Primitive forms via polyvector fields

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I. Motivations/Applications

II. Notion of primitive forms

III. Perturbative formulas

IV. Examples.

I. Motivations / Appliations

1) a primitive form is a family of hol. top. form s.t. properties - - - .

E.g.

$$\begin{array}{ccccc} Z & = & \mathbb{C} \times \mathbb{C}^2 & \xrightarrow{F} & \mathbb{C} \\ \downarrow & & \downarrow & & \\ 0 \in S & = & \mathbb{C}^2 \ni (u_0, u_1) & & \end{array} \qquad F(z, u_0, u_1) = z^3 + u_0 + u_1 z$$

$\zeta = dz \in \Omega^{top}(Z/S)$ is a primitive form

2) Introduced by Kyoji around 1979-1983

in order to generalize the theory of elliptic integral

3) Plays an important role in mirror symmetry

sympl. geo. A	Mirror symmetry conjecture	cpx geom. B
Frob. (GW theory cpt Fano mfd / FJRW theory of a wt homog. W)	$\cong$	Frob. (a choice of Prim form) (Landau-Ginzburg model $f : X \to \mathbb{C}$).

Application of LLS:

(L-L-S-Shen) If f is Arnold's 14 exceptional unimodula singularities, then FJRW theory of $(f^T, G_{max}) \sim$ Saito-Givental theory of f.

II. Notion of prim. forms

Started point: $\cdot$ $X \subset \mathbb{C}^n$ is a Stein domain

1) $f: X \to \mathbb{C}$ $\cdot$ f: hol. with $|\mathrm{Crit}(f)| < +\infty$

2) Choose a nowhere vanishing hol. n-form Ω_X. Key role

(Not essential).

Trip: $PV(X) \rightsquigarrow PV(X)[[t]] \rightsquigarrow (\mathcal{H}^{f,\Omega}_{(0)}, K^f_\Omega(\cdot,\cdot): \mathcal{H}^{f,\Omega}_{(0)} \times \mathcal{H}^{f,\Omega}_{(0)} \to \mathbb{C}[[t]])$

(coh.; higher residue pairing)

Family version - - - - - - - - - $(\mathcal{H}^{F,\Omega}_{(0)}, K^F_\Omega(\cdot,\cdot), \nabla^\Omega)$

(sheaf of coh. over a param. space S; Gauss-Manin conn)

function

$\zeta \in \Gamma(S, \mathcal{H}^{F,\Omega}_{(0)})$ s.t. Properties - - - - . ($\zeta\Omega$ original formulation)

1) polyvector fields.

$$PV(X) := \bigoplus PV^{i,j}(X), \qquad PV^{i,j}(X) := A^{0,j}(X, \Lambda^i T^{1,0}X)$$

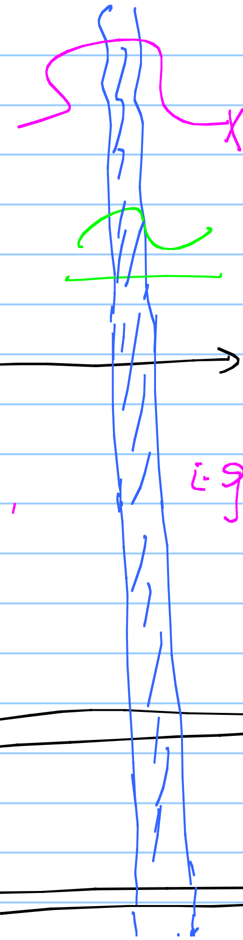
$(z_1, \cdots, z_n) \in X$.

$I = [a_1, a_2, \cdots, a_i] \subset [1, 2, \cdots, n]$. $\begin{cases} d\bar{z}^I := d\bar{z}_{a_1} \wedge d\bar{z}_{a_2} \wedge \cdots \wedge d\bar{z}_{a_i} \\ \partial_I := \frac{\partial}{\partial z_{a_1}} \wedge \frac{\partial}{\partial z_{a_2}} \wedge \cdots \wedge \frac{\partial}{\partial z_{a_i}} \end{cases}$

$$\alpha \in PV^{i,j}(X) \Rightarrow \alpha = \sum_{|I|=i, |J|=j} \alpha^I_J \, d\bar{z}^J \otimes \partial_I$$

- $PV_c(X) := \bigoplus PV_c^{i,j}(X) = \{\alpha \in PV(X) \mid \operatorname{supp}\alpha \text{ is compact}\}$ to do integration
- $\wedge$: $(d\bar{z}^J \otimes \partial_I) \wedge (d\bar{z}^L \otimes \partial_K) := (-1)^{il} (d\bar{z}^J \wedge d\bar{z}^L) \otimes (\partial_I \wedge \partial_K)$
- $|\alpha| := j - i$.

2) Polyvector fields | diff. forms



$PV(X)$ | $A(X)$ smooth diff. forms

$PV^{i,j}(X)$ | $A^{n-i,j}(X)$

Γ_Ω: $g\, d\bar{z}^J \otimes \partial_I \longmapsto g\, d\bar{z}^J \wedge (\partial_I \lrcorner \Omega_X)$

$P_\Omega(\alpha) := \Gamma_\Omega^{-1}(P(\Gamma_\Omega(\alpha)))$ e.g. $P = \bar{\partial}, \partial$.

$\bar{\partial}: PV^{i,j}(X) \to PV^{i,j+1}(X)$

$(\bar{\partial}_\Omega \alpha) \lrcorner \Omega_X = \bar{\partial}(\alpha \lrcorner \Omega_X)$

$\partial_\Omega: PV^{i,j}(X) \longrightarrow PV^{i+1,j}(X)$

$(\partial_\Omega \alpha) \lrcorner \Omega_X = \partial(\alpha \lrcorner \Omega_X)$

$$\{\alpha,\beta\}_\Omega := \partial_\Omega(\alpha\wedge\beta) - (\partial_\Omega\alpha)\wedge\beta - (-1)^{|\alpha|}\alpha\wedge(\partial_\Omega\beta)$$

$$\partial(\theta\wedge\eta) - (\partial\theta)\wedge\eta - (-1)^{|\theta|}\theta\wedge\partial\eta = 0$$

- $\{g,\beta\}_\Omega \lrcorner \Omega_X = \partial g \wedge (\beta \lrcorner \Omega_X)$ $\quad (\{\cdot,\cdot\} \not\equiv 0)$

- ∂_Ω, $\bar{\partial}_\Omega$, $\{\cdot,\cdot\}_\Omega$ preserve $PV_c(X)$

  indep. of choice of Ω_X.

3). $\left(PV_c(X),\ \bar{\partial} + \{f, \cdot\}\right) \xhookrightarrow{\ \iota\ } \left(PV(X),\ \bar{\partial} + \{f, \cdot\}\right)$

Stein $+ |\mathrm{Crit}(f)| < +\infty$

$$H^*(PV(X), \bar{\partial} + \{f, \cdot\}) = H^0(PV(X), \bar{\partial} + \{f, \cdot\}) \xleftarrow{\ \sim\ } \mathrm{Jac}(f) = \Gamma(X, \mathcal{O}_X) / (\partial_{z_1} f, \cdots, \partial_{z_n} f)$$

$$\uparrow \wr\ \iota^*$$

$$H^*(PV_c(X), \bar{\partial} + \{f, \cdot\}) \xleftarrow{\ \simeq\ } \mathrm{Jac}(f)$$

$$H^*(PV_c(X), \bar{\partial} + \{f, \cdot\}) \xrightarrow{\ Tr\ } \mathbb{C}, \qquad \mathrm{Jac}(f) \xrightarrow{\ Res\ } \mathbb{C}$$

$$Tr([\alpha]) := \int_X (\alpha \lrcorner \Omega_X) \wedge \Omega_X$$

Key observation

4) $$\begin{array}{ccc} PV_c(X) & \rightsquigarrow & PV_c(X)((t)) \\ \downarrow & & \downarrow \\ PV(X) & \rightsquigarrow & PV((t)) \end{array}$$

$$H^{f,\Omega} := H^*(PV(X)((t)),\ Q_f := \bar\partial + \{f,\cdot\} + t\partial_\Omega)$$

$$\cong \Uparrow i^*$$

$$H^*(PV_c(X)((t)), Q_f) \xrightarrow{Tr} \mathbb{C}((t))$$

$$H^{f,\Omega} = \Gamma(X,\mathcal{O}_X)((t)) \big/ \mathrm{Im}(Q_f : \Gamma(X, T^{1,0}_X)((t)) \to \Gamma(X,\mathcal{O}_X)((t)))$$

$$H^{f,\Omega}_{(0)} \subset H^{f,\Omega}$$

($[[t]]$ under $H^{f,\Omega}_{(0)}$; $((t))$ under $H^{f,\Omega}$)

$$(\cdot,\cdot) : PV_c(X)((t)) \times PV_c((t)) \to \mathbb{C}((t))$$

$$(\alpha_1 g_1(t),\ \alpha_2 g_2(t)) \longmapsto g_1(t)\, g_2(-t) \int_X \left((\alpha_1\wedge\alpha_2)\lrcorner\Omega_X\right)\wedge\Omega_X$$

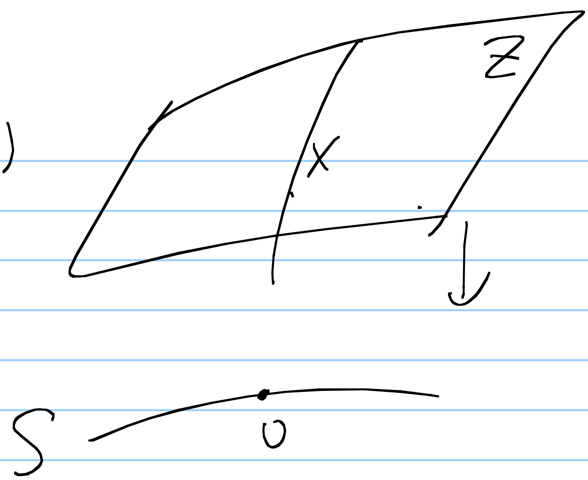
$$\rightsquigarrow \; K^f_\Omega(\cdot,\cdot) : H^{f,\Omega} \times H^{f,\Omega} \to \mathbb{C}((t))$$

$$\nabla_{t\partial_t} : PV(X)((t)) \longrightarrow PV(X)((t))$$

$$t^k \alpha \in t^k PV^{i,j} \longmapsto \nabla_{t\partial_t}(t^k \alpha) := \left(t\partial_t + i - \frac{f}{t}\right) t^k \alpha .$$

$$\rightsquigarrow \; \nabla_{t\partial_t} : H^{f,\Omega} \longrightarrow H^{f,\Omega} .$$

5)



① $\begin{array}{ccc} & & X \\ & \nearrow & \\ Z & \subset & X \times S \\ p \downarrow & & \\ S & & \end{array}$, $\Omega = \pi_X^* \Omega_X$.

② $F : Z \to \mathbb{C}$ univ. unfolding of $f : X \to \mathbb{C}$

ⓐ $F|_{Z_0 = X} = f$

ⓑ $KS : TS \xrightarrow{\sim} p_* \mathcal{O}_{Z(F)}$; $v \mapsto [\partial_v F]$

③ $p|_{Z(F)}$ is proper

$PV(Z/S) \rightsquigarrow PV(Z/S)((t))_{[[t]]}$

$PV_c(Z/S) \rightsquigarrow PV_c(Z/S)^{((t))}_{[[t]]}$

$\rightsquigarrow \left(H^{F,\Omega}, K^{F,\Omega}, \nabla^{\Omega} \right)$ ↙ $\overset{\infty}{=}$ VHS'

$\begin{cases} \nabla^{\Omega}_{v}[s] := [\partial_v s + \frac{\partial_v F}{t} s] \\ \nabla^{\Omega}_{t\partial t} \end{cases}$

Def: A primitive is a section $\zeta \in \Gamma(S, H^{F,\Omega}_{(0)})$ s.t.

(1) (primitivity) $t\nabla^{\Omega}\zeta : TS \xrightarrow{\sim} H^{F,\Omega}_{(0)} / tH^{F,\Omega}_{(0)}$; $V \mapsto [t\nabla^{\Omega}_{V}\zeta]$

(2) (orthogonality) $\forall V_1, V_2 \in TS$. $K^{F}_{\Omega}(\nabla^{\Omega}_{V_1}\zeta, \nabla^{\Omega}_{V_2}\zeta) \in t^{-2}\mathcal{O}_S$

(3) (holonomicity) $\forall V_1, V_2, V_3 \in TS$

$K^{F}_{\Omega}(\nabla^{\Omega}_{V_1}\nabla^{\Omega}_{V_2}\zeta, \nabla^{\Omega}_{V_3}\zeta) \in t^{-2}\mathcal{O}_S \oplus t^{-3}\mathcal{O}_S$

$K^{F}_{\Omega}(\nabla^{\Omega}_{t\partial_t}\nabla^{\Omega}_{V_1}\zeta, \nabla^{\Omega}_{V_2}\zeta) \in t^{-2}\mathcal{O}_S \oplus t^{3}\mathcal{O}_S$

(4) (homogeneity) $\exists r \in \mathbb{C}$ s.t. $(\nabla^{\Omega}_{t\partial_t} + \nabla^{\Omega}_{E})\zeta = r\zeta$. $E := KS^{-1}([F]) \in \Gamma(S, TS)$

III. Perturbative formula.

1) Def: Let $\mathcal{L} \subset H^{f,\Omega}$ s.t. (1) $H^{f,\Omega} = H^{f,\Omega}_{(0)} \oplus \mathcal{L}$ (2) $t^{-1}\mathcal{L} \subset \mathcal{L}$

$\mathcal{L}$ is called an opposite filtration if $K^f_\Omega(\mathcal{L}, \mathcal{L}) \subset t^{-2}\mathbb{C}[t^{-1}]$

is further called a good filtration if $\nabla^\Omega_{t\partial_t} \mathcal{L} \subset \mathcal{L}$.

Def: $\zeta_0 \in H^{f,\Omega}_{(0)}$ is called a primitive element w.r.t. $\mathcal{L}$ if

(1) (primitivity) $\zeta_0 + tH^{f,\Omega}_0$ generates $H^{f,\Omega}_{(0)}/tH^{f,\Omega}_0 \cong \mathrm{Jac}(f)$ as $\mathrm{Jac}(f)$-module

(2) (homogeneity) $\exists\, r \in \mathbb{C}$ s.t. $\nabla^0_{t\partial_t}\zeta_0 - r\zeta_0 \in \mathcal{L}$

Thm: "Good pairs" $\xrightarrow{\text{bijective}}$ Primitive forms at the germ

$$(\mathcal{L}, \zeta_0) \longmapsto \zeta(\mathcal{L}, \zeta_0) \in \Gamma(S, \mathcal{H}^{F,\Omega}_{(0)})$$

Question: How to describe $\zeta(\mathcal{L}, \zeta_0)$?

Answer: $\exists !$ splitting, $e^{\frac{f-F}{t}} \zeta_0 = \zeta_+ + \zeta_-$

(LLS) then $\zeta_+ = \zeta(\mathcal{L}, \zeta_0)$

$\frac{f-F}{t} \leftarrow \sum_{-\infty}^{+\infty} \cdot t^k$

$e^{\frac{f-F}{t}} \zeta_0 \in \mathcal{H}^{F,\Omega} \leftarrow \sum_{k=-M}^{+\infty} \cdot t^k$

$\zeta_+ \in \mathcal{H}^{F,\Omega}_{(0)}$

$\zeta_- \in e^{\frac{f-F}{t}} \mathcal{L}$

$$\mathcal{R} = \{ R_N := \mathcal{O}_{S,0}/\mathfrak{m}^N \}_{N \in \mathbb{Z}_{\geq 0}} \qquad R_N \to R_{N+1}$$

$$\check{H}^{F,\Omega} = \{ H^{F,\Omega}(R_N) := H^{F,\Omega} \otimes_{\mathcal{O}_{S,0}} R_N \}_{N \in \mathbb{Z}_{\geq 0}}$$

$$\Gamma(\mathcal{R}, \check{H}^{F,\Omega}) \ni \check{s} = \{ \check{s}_N \}_{N \in \mathbb{Z}_{\geq 0}}. \qquad \varphi_{R_N} : \mathcal{O}_{S,0} \to \mathcal{O}_{S,0}/\mathfrak{m}^N = R_N$$

Lemma: flat extension: $e^{\frac{f-F}{t}} : H^{f,\Omega} \longrightarrow \Gamma(\mathcal{R}, \check{H}^{F,\Omega})$

(1) $\qquad [s] \longmapsto \{ [\check{s}_N] \}, \quad \check{s}_N := \varphi_{R_N}(e^{\frac{F_0 - F}{t}} \pi_X^*(s))$

$$\check{\nabla}_V e^{\frac{f-F}{t}} [s] = 0 \quad \forall V \in \mathcal{T}_S.$$

- $e^{\frac{f-F}{t}}\Big|_N : H^{f,\Omega} \otimes_{\mathbb{C}} R_N \to H^{F,\Omega} \otimes_{\mathcal{O}_S} R_N$ is an isomorphism of R_N-modules
  $[s] \longmapsto [\check{s}_N]$

(2) Given an opposite filtration $\mathcal{L}$,

$$H^{f,\Omega} = H^{f,\Omega}_{(0)} \oplus \mathcal{L}.$$

$H^{F,\Omega}_{(0)}(R_N)$ $\ni$ ... $e^{\frac{f-F}{t}} \mathcal{L} \otimes_{\mathbb{C}} R_N$

$$\Rightarrow e^{\frac{f-F}{t}}[s] = [s_+(R_N)] \oplus [s_-(R_N)]$$

$$\Rightarrow \Gamma(\mathcal{R}, \check{H}^{F,\Omega}) = \Gamma(\mathcal{R}, \check{H}^{F,\Omega}_{(0)}) \oplus \Gamma(\mathcal{R}, \mathcal{L}_{\mathcal{R}}).$$

$$e^{\frac{f-F}{t}} \zeta_0 = \zeta_+ \oplus \zeta_-$$

$$\zeta(\mathcal{L}, \zeta_0) = \lim_{\leftarrow N} \zeta_+$$

$\exists! \zeta$

$$\Rightarrow e^{\frac{F-f}{t}} \zeta(R_N) \in \zeta_0 \oplus (\mathcal{L} \otimes_{\mathbb{C}} R_N) \quad \forall N.$$

IV. Examples.

$f : \mathbb{C}^* \to \mathbb{C}$; $f(z) = z + \frac{q}{z}$. $q \in \mathbb{C}^*$.

(under $\mathbb{C}^*$: X)

$\mathrm{Jac}(f) = \mathbb{C}[z, z^{-1}] / (1 - \frac{q}{z^2}) \simeq \mathbb{C}[z] / (z^2 - q)$ $\left(= QH^*(\mathbb{P}^1) \right)$

$F(z,u) = z + \frac{q e^{u_1}}{z} + u_0 : \mathbb{C}^* \times \mathbb{C}^2 \longrightarrow \mathbb{C}$.

Euler vect. field $E = u_0 \partial_{u_0} - 2\partial_{u_1} \implies \partial_E F + z \partial_z F = F \implies KS^{-1}([F]) = E$

$\Omega_X = \frac{dz}{z}$. $\Omega = \pi_X^* \Omega_X = \frac{dz}{z}$.

Let $m \in \mathbb{Z}$ and $B_m = \mathrm{Span}_{\mathbb{C}}\{z^{m-1}, z^m\} \subset H^{f,\Omega}_{(0)}$.

① Claim: $K^f_\Omega(z^{m-1}, z^{m-1}) = K^f_\Omega(z^m, z^m) = 0$, $K^f_\Omega(z^m, z^{m-1}) \in \mathbb{C}^*$.

② $\Longrightarrow$ $\mathcal{L}_m := t^{-1} B_m[t^{-1}]$ is a good opposite filtration

③ $\zeta_0^{(m)} := z^m$ is a primitive element w.r.t. $\mathcal{L}_m$ (and $r = m$)

④ $\Longrightarrow$ $e^{\frac{F - F_0}{t}} z^m - z^m \in \mathcal{L} \otimes_{\mathbb{C}} R_N$ $\forall N$

⑤ $\overset{\text{Thm}}{\Longrightarrow}$ $\zeta_+^{(m)} = z^m$ is the primitive form associated to $(\mathcal{L}_m, \zeta_0^{(m)})$

i.e. $z^{m-1} dz$ is a primitive form in the original sense