Axioms of an RCFT (on $\mathbb{P}$)

hep-th/9910156, hep-th/9810019

Data: space of states $\mathcal{H}$ (may or may not be a Hilbert space)

n-point correlation functions $\langle \quad \rangle$

$\psi_1, \dots, \psi_n \in \mathcal{H}$, $\underbrace{z_1, \dots, z_n}_{\text{distinct}} \in \mathbb{P}$

$$\langle V(\psi_1; z_1) \cdots V(\psi_n; z_n) \rangle \in \mathbb{C}$$

linear in the ψ_i, (real) smooth in the z_i, singularities only at $z_i = z_j$

symmetric with resp. to permuting the $V(\psi_i; z_i)$.

Constraints:

- n-point functions are $\overset{\text{non-deg}}{\vee}$ covariant with resp local conformal transf.
- $\mathcal{H}$ is a rep of $\mathrm{Vir} \oplus \overline{\mathrm{Vir}}$ (today $c = \bar{c}$)

  semi-simple as a rep of Vir & $\overline{\mathrm{Vir}}$ $\rightarrow$ $\bar{L}_0, L_0$ diag; homogeneous basis $\{\psi_i\}_{i \in I}$, $L_0 \psi_i = h_i \psi_i$, $\bar{L}_0 \psi_i = \bar{h}_i \psi_i$, $\overset{(-)}{h}_i$ bounded below

- ~~$\exists$ a finit dim subspace $\mathrm{Prim}(\mathcal{H}) \subset \mathcal{H}$~~
- $\exists$ subspaces $V, \bar{V}$ s.t for any $\psi_i \in \mathcal{H}$ and any $v \in V$, $\bar{v} \in \bar{V}$

$$\langle V(v; z)\, V(\psi_1; z_1) \cdots V(\psi_n; z_n) \rangle \text{ is merom. in } z$$

$$\langle V(\bar{v}; z)\, V(\psi_1; z_1) \cdots V(\psi_n; z_n) \rangle \text{ is anti-merom in } z$$

- Recursion relation for correlators

  For any $\psi_1, \dots, \psi_n \in \mathcal{H}$

$$\langle V(\psi_1; z_1)\, V(\psi_2; z_2)\, V(\psi_3; z_3) \cdots \rangle$$

$$= \sum_{k \in I} C_{1,2}^{k} (z_1 - z_2)^{h_k - h_1 - h_2} (\bar{z}_1 - \bar{z}_2)^{\bar{h}_k - \bar{h}_1 - \bar{h}_2} \langle V(\psi_k; z_2)\, V(\psi_3; z_3) \cdots \rangle$$

$\xRightarrow{\text{assoc and comm.}}$ $\psi_1, \psi_2 \in V \Rightarrow C_{1,2}^{k} = 0$ unless $\psi_k \in V$

$\Rightarrow$ V & $\bar{V}$ form a kind of algebra called a vertex operator algebra

- $V \cap \bar{V} = \mathbb{C} \cdot \Omega \leftarrow$ vacuum state

  $L_0 \Omega = \bar{L}_0 \Omega = 0$, $L_{\pm 1} \Omega = 0$

  $\langle V(\Omega; z) V(\psi_1; z_1) \cdots \rangle = \langle V(\psi_1; z_1) \cdots \rangle$

  $U(Vir)\Omega \subset V$, $U(\overline{Vir}) \subset \bar{V}$

- $Y(L_{-2}\Omega; z) = T(z)$ energy momentum tensor

  $Y(\bar{L}_{-2}\Omega; z) = \bar{T}(z)$

  $$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \text{reg}$$

  $$\bar{T}(z)\bar{T}(w) = \frac{c/2}{(\bar{z}-\bar{w})^4} + \frac{2}{(\bar{z}-\bar{w})^2}T(w) + \frac{1}{\bar{z}-\bar{w}}\bar{\partial}\bar{T}(w) + \cdots$$

~~subspace prim($\mathcal{H}$) $\subset \mathcal{H}$~~

$$Y(L_m \psi_i; z_i) = \frac{1}{2\pi i}\oint_{z_i} T(w)\, Y(\psi_i; z_i)(w - z_i)^{1+m}\, dw$$

$$Y(\bar{L}_m \psi_i; z_i) = \frac{1}{2\pi i}\oint_{\bar{z}_i} \bar{T}(w)\, Y(\psi_i; z_i)(\bar{w} - \bar{z}_i)^{1+m}\, d\bar{w}$$

- $\exists$ subspace prim($\mathcal{H}$) $\subset \mathcal{H}$ (assume $\{\psi_i\}_{i \in I} \cap$ prim($\mathcal{H}$) basis)

  for $\psi_i \in$ prim($\mathcal{H}$), $L_0 \psi_i = h_i \psi_i$, $\bar{L}_0 \psi_i = \bar{h}_i \psi_i$

  $$T(z)\, Y(\psi_i; w) = \frac{h_i}{(z-w)^2} Y(\psi_i; w) + \frac{1}{z-w}\partial Y(\psi_i; w) + \cdots$$

  $$\bar{T}(z)\, Y(\psi_i; w) = \frac{\bar{h}_i}{(\bar{z}-\bar{w})^2} Y(\psi_i; w) + \frac{1}{\bar{z}-\bar{w}}\bar{\partial} Y(\psi_i; w) + \cdots$$

  Let $f: \mathbb{R} \to \mathbb{P}$ be meromorphic

  $\hookrightarrow$ $Y(\psi_i; z)$ transforms as

  $$Y(\psi_i; z) \mapsto (f'(z))^{h_i} \left(\overline{f'(z)}\right)^{\bar{h}_i} Y(\psi_i; f(z))$$

  ψ_i are Vir, $\overline{Vir}$ hw states

  $\overset{(-)}{L}_1 \psi_i = \overset{(-)}{L}_2 \psi_i = 0$, Consequences (Propositions)

- Invariance under Möbius transformations

fixes the functional behaviour of 1, 2, 3 - point fns. (of primaries) up to scaling:

$$\langle Y(\varphi; z)\rangle = \text{const} \qquad (0 \text{ unless } h = 0 = \bar{h})$$

$$\langle Y(\varphi_1; z_1) Y(\varphi_2; z_2)\rangle = \overset{C_{1,2}}{\cancel{\text{const}}}\,(z_1 - z_2)^{-2h_1}(\bar{z}_1 - \bar{z}_2)^{-2\bar{h}_1}$$

$\uparrow$ 0 unless $h_1 = h_2$, $\bar{h}_1 = \bar{h}_2$

(singledvaluedness $\Rightarrow h_1 - \bar{h}_1 \in \mathbb{Z}$)

$$\langle Y(\varphi_1; z_1)\, V(\varphi_2; z_2)\, V(\varphi_3; z_3)\rangle$$

$$= \cancel{\text{const}}\; C_{1,2,3} \cdot \cancel{\prod_{1\le i<j\le 3}} (z_1 - z_2)^{h_3 - h_1 - h_2}(z_1 - z_3)^{h_2 - h_1 - h_3}(z_2 - z_3)^{h_1 - h_2 - h_3}$$

$$\cdot (\bar{z}_1 - \bar{z}_2)^{\bar{h}_3 - \bar{h}_1 - \bar{h}_2}(\bar{z}_1 - \bar{z}_3)^{\bar{h}_2 - \bar{h}_1 - \bar{h}_3}(\bar{z}_2 - \bar{z}_3)^{\bar{h}_1 - \bar{h}_2 - \bar{h}_3}$$

- 4pt fns are fixed up to an arbitrary ~~(local)~~ fn in the cross ratio

$$x = \frac{z_1 - z_2}{z_1 - z_3}\,\frac{z_3 - z_4}{z_2 - z_4}$$

$$\langle Y(\varphi_1; z_1) Y(\varphi_2; z_2) Y(\varphi_3; z_3) Y(\varphi_4; z_4)\rangle = f(x) \prod_{1\le i<j\le 4} (z_i - z_j)^{H/3 - h_i - h_j}(\bar{z}_i - \bar{z}_j)^{\bar{H}/3 - \bar{h}_i - \bar{h}_j}$$

$$H = \sum_i h_i, \quad \bar{H} = \sum_i \bar{h}_i$$

- If we normalise the $\varphi_i \in \text{prim}(\mathcal{H})$ s.t. $C_{i,j}$ (of 2-pt) $= \delta_{i,j}$

$\Rightarrow C_{i,j,k}$ of 3-pt is equal to $C_{i,j}^{k}$ of op. algebra:

$$Y(\varphi_1; z_1) Y(\varphi_2; z_2) = \sum_{k\in I} C_{i,j}^{k}\, Y(\varphi_k; z_2)(z_1 - z_2)^{h_k - h_1 - h_2}(\bar{z}_1 - \bar{z}_2)^{\bar{h}_k - \bar{h}_1 - \bar{h}_2}$$

- $n \geq 4$ - pt fns are determined up to arbitrary fns in $n-3$ cross ratios
- Correlations fns in non-primaries determined by cors of primaries

$\Rightarrow$ $n \geq 4$-pt correlation fns are completely determined by the $C_{i,j,k}$ for all primaries $\varphi_i, \varphi_j, \varphi_k$.

(The $C_{i,j,k}$ define/characteris a CFT they are not fixed by V)

Ex $\psi_1, \dots, \psi_n$ primary

$$\langle Y(L_{-n}\psi_1; z_1)\, Y(\psi_2; z_2) \cdots \rangle = \frac{1}{2\pi i} \oint_{z_1} (w - z_1)^{1-n} \langle T(w)\, Y(\psi_1; z_1)\, Y(\psi_2; z_2) \cdots \rangle$$

$$= -\frac{1}{2\pi i} \sum_{j=2}^{n} \oint_{z_1} (w - z_1)^{1-n} \langle Y(\psi_1; z_1) \cdots (T(w) Y(\psi_j; z_j)) \cdots Y(\psi_n; z_n) \rangle$$

$$= -\sum_{j=2}^{n} \left\{ \frac{(n-1)h_j}{(z_j - z_1)^n} - \frac{1}{(z_j - z_1)^{n-1}} \partial_{z_j} \right\} \langle Y(\psi_1; z_1) \cdots Y(\psi_n; z_n) \rangle$$

Conformal blocks Phys def ~~(Math later)~~ (will be made more rigorous)

Let $\psi_1, \psi_2, \psi_3, \psi_4$ be primary,

$$G^{21}_{34}(x) = \lim_{z_1 \to \infty} z_1^{2h_1} \bar{z}_1^{2\bar{h}_1} \langle Y(\psi_1; z_1)\, Y(\psi_2; z_2)\, Y(\psi_3; x)\, Y(\psi_4; 0) \rangle$$

cross ratio $x = \frac{z_1 - z_2}{z_1 - z_3} \frac{z_3 - z_4}{z_2 - z_4}$ in $\lim_{\substack{z_1 \to \infty \\ z_2 \to 1 \\ z_4 \to 0}}$ $x = z_3$

Perform Operator expansions in $(1,2)$ & $(3,4)$

$$G^{21}_{34}(x) = \sum_{p \in I_{prim}} C_{1,2,p}\, C_{3,4,p}\, \mathcal{F}^{21}_{34}(p; x)\, \bar{\mathcal{F}}^{21}_{34}(p; \bar{x})$$

the $\mathcal{F}^{21}_{34}(p; x)$ & $\bar{\mathcal{F}}^{21}_{34}(p; \bar{x})$ are called <u>conformal blocks.</u>

↳ Cornelius will talk about deriving fn-dep.

Due to assoc of operator alg and commutativity of corrs the $G^{21}_{34}(x)$ ~~stasify~~ satisfy many non-triv identities

$$G^{21}_{34}(x) = G^{41}_{32}(1-x) = \frac{1}{x^{2h_3} \bar{x}^{2\bar{h}_3}} G^{24}_{31}(1/x).$$

These conditions are called crossing symmetry, they over determine the $C_{i,j,k}$

$$\rightarrow \sum_p C_{2,1,p}\, C_{3,4,p}\, \mathcal{F}^{21}_{34}(p; x)\, \bar{\mathcal{F}}^{21}_{34}(p; \bar{x}) = \sum_q C_{4,1,q}\, C_{3,2,q}\, \mathcal{F}^{41}_{32}(q; 1-x)\, \bar{\mathcal{F}}^{41}_{32}(q; 1-\bar{x})$$

Refining the def of conformal blocks

In RCFT V is not "free" (i.e. Verma of Vir or other alg)

because the free module would have non-triv sub reps.

$\Rightarrow \exists$ singular vects in the free module the are set to 0 in V (Ex. $L_{-1}\Omega$)

Let N be such a singular vector

$\Rightarrow \quad \langle Y(N;z)\, Y(\psi_1;z_1) \cdots Y(\psi_n;z_n) \rangle = 0$

if $N = P \cdot \psi$ $P \in U(Vir)$, $\psi \in prim(\mathcal{H})$

$\Rightarrow \quad \langle Y(P\psi;z)\, Y(\psi_1;z_1) \cdots Y(\psi_n;z_n) \rangle$

$= \underbrace{D_P}_{\text{Diff op.}} \langle Y(\psi;z)\, Y(\psi_1;z_1) \cdots Y(\psi_n;z_n) \rangle = 0$

$\Rightarrow$ A corr must satisfy all constraints coming from sing vects for all $\psi, \psi_1, \dots \psi_n$

call these diff ops $D_1 \cdots D_m$

Refined def of conformal blocks

The conformal blocks of a chiral alg V are the spaces of local solutions of the diff ops $D_1 \cdots D_m$.

i.e. conformal blocks are local sections of ~~moduli space~~ fibre bundles over ~~moduli~~ the moduli spaces of curves with n-marked points.