Loop groups & Conformal Blocks

Goal: Explain how representations of loop groups give rise to conformal blocks.

Outline:
1) Vertex Algebras
2) Vertex algebras from loop groups
3) Conformal blocks

Reference: E. Frenkel - Vertex Algebras & Algebraic Curves
arXiv: 0007054.v4

§1. Vertex Algebras

Geometric background

$\mathcal{O} := \mathbb{C}[[z]]$ - complete topological ring.
adic topology, i.e. finest s.t. $z^n \to 0$ as $n \to \infty$

$\mathrm{Aut}(\mathcal{O})$ - topological autom's of $\mathcal{O}$

$$\cong \left\{ \rho(z) = \sum_{n=1}^{\infty} \rho_n z^n \in \mathbb{C}[[z]] \;\middle|\; \rho_1 \in \mathbb{C}^\times \right\}$$

A conformal vertex algebra is a rule which assigns to any smooth curve X a vector bundle

$$\mathcal{V}^* \longrightarrow X$$

w/
- an $\mathrm{Aut}(\mathcal{O})$ - structure
- a flat $\mathrm{Aut}(\mathcal{O})$ - connection
- $\forall\ x \in X$, a horizontal section

$$\mathcal{Y}_x : \underset{\substack{\text{formal punctured}\\ \text{disc at } x}}{\partial D_x} \longrightarrow \mathcal{V}^*\big|_{\partial D_x} \otimes \mathrm{End}(\underset{\text{dual to fiber at } x.}{\mathcal{V}_x})$$

Such a rule turns out to depend on a finite amount of algebraic data. We'll motivate this by working backwards in the construction.

X - smooth curve.

$x \in X$

$\mathcal{O}_x$ - ring of Taylor series at $x \in X$.

Def: A formal coordinate at x is a generator t_x of the max'l ideal $m_x \subseteq \mathcal{O}_x$

X smooth $\Rightarrow$ choice of coord gives

$$\mathcal{O} \xrightarrow[t_x]{\cong} \mathcal{O}_x.$$

$\hat{X} = \{ (x, t_x) \mid x \in X,\ t_x \text{ a formal coord. at } x \}$

$\hat{X}$ is an inf. dim'l scheme

$\hat{X} \longrightarrow X$ - formal frame bundle, is a principal $\mathrm{Aut}(\mathcal{O})$ - bundle.

$\hat{X} \longrightarrow X$ carries a canonical flat connection given by parallel translation along the formal coordinate.

Rmk: Frenkel in §6.1 gives a more formal construction using Harish-Chandra pairs & Beilinson - Bernstein localization.

One can also construct the connection explicitly by solving a universal ODE.

For any $\mathrm{Aut}(\mathcal{O})$-module V, the usual yoga $\Rightarrow$

$$\mathcal{V} = \hat{X} \underset{\mathrm{Aut}(\mathcal{O})}{\times} V \longrightarrow X$$

is a vector bundle w/ a flat $\mathrm{Aut}(\mathcal{O})$-connection.

So, the data of a vertex algebra is essentially:

- an $\mathrm{Aut}(\mathcal{O})$-module V
- a universal rule for constructing a horizontal section of $\mathcal{V}^*_{D^\times} \otimes \mathrm{End}(V)$ over any formal loop in a smooth curve.

Def: A <u>conformal</u> <u>vertex</u> <u>algebra</u> of central charge $c \in \mathbb{C}^\times$

1) A $\mathbb{N}$-graded v.s. $V = \bigoplus_{i=0}^{\infty} V_i$ w/ $\dim V_i < \infty$

2) "conformal vector" $\omega \in V_2 \setminus \{0\}$

3) "vertex operation", $Y \in V^*((z)) \otimes \mathrm{End}(V)$,

i.e.

$$Y(-,z) : V \longrightarrow \mathrm{End}(V)[[z, z^{-1}]],$$

s.t. $\forall A \in V_j$ $Y(A,z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-j}$

w/ α_n - homogeneous of deg $-n$

Denote $Y(\omega, z) := \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$.

Denote $|0\rangle := \frac{2}{c} Y(\omega, 0)(\omega) = \frac{2}{c} L_2 \omega$.

"Vacuum vector".

We require that:

1) (Vacuum) $Y(|0\rangle, z) = \mathrm{Id}_V$

& $\forall A \in V$, $Y(A,z)|0\rangle \in A + zV[[z]]$

(Note: $\Rightarrow L_i |0\rangle = 0$ for $i < 2$).

2) (translation) $\forall A \in V$, $[L_{-1}, Y(A,z)] = \partial_z Y(A,z)$

& $\forall A \in V_n$, $L_0 A = nA$.

3) (locality) $\forall A, B \in V$, $\exists n$ s.t.

$$(z-\omega)^n [Y(A,z), Y(B,\omega)] = 0 \in \mathrm{End}(V)[[z^{\pm}, \omega^{\pm}]]$$

Rmk: The axioms guarantee that

V is an $\mathrm{Aut}(\mathcal{O})$-module

& that $Y \in V^*((z)) \hat{\otimes} \mathrm{End}(V)$ is a horizontal section of

$$\hat{D} \underset{\mathrm{Aut}(\mathcal{O})}{\times} V^*$$

over ∂D.

For example, 1) - 3) guarantee that

$$-t^{n+1}\partial_t \longmapsto L_n \in \mathrm{End}(V)$$

induces an integrable homomorphism

$$\mathrm{Der}(\mathcal{O}) \to \mathrm{End}(V)$$

giving the $\mathrm{Aut}(\mathcal{O})$-structure.

The horizontality of Y will follow from 2) & 3).

At the moment, I am still trying to better understand the role of $|0\rangle$.

§2 Vertex Algebras from Rep$^{\underline{n}}$s of Loop Groups

Our goal here is to construct some important examples of vertex algebras so we'll start w/ a generators & rel$^{\underline{n}}$s Theorem which will speed up the constructions.

First need:

The alg. structure on $End(V)$ lets us compose vertex operators:

Let $A \in V_i$, $B \in V_j$.

Denote $Y(A,z)_{\leq l} := \sum_{m \leq l} \alpha_m z^{-m-i}$

$$Y(A,z)_{>l} := \sum_{m>l} \alpha_m z^{-m-i}$$

Define

$$:Y(A,z)Y(B,z): := Y(A,z)_{\leq -i} Y(B,z) + Y(B,z)\, Y(A,z)_{>-i}$$

Define $:Y(A_1,z) \cdot \ldots \cdot Y(A_n,z):$ for $n>2$ inductively by taking products right to left.

We can now formulate:

Thm: (Reconstruction)

Given:
- V – $\mathbb{N}$-graded v.s.
- $c \in \mathbb{C}^\times$
- $\omega \in V_2$
- $\{a^\alpha\}_{\alpha=1}^{\infty} \subseteq V$ w/ $a^\alpha \in V_{\Delta_\alpha}$
- $\omega(z) = \sum_{\mathbb{Z}} L_n z^{-n-2} \in End(V)[[z^{\pm}]]$

  & $a^\alpha(z) = \sum_{\mathbb{Z}} a_n^\alpha z^{-n-\Delta_\alpha}$

Define $|0\rangle := \frac{2}{c} \omega(0)(\omega)$.

Then, if:

a) $\forall \alpha$, $a^\alpha(z)|0\rangle \in a^\alpha + zV[[z]]$

$\omega(z)|0\rangle \in \omega + zV[[z]]$

b) $[L_{-1}, a^\alpha(z)] = \partial_z a^\alpha(z)$,

$[L_{-1}, \omega(z)] = \partial_z \omega(z)$, & $\forall A \in V_n, L_0 A = nA$

c) All $a^\alpha(z)$ & $\omega(z)$ are mutually local, i.e. $\forall$ $f(z), g(z) \in \{a^\alpha(z)\}_{\alpha=1}^\infty \cup \{\omega(z)\}$

$\exists N$ s.t. $(z-w)^N [f(z), g(w)] = 0$

d) V is spanned by $a^{\alpha_1}_{-\Delta_{\alpha_1}-j_1} \cdots a^{\alpha_n}_{-\Delta_{\alpha_n}-j_n}|0\rangle$

where $j_1 \geq \cdots \geq j_n \geq 0$ & if $j_i = j_{i+1}$, then $\alpha_i \geq \alpha_{i+1}$

& $a^0(z) := \omega$.

Thm $\exists!$ conformal vertex algebra structure on V w/ Y given by

$$Y(a^{\alpha_1}_{-\Delta_{\alpha_1}-j_1} \cdots a^{\alpha_n}_{-\Delta_{\alpha_n}-j_n}|0\rangle) :=$$

$$\frac{1}{j_1! \cdots j_n!} :\partial_z^{j_1} a^{\alpha_1}(z) \cdots \partial_z^{j_n} a^{\alpha_n}(z):$$

We'll use the Thm to exhibit vertex algebra structures on reps of loop groups.

$\mathfrak{g}$ - simple f.d. Lie alg / $\mathbb{C}$

$\langle\, ,\, \rangle$ - Killing form

$\{J^a\}_{a=1}^{\dim \mathfrak{g}}$ - orthonormal basis for $(\mathfrak{g}, \langle\, ,\, \rangle)$

$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}K$

$\left\{ J^a_n := J^a \otimes t^n \right\}_{a=1,\ n\in\mathbb{Z}}^{\dim \mathfrak{g}}$ - topological basis for $\hat{\mathfrak{g}}$

Define $[J^a_n, J^b_m] := [J^a, J^b] \otimes t^{n+m} + \delta_{n,-m}\, \delta_{a,b} K$

$(\hat{\mathfrak{g}}, [\, ,\,])$ - Kac-Moody extension of $\mathfrak{g} \otimes \mathbb{C}((t))$.

$\mathfrak{h} \subseteq \mathfrak{g}$ - Cartan sub-alg (max'l abelian)

$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}((t)) \oplus \mathbb{C}K \subseteq \hat{\mathfrak{g}}$ - Heisenberg sub-alg.

($\hat{\mathfrak{h}} \subseteq \hat{\mathfrak{g}}$ plays role of $\mathfrak{h} \subseteq \mathfrak{g}$).

For $k \in \mathbb{C} \setminus \{-h^\vee\}$ dual Coxeter #.

$\mathbb{C}_k$ = 1-dim'l rep of $\hat{\mathfrak{h}}$ s.t. $\mathfrak{h} \otimes \mathbb{C}((t))$ acts by 0 & K acts by k.

$V_k(\mathfrak{g}) := \mathrm{Ind}_{\hat{\mathfrak{h}}}^{\hat{\mathfrak{g}}} \mathbb{C}_k \cong \mathbb{C}_k \otimes_{U(\hat{\mathfrak{h}})} U(\hat{\mathfrak{g}})$

PBW $\Rightarrow$ $V_k(\mathfrak{g})$ has a basis

$$\left\{ J^{a_1}_{n_1} \cdots J^{a_\ell}_{n_\ell} v_k \;\middle|\; \begin{array}{l} v_k = 1 \otimes 1 \\ n_1 \leq \cdots \leq n_\ell < 0 \end{array} \quad \begin{array}{l} n_i = n_{i+1} \\ \Rightarrow a_i \geq a_{i+1} \end{array} \right\}$$

Define $\omega := \frac{1}{2(k+h^\vee)} \sum_{a=1}^{\dim \mathfrak{g}} (J^a_{-1})^2 v_k$

(Sugawara)

Define $Y(J^a_{-1} v_k, z) := J^a(z) := \sum_{n \in \mathbb{Z}} \mathrm{ad}_{J^a_n} z^{-n-1}$

Reconstruction Thm $\Rightarrow$ $V_k(\mathfrak{g})$ is a conformal vertex algebra.

Denote horizontal section on ∂D_x in coord. z by $J(z) := \sum_a J_a \otimes J^a(z)$

dual basis

Exercise: Compute the central charge of $V_k(\mathfrak{g})$ as a fn. of k.

Q: Why are rep^ns of loop groups vertex algebras?

Intuitively, it's natural that there should be an action of $\mathrm{Aut}(\mathcal{O})$ on them via the action on $\mathfrak{g}((t))$.

How should we understand the universal rule for the horizontal section over ∂D?

§3. Conformal Blocks

Def: V - conformal vertex algebra.

X - smooth projective curve.

$x \in X$.

Then $\varphi \in \mathcal{V}_x$ is a <u>conformal block</u>

if $\varphi(\mathcal{Y}_x \cdot A) \in \Gamma(\partial D, \mathcal{V}^*)$ extends

to a regular section of $\mathcal{V}^*$ on $X \setminus x$

$\forall A \in \mathcal{V}_x$.

$C(X, x, V)$ - space of conformal blocks

Rmk: Vacuum axiom $\Rightarrow$ $\varphi(\mathcal{Y}_x |0\rangle)$ extends to all X.

Fact: $C(X, V) \longrightarrow X$ w/ fiber

$C(X, x, V)$ at x

forms a vector bundle on X.

Further, as we deform X, $C(X, V)$ varies naturally, so

$$C(V) \longrightarrow \mathcal{M}$$ - moduli of smooth curves

is a vector bundle on moduli space.

Example: $V = V_k(\mathfrak{g})$

In this case $C(X, x, V_k(\mathfrak{g}))$ simplifies.

$$\mathfrak{g}_{x,out} := \mathfrak{g} \otimes \mathcal{O}_{X \setminus x}$$

$\mathfrak{g}_{x,out} \circlearrowright V_k(\mathfrak{g})$ by

$$f \mapsto \tilde{f} = \operatorname{Res}_{z=0} \langle f, J(z) \rangle dz \in \operatorname{End}(V_k(\mathfrak{g}))$$

Can show $\forall\ \varphi \in C(X, x, V_k(\mathfrak{g}))$, $A \in V_k(\mathfrak{g})$, $f \in \mathfrak{g}_{x,out}$, we have $\varphi(\tilde{f} \cdot A) = 0$.

i.e. we have

invariants

$$(\dagger) \quad C(X, x, V_k(\mathfrak{g})) \hookrightarrow (\mathcal{V}_k(\mathfrak{g})^*_x)^{\mathfrak{g}_{x,out}}$$

Prop: $(\dagger)$ is an isomorphism.

i.e. conformal blocks for $V_k(\mathfrak{g})$ are invariants of $\mathcal{V}_k(\mathfrak{g})^*_x$ under $\mathfrak{g}_{x,out}$-action.

We can also twist $V_k(\mathfrak{g})$ by a G-bundle $P \to X$.

Get P-twisted conformal blocks
$$C^P(X, x, V_k(\mathfrak{g}))$$

Fact: $C^P(X, x, V_k(\mathfrak{g})) = \operatorname{Hom}_{\mathfrak{g}^P_{x,out}}(\mathcal{V}_x(\mathfrak{g})^P, \mathbb{C})$

$C^P(X, x, V_k(\mathfrak{g}))$ fit together to give a vector bundle

$$C(V_k(\mathfrak{g})) \longrightarrow \mathcal{M}_G$$

- moduli of marked curves w/ G-bundles.

Fact: $C^P(X, x, V_0(\mathfrak{g})) \cong \Gamma(\mathcal{O}_{\mathcal{M}_{G,(X,p)}})$

More generally for $k \in \mathbb{N}$

$$C^P(X, x, L_k(\mathfrak{g})) = \Gamma(\mathcal{L}^k_{\mathcal{M}_{G,(X,p)}})$$

determinant line on $\mathcal{M}_G$

So, the conformal blocks give us a construction of the local graded coordinate ring of $\mathcal{M}_G$.

Proj of this ring gives us a natural compactification of $\mathcal{M}_G$

So the conformal blocks encode how G-bundles on curves can degenerate.