Equiangular lines in Euclidean spaces

Ferenc Szöllősi
(joint work with G. Greaves and A. Munemasa)
szoferi@gmail.com

Research Center for Pure and Applied Mathematics
Graduate School of Information Sciences
Tohoku University

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Prologue

- This talk is about algebraic and combinatorial matrix theory.
- We discuss some old and new results on equiangular lines.
- We consider our work as basic/fundamental research.[1]
  - We ask and answer our own questions;
  - pose open problems; and
  - challenge and occasionally correct earlier results.

[1] Of course, we state nothing about its significance.

Introduction

Let $d > 1$ be an integer, and let $\mathbb{R}^d$ denote the Euclidean space with the usual inner product $\langle ., . \rangle$.

Definition:

A set of $n > 1$ lines, represented by the unit vectors $v_1, \ldots, v_n \in \mathbb{R}^d$ is called equiangular, if there exists a constant α such that $\langle v_i, v_j \rangle = \pm\alpha$ for all $1 \leq i < j \leq n$. The constant is called the common angle.

Example:

Any orthonormal basis with $n \leq d$ and $\alpha = 0$.

Of course, we are interested in the case $n > d$ and hence $\alpha \neq 0$.

Example:

Three lines in $\mathbb{R}^2$ with common angle $1/2$.

An example: Six diagonals of the icosahedron in $\mathbb{R}^3$

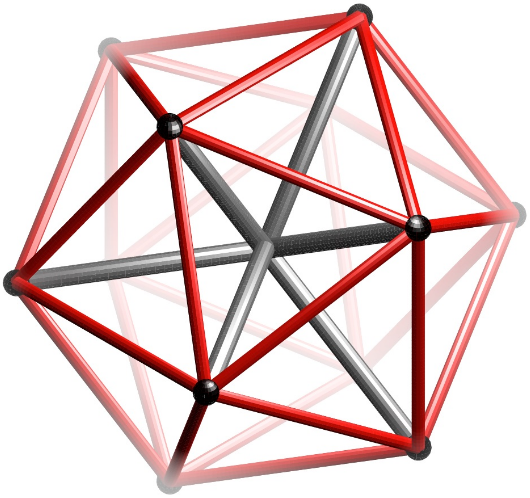


Figure : Image courtesy of Mahdad Khatirinejad

What was known in 1973?

Problem:

Determine $N(d)$, the maximum number of equiangular lines in $\mathbb{R}^d$.

This is a packing problem.

Theorem[Seidel et al., 1973]:

Lower and upper bounds for $N(d)$ for $d \leq 23$:

d	2	3	4	5	6	7	8-13	14	15	16
$N(d)$	3	6	6	10	16	28	28	28-30	36	40-42
$1/\alpha$	2	$\sqrt{5}$	$\sqrt{5}, 3$	3	3	3	3	$3(?), 5$	5	5

d	17	18	19	20	21	22	23
$N(d)$	48-51	48-61	72-76	90-96	126	176	276
$1/\alpha$	5	5	5	5	5	5	5

There are two distinct constructions in $\mathbb{R}^{14}$. All cases $d \geq 24$ are open.

What was claimed in 1995?

Problem:

Determine $N(d)$, the maximum number of equiangular lines in $\mathbb{R}^d$.

This is a packing problem.

Claim[Seidel, 1995]:

Lower and upper bounds for $N(d)$ for $d \leq 23$:

d	2	3	4	5	6	7	8-13	14	15	16
$N(d)$	3	6	6	10	16	28	28	28	36	40
$1/\alpha$	2	$\sqrt{5}$	$\sqrt{5}, 3$	3	3	3	3	3, 5	5	5

d	17	18	19	20	21	22	23
$N(d)$	48	48	72-76	90-96	126	176	276
$1/\alpha$	5	5	5	5	5	5	5

There are only two open cases for $d \leq 43$. All cases $d \geq 44$ are open.
Reference: Seidel et al., 1973. **?!**

What is the state-of-the-art today?

Consensus by the experts (A. Brouwer and D. Taylor) is as follows:

Seidel is likely to be mistaken.

Therefore the problem of equiangular lines is wide open.

Our contribution:

- Sorting out the contradicting literature; moreover we present
- a new general lower bound on the number of equiangular lines;
- the full classification of $n \leq 12$ equiangular lines; and
- analysis of line systems with “high degree of symmetry”.

Upper bounds on $N(d)$

Theorem[Gerzon, 1973]:

$$N(d) \leq \frac{1}{2}d^2 + \frac{1}{2}d,$$

if equality holds then $d = 2,3$, or $d + 2$ is the square of an odd integer.

Equality indeed holds for $d = 2,3,7$, and 23.

- Makhnev proved that $N(47) < 24 \times 47$;
- Bannai, Munemasa and Venkov exluded infinitely many cases;
- Barg et al. recently improved on this bound for $d \leq 42$.

These are three fundamentally different approaches.

Lower bounds on $N(d)$

Lemma:

For all $d \geq 1$ we have
$$N(d) > cd\sqrt{d}$$
for some $c > 0$.

- It was conjectured that this is the correct order of magnitude; until
- de Caen came up with a surprising quadratic construction in specific dimensions.

Theorem[de Caen, 2000]:

For all $d \geq 1$ we have
$$N(d) \geq \frac{1}{72}(d+1)^2. \qquad (1/72 \approx 0.013)$$

New results

Theorem[G–M–Sz, 2013+]:

Let $d \geq 25$ and let $m = 4^i$, $i \geq 2$ be the unique integer number for which $3m/2 + 1 \leq d \leq 6m$. Then

$$N(d) \geq \begin{cases} m(m/2+1) & \text{for} \quad 3m/2+1 \leq d \leq 33m/8 - 1; \\ 4m(d-4m+1) & \phantom{\text{for}} \quad 33m/8 \leq d \leq 6m-1; \\ 4m(2m+1)-1 & \phantom{\text{for}} \quad d = 6m. \end{cases}$$

Corollary:

For every $d \geq 2$ we have

$$N(d) \geq \left\lceil \frac{1}{1089}\left(32d^2 + 328d + 296\right) \right\rceil. \qquad (32/1089 \approx 0.029)$$

Sketch of the proof

The idea is to transform a collection of pairwise mutually unbiased bases into equiangular line systems.

Definition:

Two orthonormal bases of $\mathbb{R}^m$, represented by the orthogonal matrices U and V, are called unbiased, if all entries of UV^T are $\pm 1/\sqrt{m}$.

Example:

Let $U = I_m$ and $V = H/\sqrt{m}$ for any real Hadamard matrix of order m.

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Sketch of the proof (ctd.)

Theorem[Kantor et al., 1996]:

For $m = 4^i$, $i \geq 1$ there exist $m/2 + 1$ pairwise mutually unbiased bases.

- Therefore we have $m(m/2 + 1)$ lines (the column vectors of the bases) with three angles: $\{0, \pm 1/\sqrt{m}\}$.
- The goal is to transform this line system into equiangular lines.
- We can do this by embedding the lines into a $d = m + m/2 + 1$-dimensional space.
- The common angle will be $\alpha = 1/(\sqrt{m} + 1)$.

Illustration

Let $m = 4$. We construct:
$n = m(m/2+1) = 12$ vectors in dimension $d = m + m/2 + 1 = 7$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

Illustration

Let $m = 4$. We construct:
$n = m(m/2 + 1) = 12$ vectors in dimension $d = m + m/2 + 1 = 7$.

$$\left[\begin{array}{cccc|cccc|cccc} 2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{array}\right]$$

Illustration

Let $m = 4$. We construct:
$n = m(m/2+1) = 12$ vectors in dimension $d = m + m/2 + 1 = 7$.

$$\frac{1}{\sqrt{6}}\left[\begin{array}{cccc|cccc|cccc} 2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ \hline \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \end{array}\right]$$

Illustration

Let $m = 4$. We construct:
$n = m(m/2 + 1) = 12$ vectors in dimension $d = m + m/2 + 1 = 7$.

$$\frac{1}{\sqrt{6}}\left[\begin{array}{cccc|cccc|cccc} 2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ \hline \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \end{array}\right]$$

- Observe that we do not need to use all $m/2 + 1$ bases.
- In particular, we have 8 lines in $\mathbb{R}^6$ automatically.
- Even better: 11 lines in $\mathbb{R}^6$ and 8 lines in $\mathbb{R}^5$!

Seidel matrices

- Let $[G]_{i,j} = \langle v_i, v_j \rangle$ be the Gram matrix of the equiangular line system; and
- consider instead the Seidel matrix $S := (G - I)/\alpha$.
- S is a symmetric ± 1 matrix of order n with zero diagonal.
- The smallest eigenvalue of S, $\lambda_{\min} = -1/\alpha$ of multiplicity $n - d$.

The Seidel matrix carries complete information about the equiangular line system.

Study via linear algebra and spectral graph theory is possible.

Seidel matrices are the central objects of our work.

Equivalence of Seidel matrices

Definition:

Two Seidel matrices, S_1 and S_2 are switching-equivalent, if there exist permutation matrices P and diagonal ± 1 matrices D such that $S_2 = PDS_1DP^T$. The switching-class of a Seidel matrix S is the set of all Seidel matrices equivalent to it.

This equivalence corresponds to permutation of the vectors $v_1, \ldots, v_n$ and replacing some of them by their negative.

Example[Switching w.r.t. the 4th vector]:

$$S_1 = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \rightsquigarrow S_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix}.$$

Determining the Seidel matrices up to equivalence

Problem:

Determine the Seidel matrices up to switching-equivalence.

Solved by:

- van Lint and Seidel for $n \leq 7$ in 1966;
- Bussemaker, Mathon, and Seidel for $n = 8, 9$ in 1981;
- Spence for $n = 10$ in early 1990s;
- McKay for $n = 11$ in late 1990s;
- G–M–Sz for $n = 12$ in 2013+.

The number of classes (#) and self-complementary classes (s):

d	1	2	3	4	5	6	7	8	9	10	11	12
#	1	1	2	3	7	16	54	243	2.038	33.120	1.182.004	87.723.296
s	1	1	0	1	1	4	0	19	10	320	0	25.112

The number of distinct switching classes

The number of switching classes are predicted by the following spectacular equicardinality result. Recall that an Euler graph is a (not necessarily connected) graph whose vertices are of even degree.

Theorem[Mallows–Sloane 1975]:

The number of switching classes of Seidel matrices of size n and the number of Euler graphs on n vertices are equal in number.

There is an explicit formula for computing the latter due to Robinson and Liskovec (1969–1970).

Theorem[Seidel]:

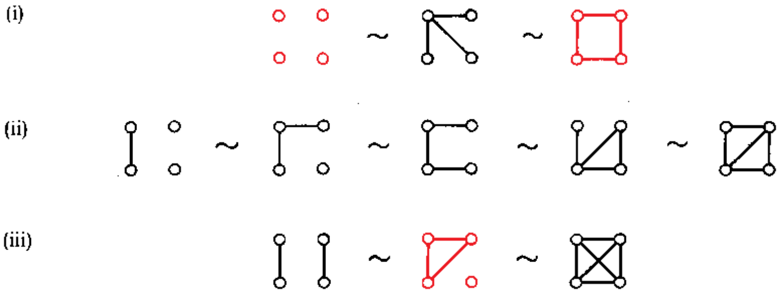
For n odd each switching class contains a unique Euler graph. (Where the adjacency matrix is $A = (J - S - I)/2$).

The observation above does not hold for n even.

Example: 4 lines forming 3 classes

$$A = (J - S - I)/2$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix}.$$



Class (ii) does not contain any Euler graphs!

Invariants

Determining whether two Seidel matrices are equivalent is a non-trivial, time-consuming process. However, we have various invariants for differentiating in certain cases. This is a kind of *canonical form*, and a one-way criterion in general.

- The determinant;
- the spectrum;
- the induced subgraph profile;

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix}.$$

$\det = -3$ $\det = 5$ $\det = -3$

$\Lambda = \{[-1]^3, [3]^1\}$ $\Lambda = \{[-3]^1, [1]^3\}.$

$$\Lambda = \{[-\sqrt{5}]^1, [-1]^1, [1]^1, [\sqrt{5}]^1\}$$

How to reach a classification of order 12?

- The Mallows–Sloane theorem predicts that there are exactly 87.723.296 switching classes; then
- one generates at least this number of matrices; then
- one *carefully* chooses an invariant and evaluate it on the candidate matrices; then
- if the range of the invariant is of cardinality 87.723.296 then we are done!

Let $\varphi_{11}(.)$ be any complete invariant of 11×11 Seidel matrices.

Theorem[G–M–Sz, 2013+]:

The Seidel matrices of order 12, S_1 and S_2 are equivalent if and only if $\varphi_{12}(S_1) = \varphi_{12}(S_2)$, where $\varphi_{12}(.)$ is the following multiset:

$$\varphi_{12}(S) = \{\varphi_{11}(S') \colon S' \text{ is a } 11 \times 11 \text{ principal submatrix of } S\}.$$

The tight frame bound

Theorem[Seidel et al., 1966]:

Assume that there exist n equiangular lines in $\mathbb{R}^d$ with common angle $\alpha \leq 1/\sqrt{d+2}$. Then

$$n \leq \frac{d(1-\alpha^2)}{1-d\alpha^2},$$

and equality holds if and only if the corresponding Seidel matrix has exactly two distinct eigenvalues.

- The r.h.s.$\to n$ as $\alpha \to 0$; only odd numbers $1/\alpha$ are possible.
- The philosophy during the 1970-1980s was that “two eigenvalues” is the answer to the equiangular line problem.

Example:

In $\mathbb{R}^{95}$ we have $n \leq 438$ for $\alpha \leq 1/11$. However, for $\alpha = 1/9$, for which the previous result is vacuous, we can construct 2048 lines. On the other hand, $n = 188$ is the maximum number for $\alpha = 1/3$.

Using too small or too large angles are both unsatisfactory in general.

A case study

Problem:

Determine $N(14)$.

- For $\alpha = 1/3$, we have $n \leq 28$ in $\mathbb{R}^{14}$. Equality is possible.
- There is another construction of $n = 28$ lines with $\alpha = 1/5$.
- The relative bound gives us $n \leq 336/11$, that is $n \leq 30$.
- Since the upper bound is not an integer, there are no Seidel matrices with two eigenvalues.

What is next best thing after Seidel matrices with two eigenvalues?

- We don't know.
- Nevertheless we study Seidel matrices with three eigenvalues.

Motivating examples

Example[28 lines in $\mathbb{R}^{14}$ with angle $1/5$ by J. C. Tremain]:

By removing a $J_8 - I_8$ submatrix from a H_{36} real Hadamard matrix.

$$\Lambda = \left\{[-5]^{14}, [3]^{7}, [7]^{7}\right\}.$$

Example[40 lines in $\mathbb{R}^{16}$ with angle $1/5$]:

From a "rank-3" graph on 40 vertices $\left(\Lambda(G) = \{[12]^1, [2]^{24}, [-4]^{15}\}\right)$.

$$\Lambda = \left\{[-5]^{24}, [7]^{15}, [15]^{1}\right\}.$$

Example[48 lines in $\mathbb{R}^{17}$ with angle $1/5$]:

From the residual design of a $\mathrm{STS}(19)$.

$$\Lambda = \left\{[-5]^{31}, [7]^{8}, [11]^{9}\right\}.$$

Seidel matrices with three eigenvalues

Theorem[G–M–Sz, 2013+]:

Let S be a Seidel matrix of order $n \geq 2$ with smallest eigenvalue $\lambda_0 < 0$ of multiplicity $n - d \geq 1$. Let $\beta \neq \lambda_0$ another eigenvalue of S with multiplicity $1 \leq m \leq d$. Then

$$\left|\beta + \frac{\lambda_0(n-d)}{d}\right| \leq \frac{\sqrt{n(d(n-1) - \lambda_0^2(n-d))}}{d} \cdot \sqrt{\frac{d-m}{m}}.$$

Equality if and only if S has at most 3 distinct eigenvalues.

Proof: Spectral analysis of S and S^2, and Cauchy–Schwartz.

$$Tr(S) = (n-d)\lambda_0 + m\beta + \sum_{i=1}^{d-m} \lambda_i = 0$$

$$Tr(S^2) = (n-d)\lambda_0^2 + m\beta^2 + \sum_{i=1}^{d-m} \lambda_i^2 = n(n-1)$$

List of feasible parameter sets

n	d	λ	μ	ν	Existence
28	14	$[-5]^{14}$	$[3]^{7}$	$[7]^{7}$	Yes
30	14	$[-5]^{16}$	$[5]^{9}$	$[7]^{5}$	?
40	16	$[-5]^{24}$	$[5]^{6}$	$[9]^{10}$	?
40	16	$[-5]^{24}$	$[7]^{15}$	$[15]^{1}$	Yes
42	16	$[-5]^{26}$	$[7]^{7}$	$[9]^{9}$	?
48	17	$[-5]^{31}$	$[7]^{8}$	$[11]^{9}$	Yes
49	17	$[-5]^{32}$	$[9]^{16}$	$[16]^{1}$	?
48	18	$[-5]^{30}$	$[3]^{6}$	$[11]^{12}$	?
48	18	$[-5]^{30}$	$[7]^{16}$	$[19]^{2}$	?
54	18	$[-5]^{36}$	$[7]^{9}$	$[13]^{9}$	?
60	18	$[-5]^{42}$	$[11]^{15}$	$[15]^{3}$	?
72	19	$[-5]^{53}$	$[13]^{16}$	$[19]^{3}$	Yes
75	19	$[-5]^{56}$	$[10]^{1}$	$[15]^{18}$	?
90	20	$[-5]^{70}$	$[13]^{5}$	$[19]^{15}$	?
95	20	$[-5]^{75}$	$[14]^{1}$	$[19]^{19}$	?

Existence of Seidel matrices

Existence is completely determined in the following sense.

Theorem[G–M–Sz, 2013+]:

There exists a Seidel matrix with exactly three distinct eigenvalues if and only if $n \neq 4$ or $n \neq p \equiv 3 \pmod 4$ for some prime number p.

Lemma:

Let $n \geq 6$ be a composite number. There exists a Seidel matrix with exactly three distinct eigenvalues of order n.

Proof: Write $n = ab$ with $a \geq 3$ and $b \geq 2$. Then $J_b \otimes (J_a - 2I_a) + I_{ab}$ has spectrum $\Lambda = \{[1-2b]^{a-1}, [1]^{a(b-1)}, [(a-2)b+1]^1\}$.

Example[Paley matrices]:

The Paley matrices of prime order $p \equiv 1 \pmod 4$ has spectrum $\Lambda = \{[-\sqrt{p}]^{(p-1)/2}, [0]^1, [\sqrt{p}]^{(p-1)/2}\}$.

Nonexistence for $n = 4$ is clear, otherwise technical.

A construction from regular two-graphs

We start with n equiangular lines in $\mathbb{R}^d$...

Theorem[G–M–Sz, 2013+]:

Let S be a Seidel matrix of order $n \geq 2$ with spectrum $\{[\lambda_0]^{n-d}, [\lambda_1]^d\}$ with $\lambda_1 \leq \{d-1, n-d-1\}$ and assume that it admits the block partition

$$S = \left[\begin{array}{cc} J_{\lambda_1+1} - I_{\lambda_1+1} & * \\ * & S' \end{array}\right].$$

Then the spectrum of S' is $\Lambda = \{[\lambda_0]^{n-d-\lambda_1}, [\lambda_1]^{d-\lambda_1-1}, [\lambda_0+\lambda_1+1]^{\lambda_1}\}$.

...and end up with $n - \lambda_1 - 1$ equiangular lines in $\mathbb{R}^{d-1}$.

- MANY examples arise this way. But...
- the bounds on λ_1 are somewhat restrictive; and
- no control over the existence of $J_c - I_c$ submatrices: for Paley matrices even the order of magnitude of c is unknown!

Let's do an experiment ("baby" example)

- Begin with a symmetric Hadamard matrix H with constant diagonal -1 of order 36;
- since $HH^T = 36I_{36}$ observe that $\Lambda(H) = \{[-6]^{21}, [6]^{15}\}$;
- remove the main diagonal and consider the Seidel matrix $S = H + I$ with spectrum $\Lambda(S) = \{[-5]^{21}, [7]^{15}\}$.
- This is our regular two-graph. It describes 36 lines in $\mathbb{R}^{15}$.
- Let's remove a $J_c - I_c$ submatrix from H for $c = 1, 2, \ldots$

$$\begin{aligned}
\Lambda(S \setminus K_1) &= \{[-5]^{20}, [2]^1, [7]^{14}\} \\
\Lambda(S \setminus K_2) &= \{[-5]^{19}, [1]^1, [3]^1, [7]^{13}\} \\
\Lambda(S \setminus K_3) &= \{[-5]^{18}, [0]^1, [3]^2, [7]^{12}\} \\
\Lambda(S \setminus K_4) &= \{[-5]^{17}, [-1]^1, [3]^3, [7]^{11}\} \\
\Lambda(S \setminus K_5) &= \{[-5]^{16}, [-2]^1, [3]^4, [7]^{10}\} \\
\Lambda(S \setminus K_6) &= \{[-5]^{15}, [-3]^1, [3]^5, [7]^9\} \\
\Lambda(S \setminus K_7) &= \{[-5]^{14}, [-4]^1, [3]^6, [7]^8\} \\
\Lambda(S \setminus K_8) &= \{[{\color{red}-5}]^{13}, [{\color{red}-5}]^1, [3]^7, [7]^7\}
\end{aligned}$$

We end up with 28 lines in $\mathbb{R}^{14}$. Two things can go wrong...

Interlacing

A fundamental tool in spectral graph theory is interlacing.

Theorem:

Let

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

be a symmetric matrix of order n with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ and A of size m with eigenvalues $\mu_1, \mu_2, \ldots, \mu_m$. Then, we have $\lambda_i \leq \mu_i \leq \lambda_{n-m+i}$ for $1 \leq i \leq m$.

In other words, the eigenvalues of the principal submatrices are 'trapped' in-between the initial eigenvalues.

Some exercises

Lemma:

Let S be a Seidel matrix of order n with spectrum $\{[\lambda_0]^{n-d}, [\lambda_1]^d\}$ and let S' be its principal $(n-1) \times (n-1)$ submatrix. Then $\Lambda(S') = \{[\lambda_0]^{n-d-1}, [\lambda_0 + \lambda_1]^1, [\lambda_1]^{d-1}\}$.

Proof: By interlacing $\Lambda(S') = \{[\lambda_0]^{n-d-1}, [x]^1, [\lambda_1]^{d-1}\}$ for some unknown real number x. Since $\mathrm{Tr}(S') = 0$, we find that $x = \lambda_0 + \lambda_1$, as claimed.

Lemma:

Let S be a Seidel matrix of order n with spectrum $\{[\lambda_0]^{n-d}, [\lambda_1]^d\}$ and let S' be its principal $(n-2) \times (n-2)$ submatrix. Then $\Lambda(S') = \{[\lambda_0]^{n-d-2}, [\lambda_0 + \lambda_1 - 1]^1, [\lambda_0 + \lambda_1 + 1]^1, [\lambda_1]^{d-2}\}$.

Proof: By interlacing $\Lambda(S') = \{[\lambda_0]^{n-d-2}, [x]^1, [y]^1, [\lambda_1]^{d-2}\}$ for some unknown real numbers x and y. Since $\mathrm{Tr}(S') = 0$ and $\mathrm{Tr}((S')^2) = (n-2)(n-3)$, we find that $x, y = \lambda_0 + \lambda_1 \mp 1$, as claimed.

A technical proposition

Proposition[G–M–Sz, 2013+]:

Let S be a Seidel matrix of order $n \geq 2$ with spectrum $\{[\lambda_0]^{n-d}, [\lambda_1]^d\}$ and assume that for some $1 \leq c \leq d$ it admits the block partition

$$S = \begin{bmatrix} J_c - I_c & * \\ * & S' \end{bmatrix}. \qquad \text{Then}$$

$$\mathrm{Tr}((S')^3) = (n-3c)(n-1)(\lambda_0 + \lambda_1) + c(c-1)(3\lambda_0 + 3\lambda_1 - c + 2).$$

Proof: Follows for $c = 1, 2$ from previous Lemmata. For $c \leq 3$ one should understand the distribution of 3×3 minors and apply the following identity:

$$\mathrm{Tr}(S^3) = 3 \sum_M \det M,$$

where M runs through all 3×3 minors of S. The right hand side contains $\mathrm{Tr}((S')^3)$ as a subsum. We omit the details.

Proof of the $J_{\lambda_1+1} - I_{\lambda_1+1}$ removal construction

Theorem[G–M–Sz, 2013+]:

Let S be a Seidel matrix of order $n \geq 2$ with spectrum $\{[\lambda_0]^{n-d}, [\lambda_1]^d\}$ and assume that for some $1 \leq c \leq d$ it admits the block partition

$$S = \begin{bmatrix} J_c - I_c & * \\ * & S' \end{bmatrix}.$$

Then $\Lambda(S') = \{[\lambda_0]^{n-d-c}, [\lambda_1]^{d-c}, [\lambda_0 + \lambda_1 + 1 - c]^1, [\lambda_0 + \lambda_1 + 1]^{c-1}\}$.

Proof: Induction on c. Cases $c = 1, 2$ are trivial, so assume $c \geq 3$. By removing the first $c - 1$ rows and columns of S we find the spectrum $\{[\lambda_0]^{n-d-c+1}, [\lambda_1]^{d-c+1}, [\lambda_0 + \lambda_1 + 2 - c]^1, [\lambda_0 + \lambda_1 + 1]^{c-2}\}$. Finally, we remove the cth column and use interlacing to find the spectrum of S': $\{[\lambda_0]^{n-d-c}, [\lambda_1]^{d-c}, [\lambda_0 + \lambda_1 + 1]^{c-3}, [x]^1, [y]^1, [z]^1\}$. Since we know $\mathrm{Tr}(S')$, $\mathrm{Tr}((S')^2)$, and $\mathrm{Tr}((S')^3)$, we have three equations. Done.

- The Theorem follows by setting $c = \lambda_1 + 1$ above.

The complex world

Complex equiangular: $|\langle v_i, v_j\rangle| = \alpha$.

Conjecture:

There exists d^2 equiangular lines in $\mathbb{C}^d$ for every d.

- Examples up to $d \leq 16$; sporadic ones in higher dimensions.
- Non-trivial (yet, so far fruitless...) progress in square dimensions.
- Results of computer algebra (Gröbner bases).
- Only nonexistence results came out from theory.
- Actual applications in quantum tomography (SIC-POVMs)

Example[Belovs]:

$$\Phi = \left[\begin{array}{cccc|cccc|cccc|cccc} x & x & x & x & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -\mathbf{i} \\ 1 & 1 & -1 & -1 & x & x & x & x & \mathbf{i} & -\mathbf{i} & \mathbf{i} & -\mathbf{i} & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & x & x & x & x & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} \\ 1 & -1 & -1 & 1 & -\mathbf{i} & \mathbf{i} & \mathbf{i} & -\mathbf{i} & -1 & 1 & 1 & -1 & x & x & x & x \end{array}\right],$$

where $(x^2-1)^2 = |x+1+\mathbf{i}(x-1)|^2$, in particular $x = \sqrt{2+\sqrt{5}}$.

How to construct d^2 complex equiangular lines?

- Let G be a group of order M^2 and $\psi \in \mathbb{C}^d$ a so-called *fiducial* vector.
- Choose ψ so that $\{A\psi\colon A \in G\}$ spans a set of equiangular lines.

Example[Hoggar]:

Let

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Y = \mathbf{i}XZ = \begin{bmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}$$

and let $G = \{I, X, Y, Z\}^{\otimes 3}$, $\psi = \frac{\sqrt{3}}{12}\left[0, 0, \tau, \overline{\tau}, \tau, -\tau, 0, \sqrt{2}\right]^T$ where $\tau = (1+\mathbf{i})/\sqrt{2}$. Then $\Phi = \{A\psi\colon A \in G\}$ is a set of 64 lines.

Problem:

How to choose G and ψ?

Grassl attacked this problem with Gröbner basis techniques.

The Hungarian alphabet (44 “letters”)

A	Á	B	C	CS	D	DZ	DZS			
E	É	F	G	GY	H					
I	Í	J	K	L	LY	M	N	NY		
O	Ó	Ö	Ő	P	(Q)	R	S	SZ	T	TY
U	Ú	Ü	Ű	V	(W)	(X)	(Y)	Z	ZS	

Köszönöm a figyelmet!

Thank you for your attention.