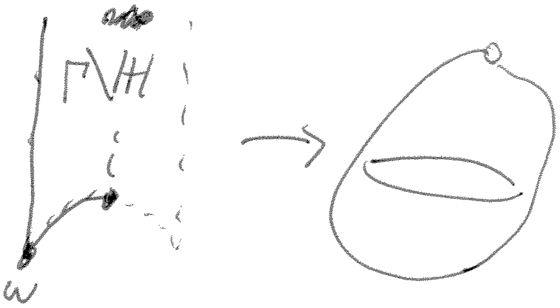
① Group Actions

$\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, $\Gamma = SL_2(\mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, ad-bc=1\}$

$\Gamma \times \mathbb{H} \to \mathbb{H}$, $(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau) \mapsto \gamma\tau = \dfrac{a\tau + b}{c\tau + d}$

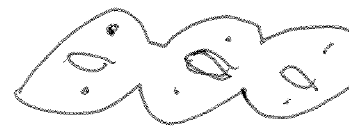
$\Gamma = \langle S, T \mid S^2 = (ST)^3, S^4 = \mathbb{1} \rangle$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$T\tau = \tau + 1$, $S\tau = -\dfrac{1}{\tau} = -\dfrac{\bar{\tau}}{|\tau|^2}$ →



$\gamma\infty = \dfrac{a}{c}$ is trans., $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ → compact.

$G \leq \Gamma$, $[\Gamma : G] < \infty \Rightarrow G\backslash\mathbb{H}^* =$



$\mathcal{F} = \{\text{mer. } f : \mathbb{H} \to \mathbb{C}\}$, $k \in \mathbb{Z}$,

$|_k : \mathcal{F} \times \Gamma \to \mathcal{F}$, $(f, \gamma) \mapsto f|_k\gamma$, where

$f|_k\gamma(\tau) = (c\tau + d)^{-k} f\left(\dfrac{a\tau + b}{c\tau + d}\right)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\tau \in \mathbb{H}$.

② Modular forms

f is modular wt k for $G \leq \Gamma$ if

① $f|_k\gamma = f$ $\forall \gamma \in G$

② Fourier expansion $f(\tau) = \sum_{n \in \mathbb{Z}} a(n) q^{\frac{n}{N}}$, $q = e^{2\pi i \tau}$

is left-finite, where $T^N \in G$.

$\underline{k=0}$: Modular functions = function field of $G\backslash\mathbb{H}^*$
$(G=\Gamma)$ $= \mathbb{C}(J)$, $J = \frac{1}{q} + 196884q + \ldots$

$\underline{k=2}$: $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \gamma'(\tau) = \frac{1}{(c\tau+d)^2}$, so $f|_2\gamma = f \;\forall G$

$\Leftrightarrow$ $f(\gamma\tau)\,d(\gamma\tau) = f(\tau)\,d\tau \quad \forall \gamma$

$\Leftrightarrow$ $f d\tau$ is a $\underline{\text{mer. diff.}}$ on $G\backslash\mathbb{H}^*$.

etc. $\underline{\text{for } G=\Gamma}$:

$\underline{\text{Note}}$: $S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, so $f|_k S^2(\tau) = (-1)^{-k} f(\tau) = f(\tau) \Leftrightarrow f=0$ or $2|k$.

$\underline{\text{Holomorphic}}$ modular forms are $\mathcal{M} = \mathbb{C}[E_4, E_6]$
$= \bigoplus_{k\geq 0} \mathcal{M}_{2k}$

$E_{2k} = 1 -$

$\mathcal{M}_0 = \mathbb{C}$, $\mathcal{M}_2 = \{0\}$, $\mathcal{M}_4 = \mathbb{C}E_4$, $\mathcal{M}_6 = \mathbb{C}E_6$, $\mathcal{M}_8 = \mathbb{C}E_4^2$, etc.

$$J = \frac{E_4^3}{E_4^3 - E_6^2} = \frac{E_4^3}{\Delta}, \text{ where } \Delta = q\prod_{n\geq 1}(1-q^n)^{24} \in \mathcal{M}_{12}$$

has no zero in $\mathbb{H}$.

$\underline{\text{weakly holomorphic}}$ $= \mathcal{M}\left[\frac{1}{\Delta}\right]$ = hol. in $\mathbb{H}$ w/ pole @ ∞.

③ VVMFs

Let $\rho : \Gamma \to GL_d(\mathbb{C})$, $\rho(T) = \rho\begin{pmatrix}1&1\\0&1\end{pmatrix} = \text{diag}\{e(r_1), \dots, e(r_d)\}$
$r_j \in \mathbb{R}$.

$F = \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} \in \mathcal{F}^d$ is a <u>VVMF</u> for ρ of wt k

if ① $F|_k \gamma = \begin{pmatrix} f_1|_k\gamma \\ \vdots \\ f_d|_k\gamma \end{pmatrix} = \rho(\gamma) F \quad \forall \gamma \in \Gamma$.

② $f_j = \sum_{n \geq m_j} a_j(n) q^{n+r_j}$ for $1 \leq j \leq d$.

<u>Holomorphic</u> if $f_j : \mathbb{H} \to \mathbb{C}$ is hol. and $m_j \geq 0 \ \forall j$.

Assume $\rho(S^2) = \pm 1$.

Holomorphic VVMFs for ρ are $\mathcal{H}(\rho) = \bigoplus_{k \geq 0} \mathcal{H}(k_0 + 2k, \rho)$,

where $k_0 \in \mathbb{Z}$ satisfies $k_0 \geq \frac{12}{d}(r_1 + \dots + r_d) + 1 - d$.

<u>Thm</u>: (Marks–Mason) $\mathcal{H}(\rho)$ is a <u>free</u> $\mathcal{M}$-module of rank d.

Det. wts. of generators $F_1, \dots, F_d$ is nontrivial!

④ The modular derivative

Let $\eta = \sqrt[24]{\Delta} = q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n)$.

Then $E_2 = \frac{\eta'}{\eta} = 1 - 24 \sum_{n\geq 1} \sigma_1(n) q^n$ is

quasi-modular of weight two, i.e.

$$E_2|_2\gamma(\tau) = E_2(\tau) + \frac{6c}{\pi i}(c\tau+d)^{-1}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

this "defect" cancels the defect of $\theta = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$,

so that $D_k = \theta - \frac{k}{12}E_2$ takes M_k to M_{k+2}. This derivative is covariant w.r.t. $|_k$, i.e. $\forall f \in \mathcal{F}$, $(D_k f)|_{k+2}\gamma = D_k(f|_k\gamma)$. Every vector-valued modular form satisfies a modular differential equation $D_k^n f + M_2 D_k^{n-1} f + \dots + M_{2n} f = 0$, where M_j is modular of weight j. $\rho(T)$ is then the monodromy of the solutions around $q=0$.

The Bounded Denominator Conjecture for Vector-Valued Modular Forms

Christopher Marks

University of Alberta

November 21, 2013

Representations of finite level

Let $\rho : \Gamma \to GL_d(\mathbb{C})$ be a d-dimensional representation of the modular group $\Gamma = SL_2(\mathbb{Z})$, such that

$$\rho(T) = \rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \operatorname{diag}\left\{ e^{2\pi i \frac{A_1}{N}}, \cdots, e^{2\pi i \frac{A_d}{N}} \right\}$$

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Then $\ker \rho$ contains the normal closure of T^N in Γ,

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is the *principal congruence subgroup* of level N, and $\ker \rho$ is a congruence subgroup of level N (thus of finite index in Γ).
If $N \geq 6$, then $[\Gamma(N) : \Delta(N)] = \infty$, and ρ may have finite or infinite image.

VVMFs with bounded denominators

We denote by $\mathcal{H}_{\mathbb{Q}}(\rho)$ the space of holomorphic vector-valued modular forms

$$F = \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} = \begin{pmatrix} \sum_{n\geq 0} a_1(n) q^{n+\frac{A_1}{N}} \\ \vdots \\ \sum_{n\geq 0} a_d(n) q^{n+\frac{A_d}{N}} \end{pmatrix}$$

for ρ such that $a_j(n) \in \mathbb{Q}$ for all $1 \leq j \leq d$, $n \geq 0$. (Here $q = q(\tau) = e^{2\pi i \tau}$ for $\tau \in \mathbb{H}$, the complex upper half-plane).

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Corollary: If $N \leq 5$, then all $F \in \mathcal{H}_{\mathbb{Q}}(\rho)$ have bounded denominators.

The bounded denominator conjecture for VVMFs

On the other hand, if $\ker\rho$ is *noncongruence* (i.e. $\ker\rho \not\supseteq \Gamma(M)$ for any $M \geq 1$), then it is expected that *no* nonzero vector in $\mathcal{H}_{\mathbb{Q}}(\rho)$ has bounded denominators.

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One expects there to be at least one prime $p | [\Gamma : \ker\rho]$ such that p appears to an arbitrarily high power in the denominators of the $a_j(n)$, for at least one fixed j.

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Classic example: Fricke pointed out in an 1886 article that

$$u(\tau) = \int_{i\infty}^{\tau} \eta^4(z)\, dz = \sum_{n\geq 0} \frac{\Psi(n)}{6n+1} q^{n+\frac{1}{6}}$$

is modular for $\Delta(6)$, and $\Psi(n) = \sum \left(\frac{b}{3}\right) b$ sums over all $b \geq 1$ such that $n = 3a^3 + b^2$ for some $a \in \mathbb{Z}$.

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Here $\eta = q^{\frac{1}{24}} \prod_{n \geq 1}(1 - q^n)$ denotes Dedekind's eta function, a modular form of weight $\frac{1}{2}$ for $SL_2(\mathbb{Z})$.

Motivating example: rational vertex operator algebras

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A VOA is an infinite-dimensional complex vector space V, such that each $v \in V$ acts on V as a family of endomorphisms.

In particular, the *conformal vector* $\omega \in V$ defines a family of endomorphisms $\{L_n \mid n \in \mathbb{Z}\}$ that is isomorphic to the Virasoro Lie Algebra, i.e.

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}c,$$

where $c \in \mathbb{C}$ is the *central charge* associated to V.

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- Each such module has an $\mathbb{N}$-grading $M^{(j)} = \bigoplus_{n\geq 0} M_n^{(j)}$, with $\dim M_n^{(j)} < \infty$ for each j, n, such that the Virasoro element L_0 acts on the summand $M_n^{(j)}$ as the scalar $n + h_j$, where $h_j \in \mathbb{Q}$ is the *conformal weight* of the module $M^{(j)}$.

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Zhu proved in the 1990s that if one now makes a formal vector

$$F = \begin{pmatrix} tr|_{M_1} q^{L_0 - \frac{c}{24}} \\ \vdots \\ tr|_{M_d} q^{L_0 - \frac{c}{24}} \end{pmatrix} = \begin{pmatrix} q^{h_1 - \frac{c}{24}} \sum_{n\geq 0} \dim M_n^{(1)} q^n \\ \vdots \\ q^{h_d - \frac{c}{24}} \sum_{n\geq 0} \dim M_n^{(d)} q^n \end{pmatrix},$$

then interpreting q as $e^{2\pi i\tau}$ makes F a vector-valued modular form for $SL_2(\mathbb{Z})$, which evidently has nonnegative integral q-coefficients.

The congruence property for rational VOAs

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We note, however, that such a proof is far from imminent, and will (most likely) require deep results from algebraic geometry!

Pursuing a proof

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This may be hopeless as a viable strategy, but has the advantage of being accessible in low dimension!

Dimension one

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In fact, Γ' is congruence of level 12 and $\Gamma/\Gamma' \cong \mathbb{Z}/12\mathbb{Z}$, so $\rho = \chi^N$ for some integer N, where

$$\chi(T) = \chi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = e^{\frac{2\pi i}{12}}, \quad \chi(S) = \chi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = e^{-\frac{2\pi i}{4}}.$$

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In particular, any $F = f_1 \in \mathcal{H}_{\mathbb{Q}}(\rho)$ (i.e. scalar modular form with character $\rho = \chi^N$) has bounded denominators.
In fact, this follows immediately from observing that

$$\mathcal{H}_{\mathbb{Q}}(\chi^N) = \mathcal{M}_{\mathbb{Q}}\eta^{2N} = \mathbb{Q}[E_4, E_6]\eta^{2N}$$

is a free module of rank 1 over the polynomial ring $\mathbb{Q}[E_4, E_6]$ of modular forms for Γ with rational q-expansions, together with the fact that η, E_4, E_6 have integral expansions.

Dimension two

Suppose $\rho : \Gamma \to GL_2(\mathbb{C})$ is irreducible with

$$\rho(T) = \begin{pmatrix} e^{2\pi i \frac{A}{N}} & 0 \\ 0 & e^{2\pi i \frac{B}{N}} \end{pmatrix}, \ N \geq 1, \ 0 \leq A, B \leq N{-}1, \ (A, B, N) = 1.$$

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Mason proved that $\mathcal{H}_{\mathbb{Q}}(\rho)$ is a free module of rank two over $\mathbb{Q}[E_4, E_6]$ with generators $F, D_k F$, where F has weight $k = \frac{6(A+B)}{N} - 1 \in \mathbb{Z}$ and $D_k = q\frac{d}{dq} - \frac{k}{12}E_2$ denotes the modular derivative in weight k.

The modular differential equation

The minimal weight vector

$$F = \begin{pmatrix} q^{\frac{A}{N}} + \sum_{n \geq 0} a(n) q^{n + \frac{A}{N}} \\ q^{\frac{b}{N}} + \sum_{n \geq 0} b(n) q^{n + \frac{b}{N}} \end{pmatrix}$$

for $\mathcal{H}_{\mathbb{Q}}(\rho)$ has components which span the solution space of a second order *modular differential equation*

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where $D_k^2 := D_{k+2} \circ D_k$ and $\alpha_4 = \frac{AB}{N^2} - \frac{k(k+2)}{144}$ is uniquely determined by ρ.

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Rewriting the equation in terms of q shows that $q = 0$ is a regular singular point in the sense of Fuchs, so the coefficients $a(n), b(n)$ are obtainable via a standard recursive formula.

Mason's result

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Very recently, Mason has closed these cases, by proving that if ρ has infinite image, then in fact there are infinitely many primes dividing the denominators of F.

Dimension three

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Again there is a free module structure over $\mathbb{Q}[E_4, E_6]$, of rank three this time, with generators $F, D_k F, D_k^2 F$, and the minimal weight vector F satisfies a third order modular differential equation

$$D_k^3 f + \alpha_4 E_4 D_k f + \alpha_6 E_6 f = 0,$$

where again $k, \alpha_4, \alpha_6 \in \mathbb{Q}$ are uniquely determined by the eigenvalues of $\rho(T)$.

3-d theorem

Again one may analyze the (substantially more delicate!) recursive formula to obtain the following

Theorem (CM)

Let N be the level of ρ and suppose there is a prime p dividing $\frac{N}{(2^8 \cdot 3^4 \cdot 5^2 \cdot 7^2, N)}$. Then p occurs to unbounded powers in the denominators of the Fourier coefficients of F, and every nonzero vector in $\mathcal{H}_{\mathbb{Q}}(\rho)$ has unbounded denominators. □

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Unlike in dimension two, here many examples are found where $\ker\rho$ is a finite index, noncongruence subgroup, so the components of F provide examples of noncongruence modular forms with provably unbounded denominators; there have been precious few examples of this in the literature!

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Again there are a (much larger) number of open cases, and it appears some sort of "generalized hypergeometric" analysis might close these cases.

Higher dimension

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It is a result of Mason and the speaker that if $\rho : \Gamma \to GL_d(\mathbb{C})$ is irreducible of arbitrary dimension d and finite level, then the space $\mathcal{H}(\rho)$ of vector-valued modular forms for ρ is again a free module over $\mathbb{C}[E_4, E_6]$, of rank d.

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Clearly new ideas are needed!

Logarithmic VVMFs

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The free module theorem continues to hold here, and the generators may be taken to be

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Tensoring with a power χ^N of Γ similarly gives generators

$$F_N = \begin{pmatrix} \tau\eta^{2N} \\ \eta^{2N} \end{pmatrix}, \; 12D_{N-1} = \begin{pmatrix} 2\pi i\eta^{2N} + \tau\eta^{2N}E_2 \\ \eta^{2N}E_2 \end{pmatrix},$$

which "nearly" have integral Fourier expansions.

Logarithmic VOAs

Such vector-valued modular forms appear in logarithmic conformal field theory (so-called essentially because of this phenomenon). Such field theories are again modeled by vertex operator algebras, where now the modules associated to the VOA are not necessarily completely reducible under the action of the Virasoro element L_0.

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In particular, one would like to identify vectors whose "pure" q-expansions have integral coefficients. The interpretation of the coefficients of expansions carrying $\log q$ terms has not been made clear by physicists(?).

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One way to obtain a desirable vector is to look at modular differential equations that "factor" into interesting pieces. Here is a recent example observed by the speaker:

$$D_0^3 f - \frac{1}{18} E_4 D_0 f + \frac{1}{54} E_6 f = D_4 L f,$$

where the solutions of $L[f] = D_0^2 - \frac{1}{18} E_4 f = 0$ yield a vector-valued modular function

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} q^{\frac{1}{3}} + \sum_{n \geq 0} a_1(n) q^{n+\frac{1}{3}} \\ q^{-\frac{1}{6}} + \sum_{n \geq 0} a_2(n) q^{n-\frac{1}{6}} \end{pmatrix}$$

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with positive integer q-expansions.
To obtain the third solution, one sets $f_3 = g + \log(q) f_1$ for some $g = \sum_{n \geq 0} b(n) q^{n+\frac{1}{3}}$, and determines the $b(n)$ from the inhomogeneous equation $L[g] = \eta^8 - 2 D_1 f_1$.

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The $b(n)$ are evidently rational, but appear to have unbounded denominators...

Thanks very much!