Triple Intersection Formulas for Isotropic Grassmannians

I. Triple Intersections

$$[X_\lambda]\cdot[X_{(r)}] = \sum_{\nu} c^{\nu}_{\lambda,r}\,[X_\nu] \qquad c^{\nu}_{\lambda,r} \in \{0,1\}$$

"special Schubert class" — $[X_{(r)}]$

Pieri coef. — $c^{\nu}_{\lambda,r}$

$$\#\left([X_\lambda]\cdot[X_{(r)}]\cdot[X_{\nu^\vee}]\right) = c^{\nu}_{\lambda,r}$$

① I begin with an equation that might be familiar to some of you. Express product of Schubert classes in terms of Schubert basis.

② In [Pieri 1893], Pieri gives rule for calculating $c^{\nu}_{\lambda,r}$

③ Geometrically the Pieri coefs are "triple intersection numbers". They calculate the # of int. pts of 3 Schubert varieties in general position. This view pt. was used by Hodge to give a geometric pf.

$H^*(Gr)$ — [Hodge 1942]

[Buch, R 2011] — $K(Gr)$, $K(OG_{max})$, $K(LG)$

$H^*(IG)$ — [Buch, Kresch, Tamvakis 2009] ss [BKT 2009]

$K(IG)$ — what we'll discuss today

II. Changes for K-theory

$X = \mathbb{IG}_\omega(m, N)$

$\chi_X([X_P] \cdot [X_{T^\vee}] \cdot [X_{(r)}])$

(③ technically should write $[\mathcal{O}_{X_P}]$ not $[X_P]$)

① sheaf euler characteristic

② Schubert symbols

(④ Pieri coefs now sums of triple intersection numbers)

$$[X_P] \cdot [X_{T^\vee}] = [X_P \cap \underset{\text{opp. flag}}{\tilde{X}_{T^\vee}}] = [Y_{P,T}]$$

$Y_{P,T} := X_P \cap \tilde{X}_{T^\vee}$ (Richardson variety)

$\Rightarrow \chi_X([Y_{P,T}] \cdot [X_{(r)}])$ is K-theoretic triple int #.

Having looked at the changes in $K(G)$, let's go back to the common features to all arguments:

III. Main Result

Let $X := IG_\omega(m,N)$ be iso. Grass. of type B, C, or D.

$$\begin{array}{ccccc} IF_\omega(1,m,N) & \xrightarrow[\varphi]{} & IG_\omega(1,N) & \xhookrightarrow[\iota]{} & \mathbb{P}^{N-1} \\ \downarrow \pi & & & & \\ IG_\omega(m,N) = X & & & & \end{array}$$

$[\mathbb{P}(L_{(r)})]\cdot$

<u>Lemma</u>: $\chi_X\left([Y_{P,T}]\cdot[X_{(r)}]\right) = \chi_{\mathbb{P}^{N-1}}\left(\underbrace{[\varphi(\pi^{-1}(Y_{P,T}))]}_{\text{projected Richardson variety}}\cdot[\mathbb{P}(L_{(r)})]\right)$

uses projection formula,
flat maps, + cohomologically trivial maps
[Knutson, Lam, Speyer 2010]
[Billey, ~~Coks~~ ~~Cosk~~ Coskun 2012]

<u>Thm</u>: The projected Richardson variety $\varphi(\pi^{-1}(Y_{P,T}))$ $\forall\, P,T$
[R 2013] is a complete intersection in $\mathbb{P}^{N-1}$
and is defined by

$$\#\{c \in Q_{P,T} : c > 0,\ c-1 \notin Q_{P,T}\} \text{ quadratic eqns}$$

and $\#\{c \in \mathcal{L}_{P,T}\}$ linear eqns.

<u>Upshot</u>: Need only sheaf euler characteristic of complete intersection in projective space to compute triple int. #'s. Well studied!

IV. Equations for the projected Richardson variety in types B/C

$\mathbb{C}^N$, basis $\{e_1, \dots, e_N\}$

flag $\mathbb{C}^1 \subset \mathbb{C}^2 \subset \dots \subset \mathbb{C}^N$

non deg. bilinear form ω s.t. $\omega(e_i, e_j) = \delta_{i+j, N+1}$ $\forall\, i \leq j$

$\rightarrow$ make ω symmetric or skew symmetric.

$\boxed{\text{EG}}$ $\mathbb{C}^4$, ω skew symm.

$$\omega\big((a,b,c,d),(e,f,g,h)\big) = ah + bg - cf - de$$

$\boxed{\text{DEF}}$ $IG_\omega(m,N) := \{\Sigma \in Gr(m,N) : \omega(v_1, v_2) = 0 \ \forall\, v_1, v_2 \in \Sigma\}$

ω sym — $OG(m,N)$ (B)

ω skew — $SG(m,N)$ (C)

symplectic vector spaces must be even dimensional!

$\boxed{\text{DEF}}$ $P = \{p_1 < \dots < p_m\}$

$X_P :=$ closure of Schubert cell X_P°

Any subspace $\mathbb{C}^N$ has a "Schubert symbol"
Schubert cell is set of all Σ w/ given Schubert symbol

$\boxed{\text{OBS}}$ $X_P = \{\Sigma \in IG_\omega(m,N) : \dim(\Sigma \cap \mathbb{C}^{p_j}) \geq j\}$

- types B/C only
- "Schubert variety"
- type D problems not obvious until you start working experimentally

$\boxed{\text{EG}}$ $OG(2,7)$

$P = \{3,6\}$

$$\begin{vmatrix} a & b & c & 0 & 0 & 0 & 0 \\ d & e & f & g & h & i & 0 \end{vmatrix}$$

$bi + ch = 0$

$2ei + 2fh + g^2 = 0$

$\boxed{\text{DEF}}$ $Y_{P,T} := X_P \cap \underbrace{\tilde{X}_{T^\vee}}_{}$ — opp. flag

$T \leq P \Rightarrow Y_{P,T} \neq \emptyset$

$\boxed{\text{DEF}}$ $T \leq P$. $D(P,T) := \{(j,c) : t_j \leq c \leq p_j\}$ "<u>Richardson Diagram</u>"

$P = \{p_1 < \dots < p_m\}$ $t = \{t_1 < \dots < t_m\}$

$\boxed{\text{EG}}$ $SG(3,8)$

$P = \{2,3,8\}$

$T = \{1,3,5\}$

$$\begin{array}{cccccccc} x_1 & x_2 & 3 & 4 & 5 & 6 & 7 & x_8 \\ \end{array}$$
$$\begin{vmatrix} * & * & & & & & & \\ & & * & 0 & & & & \\ & & & 0 & * & * & * & * \end{vmatrix}$$

<u>most</u> elements of $Y_{P,T}$ can be rep. by matrices of ~~shape~~ this shape

$D(P,T)$ helps

- us figure out what equations the projected Richardson variety satisfies

$$\left.\begin{array}{l} x_4 = 0 \\ x_5 = 0 \\ x_1 x_8 + x_2 x_7 = 0 \end{array}\right\} \text{isotropic condition}$$

$\boxed{\text{DEF}}$ $c \in [1,N]$ is a <u>zero column</u> if $p_j < c < t_{j+1}$ for some j.

$c \in [1,N]$ is a <u>cut</u> if $p_j \leq c < t_{j+1}$ some j.

$(i,c) \in D(P,T)$ <u>lone star</u> if $t_i = p_i = c$

WAIT TILL NEXT PAGE

$\boxed{\text{DEF}}$ $1 \leq c \leq n$: $f_c = x_1 x_N + x_2 x_{N-1} + \dots + x_c x_{N+1-c}$

type B $\longrightarrow$ $f_{n+1} = x_1 x_N + \dots + x_n x_{n+2} + x_{n+1}^2$

$N = 2n+1$

Turns out these features of $D(P,T)$ define the projected Richardson variety completely.

Let us define these sets from the main thm.

$\boxed{\text{DEF}}$ $$\mathcal{Y}_{P,T} := \{ \overset{[1,N]}{\overset{\in}{c}} : c \text{ is a zero column or } N+1-c \text{ is a lone star} \}$$

$$\mathcal{Q}_{P,T} := \{ c \in [1,n] : c \text{ is a cut} \} \cup \underbrace{\{n+1\}}_{\text{if type is B}}$$

$$Z_{P,T} := Z\left(\{ f_c : c \in \mathcal{Q}_{P,T} \} \cup \{ x_c : c \in \mathcal{Y}_{P,T} \} \right)$$

define this here ?

Explain #'s of quadratic and linear eqn's in main thm.

$Z_{P,T}$ $\underline{\text{is}}$ $\varphi(\pi^{-1}(Y_{P,T}))$ (need to prove it...)

V. Idea of Proof that $Z_{P,T} = \varphi(\pi^{-1}(Y_{P,T}))$

$\boxed{\text{DEF}}$ $P \to T$ iff

1. $T \le P$
2. $p_i \le t_{i+1}$ $\forall i$
3. $p_i = t_{i+1} \Rightarrow N+1-p_i$ <u>not</u> a zero column

$\boxed{\text{Prop 1}}$ $\varphi(\pi^{-1}(Y_{P,T})) \subset Z_{P,T}$ in types B/C

[BKT 2009]

(and in type D [R 2013])

$\boxed{\text{Prop 2}}$ $P \to T \Rightarrow \pi^{-1}(Y_{P,T}) \overset{\text{birational}}{\underset{\varphi}{\dashrightarrow}} Z_{P,T}$ in types B/C/D

$\Rightarrow Z_{P,T} = \varphi(\pi^{-1}(Y_{P,T}))$ [BKT 2009]

$\boxed{\text{Thm}}$ $P \not\to T \Rightarrow \exists\ \tilde{P}$ s.t. $T \le \tilde{P} \le P$

[R 2013]

$$Z_{\tilde{P},T} = Z_{P,T}$$

$$\tilde{P} \to T$$

Pf: ① by explicit construction of $\tilde{P}$

② uses experimental math!

$\exists$ smaller Rich var with same Pick proj.

$\boxed{\text{COR}}$ $P \not\to T \Rightarrow \varphi(\pi^{-1}(Y_{P,T})) \supset Z_{P,T}$

$\Rightarrow Z_{P,T} = \varphi(\pi^{-1}(Y_{P,T}))$ in <u>general</u>

<u>Upshot</u>: Pieri coef. in K-theory are computable!