solution for $\varepsilon_1 = -\varepsilon_2$

In the case of the Ω–deformation of $\mathcal{N} = 4$ SYM it is first of all necessary to analytically continue the undeformed solution (type II*). The undeformed background is then $dS_5 \times H^5$ and the deformation at first order is given by

$$B = \left(V \wedge dx^8 - \frac{Q}{Q + ar^4} x^8 \omega \right),$$

$$C_2 = -\frac{Q}{Q + ar^4} x^8 \omega .$$

The structure of the solution not quite the same as the previous one.

Again we can compute the dilaton and C_0 fields:

$$\begin{cases} \Phi = \left(-\frac{1}{2} + \frac{Q}{4ar^4} \right) V^2 + \left(\frac{aQ}{Q+ar^4} - \frac{Q^2}{ar^8} \right) \varepsilon^2 (x^8)^2 \\ C_0 = \frac{\mathrm{i}\, Q}{4ar^4} V^2 + \mathrm{i} \left(\frac{aQ}{Q+ar^4} - \frac{Q^2}{ar^8} \right) \varepsilon^2 (x^8)^2 \end{cases}$$

Outline

What I did not tell you

- Where does non-commutativity come from?
- How do we understand and realize BPS states in this framework?
- How is our work related to parallel attempts at understanding the Ω–deformation from String Theory?
  - Start from the fluxbrane in M–theory and reduce on the Melvin circle
  - Insertion of graviphoton operators in the heterotic description
- What is the relationship with the topological string description?

Open issues

- Is it possible to realize the gauge theory on S^4?
- Can we realize explicitly Liouville theory?
- Can we construct a non-perturbative (in ε) gravity dual?

Thank you for your attention

Outline

Taub–nut

In order to make our construction more transparent it is convenient to start from a **Taub–nut space** and put a fluxtrap in $\mathrm{TN}_Q \times S^1 \times \mathbb{R}^5$.

A Taub–nut space is a singular S^1 fibration over $\mathbb{R}^3$

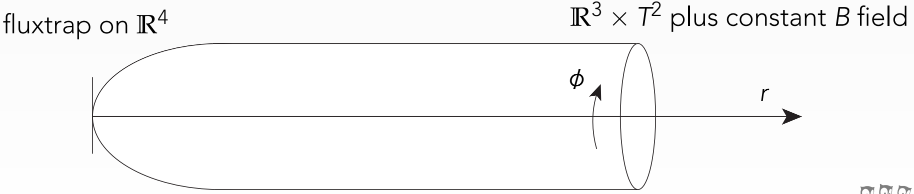
$$\begin{array}{ccc} S^1(\theta) & \longrightarrow & \mathrm{TN} \\ & & \downarrow \\ & & \mathbb{R}^3(\mathbf{r}) \end{array}$$

It interpolates between $\mathbb{R}^4$ for $r \to 0$ and $\mathbb{R}^3 \times S^1$ for $r \to \infty$.

Flux–trap

$$
\begin{aligned}
ds^2 &= V(r)\, d\mathbf{r}^2 + \frac{1}{V(r) + \varepsilon^2} (d\phi + Q \cos\omega\, d\psi)^2 + \frac{V(r)}{V(r) + \varepsilon^2} (dx^9)^2 + d\mathbf{x}^2_{4\ldots8}\,, \\
B &= \frac{\varepsilon}{V(r) + \varepsilon^2} (d\phi + Q \cos\omega\, d\psi) \wedge dx^9\,, \\
e^{-\Phi} &= \sqrt{1 + \frac{\varepsilon^2}{V(r)}}\,.
\end{aligned}
$$

This interpolates between the fluxtrap in flat space that we used to reproduce Nekrasov's action and $\mathbb{R}^3 \times T^2$ with a constant B field.



The alternative description

In the limit $r \to \infty$ the Taub–nut becomes $\mathbb{R}^3 \times S^1$ and the fluxtrap is the result of a **T–duality on a torus with shear**, *i.e.* a constant B field. Putting a D4–brane wrapping the Taub–nut space we obtain the **alternative description of the Ω deformation** proposed by Witten and Nekrasov.

We can calculate the Killing spinors before and after the duality and we find (from the analysis of the bulk) that the supersymmetry generators in the four–dimensional theory are "rotated"

$$\eta_{\,\varepsilon} = \exp[\frac{\vartheta}{2} r_{39}]\,\eta_{\,\varepsilon=0}\,.$$

where

$$\tan\frac{\vartheta}{2} = \varepsilon\,\lambda$$

Non-commutativity

The Ω deformation for $\varepsilon_1 = -\varepsilon_2$ is related to **topological strings**. It has been observed that

> “ *the **Riemann surface** Σ behaves for many purposes as a **subspace of a quantum mechanical** (s, v) **phase space** where $g_s = \hbar$.* [Aganagic, Dijkgraaf, Klemm, Marino, Vafa] ”

> “ *this gauge theory provides the **quantization of the classical integrable system** underlying the moduli space of vacua of the ordinary four dimensional N = 2 theory* [Nekrasov, Shatashvili] ”

Our construction gives a **precise geometrical interpretation** for this observation in terms of Riemann surface on a non-commutative plane.

Reduction

Lift the background to M-theory... and **reduce** it on ϕ

$$
\begin{aligned}
ds^2 &= V(r)^{1/2}\, d\mathbf{r}^2 + V(r)^{-1/2} \left[d\mathbf{x}_{4\dots10}^2 - \frac{\varepsilon^2}{V(r) + \varepsilon^2} \left((dx^9)^2 + (dx^{10})^2 \right) \right] , \\
B &= \frac{\varepsilon}{V(r) + \varepsilon^2}\, dx^9 \wedge dx^{10} , \\
e^{-\Phi} &= V(r)^{1/4} \sqrt{V(r) + \varepsilon^2} , \\
A_1 &= Q \cos\omega \, d\psi , \\
A_3 &= B \wedge A_1 .
\end{aligned}
$$

These are Q **D6–branes extended in** $(x^4, \dots, x^{10})$ in presence of an Ω–deformation.

The Seiberg–Witten map

An equivalent description is obtained by applying the **Seiberg–Witten map** to the D6–brane theory in order to turn the B–field into a non-commutativity parameter:

$$\left(\hat{g} + \hat{B}\right)^{-1} = \tilde{g}^{-1} + \Theta \,,$$

where $\hat{g}$ and $\hat{B}$ are the pullbacks of metric and B–field on the brane and $\tilde{g}$ is the new effective metric for a non-commutative space satisfying

$$[x^i, x^j] = \mathrm{i}\, \Theta^{ij} \,.$$

Applying this map to our case:

$$\tilde{g}_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j = \mathrm{d}\mathbf{x}^2_{4\dots10} \,,$$
$$[x^9, x^{10}] = \mathrm{i}\ \varepsilon \ .$$

All dependence on ε disappears from the D6–brane theory and is turned into a constant non-commutativity parameter.

A non-commutative Riemann surface

Let's follow the fate of the **branes** whose dynamics reproduce the Ω–deformed gauge theory.

Start from the configuration of **D4–NS5s, with the D4 wrapping the Taub–nut space**.

In the M–theory lift this configuration turns into a **single M5–brane** extended in the directions $(x^0, \dots, x^3)$ and wrapped **on a Riemann surface** Σ embedded in the (s, v) plane.

Reduction on ϕ turns the M5–brane into an **D4–brane** extended in **r** and **wrapped on** Σ, which is now embedded in the worldvolume of the D6–brane.

For finite ε this picture remains the same, but this time **the Riemann surface** Σ **is embedded in a non-commutative complex plane** where

$$[s, v] = \mathrm{i}\ \varepsilon\ .$$

The point

- We repeat our construction starting from a Taub–nut space in the bulk
- The Taub–trap solution **interpolates** between Nekrasov's **original description** and Nekrasov–Witten's **"alternative" description**
- We lift the IIA background to **M–theory**
- We **reduce** it on the isometry circle.
- The resulting D6 background has a natural **non–commutativity** ϵ
- The gauge theory describes the dynamics of a D4 wrapped on a Riemann surface living on a non-commutative $\mathbb{C}^2$ plane. This is the geometric interpretation of the **"quantum spectral curve"**.

Outline

Oscilloids and DOZZoids

We want to understand the BPS object that enter the study of the quantum effective action in the Ω–deformation:

1. BPS instantons localized at the origin (**oscilloids**);
2. perturbative modes of the fundamental fields, with momentum along the $U(1) \times U(1)$ symmetry (**DOZZoids**).

In **AGT**:

1. Give the modular form defining the holomorphic factor of the Liouville field theory;
2. When resumming their virtual effects we obtain the holomorphic DOZZ factors.

Gauge theory (in five dimensional Melvin spacetime):

1. particles, static in the new time direction;
2. BPS excitations of the vector multiplet.

The BPS instantons of four-dimensional gauge theory (oscilloids) are **D-(-1)-branes** of type iib string theory bound to D3–branes in the type iib fluxtrap solution

$$\mathscr{L} = -\mu_0 e^{-\Phi} = -\frac{1}{g^{\Omega}_{\mathrm{iia}} \ell_\Omega} \sqrt{\left(1 + \varepsilon_1^2 \rho_1^2\right)\left(1 + \varepsilon_2^2 \rho_2^2\right)} \,,$$

The energy for n_{D0} branes is

$$E_{\mathrm{D0}} = \frac{n_{\mathrm{D0}}}{g^{\Omega}_{\mathrm{iia}} \ell_\Omega} \sqrt{\left(1 + \varepsilon_1^2 \rho_1^2\right)\left(1 + \varepsilon_2^2 \rho_2^2\right)} \,,$$

which is minimized for $\rho_1 = \rho_2 = 0$. These particles are localized to the origin by the spatial profile of the dilaton.

DOZZoids in the fluxtrap

The DOZZoids are **open string states stretching between two D4–branes** carrying angular momentum in the $U(1) \times U(1)$ rotational isometry directions. To compute the energy:

1. write the worldsheet action for an open string
2. impose equations of motion and classical Virasoro constraints
3. make an ansatz in which the string rotates in the two angular directions and is stretched between two D3 branes in the dual-Melvin direction

The (mildly) surprising result is that the **angular momentum and the stretching combine linearly**, as expected from the gauge theory result:

$$E = \left| \varepsilon_2 J_2 - \frac{L_2}{2\pi\alpha'} \right|$$

The reciprocal frame

Bulk	Probe	Gauge Theory
type iib in Melvin space	D5	six-dimensional gauge theory with Wilson line boundary conditions in two directions
	T–duality in $\tilde{x}_8$ *and* $\tilde{x}_9$	
type iib in complex fluxtrap	D3	Ω–deformed $\mathcal{N} = 4$ SYM
	T–duality in $\tilde{x}_6$ *and lift*	
M–theory fluxtrap	M5	$(2,0)$ six-dimensional theory
	reduction in σ_1 *and* σ_2	
type iib in deformed D5/NS5	D3	reciprocal gauge theory

The reciprocal frame

frame	object	ρ_1	ϕ_1	ρ_2	ϕ_2	ρ_3	ψ	x_6	x_7	x_8	x_9	x_{10}
fluxtrap	D4	$\times$	$\times$	$\times$	$\times$			$\times$				$\blacksquare$
	F1				$\circlearrowleft$			$\times$			$\times$	$\blacksquare$
	D0							$\times$				$\blacksquare$
M–theory	M5	$\times$	$\times$	$\times$	$\times$			$\times$				$\times$
	M2				$\circlearrowleft$			$\times$			$\times$	$\times$
	P							$\nearrow$				$\nearrow$
reciprocal	D3	$\times$	$\blacksquare$	$\times$				$\times$				$\times$
	D3		$\blacksquare$		$\times$			$\times$			$\times$	$\times$
	P		$\blacksquare$					$\nearrow$				$\nearrow$
	D5		$\blacksquare$	$\times$				$\times$	$\times$	$\times$	$\times$	$\times$
	NS5	$\times$	$\blacksquare$					$\times$	$\times$	$\times$	$\times$	$\times$

The reciprocal gauge theory

The effective theory for the D3-brane in the reciprocal frame has the following properties:

- Four-dimensional supersymmetric gauge theory
- The **gauge coupling** is

$$\frac{1}{g_{\text{rec}}^2} \xrightarrow[\rho_1,\rho_2\to\infty]{} 2\pi\,\frac{\varepsilon_2}{\varepsilon_1} = 2\pi\,b.$$

- Under **S-duality** it transforms as $\varepsilon_1 \leftrightarrow \varepsilon_2$

We have found a gauge theory that has the same coupling as Liouville in AGT and in which the equivalent of Liouville duality $b \mapsto 1/b$ is realized by S-duality of type IIB string theory.

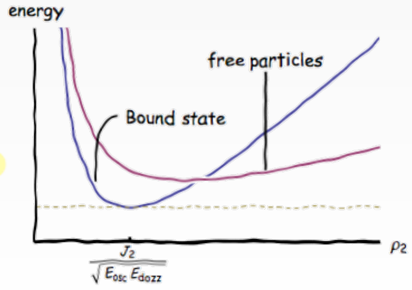
Bound states in the reciprocal theory

Both the oscilloids and the DOZZoids are mapped to D3-branes in the reciprocal theory and we can describe them in a unified way.
Bound state with both charges (here linear and angular momentum):

$$E = \sqrt{\left[\left(\frac{\varepsilon_1 R_{10}}{2\pi\alpha'} L_2 - \varepsilon_2 J_2\right)^2 + \left(\frac{J_2}{\rho_2}\right)^2\right]\left[1 + \rho_2^2 \left(1 + \varepsilon_1^2 \rho_1^2\right)\left(\frac{P}{J_2}\right)^2\right]}$$

Minimum for a finite value of ρ_2. All **momenta combine linearly**:

$$E_{\text{min}} = \left|\frac{\varepsilon_1 R_{10}}{2\pi\alpha'} L_2 - \varepsilon_2 J_2 + P\right|$$



The Point

- **Identify the BPS states** that appear in AGT and lead to the Liouville correlators;
- Construct these states in terms of **D0-branes and fundamental strings** in the fluxtrap;
- We dualize these objects to a new (reciprocal) frame, where we find a **novel gauge theory** that has some of the key properties observed by AGT;
- The BPS states turn out to be bound in this new theory.