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Stabilization of Linear Higher Derivative Gravity with Constraints

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Outline

- Introduction
  - Why higher derivative gravity?
  - Toy model: HD scalar field theory
- Linear higher derivative gravity (with constraints)
  - Helicity-2 sector
  - Helicity-1 sector
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- Summary

Introduction: Why HD gravity?

- **Consider higher derivative gravity with action (Stelle 77')**

$$S = \frac{M_P^2}{2} \int d^4x \sqrt{g} (R - 2\Lambda + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu}).$$

- **Most general action with quadratic curvature invariants in 4D.**

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- Most general action with quadratic curvature invariants in 4D.
- It is a renormalizable theory of gravity.
- It suffers from Ostrogradski’s instability.

Toy model: HD scalar field theory

- **Consider the action**

$$S = \int d^4x - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - \frac{1}{2M^2}(\nabla^\mu\nabla_\mu\phi)^2,$$

Toy model: HD scalar field theory

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with equation of motion

$$\left[\nabla^\mu\nabla_\mu - \frac{1}{M^2}(\nabla^\mu\nabla_\mu)^2\right]\phi = 0.$$

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- **Propagator:**

$$\hat{P} = \left[\nabla^\mu\nabla_\mu - \frac{1}{M^2}(\nabla^\mu\nabla_\mu)^2\right]^{-1} \propto k^{-4}.$$

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Improved renormalization property!

What is Ostrogradski’s instability?

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Any non-degenerate theory whose dynamical variable is higher than 2nd order in time derivative there exist a linear instability.

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- E.g.

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Degenerate!

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I. Non-degeneracy

- **E.g.**

$$L = -\frac{\ddot{q}q}{2} - \frac{m^2q^2}{2} \Longleftrightarrow L = \frac{\dot{q}^2}{2} - \frac{m^2q^2}{2}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

Degenerate! ***Stable!***

What is Ostrogradski's instability?

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II. In the literature there are several HD theories claimed to be stable,

they DO NOT violate the Ostrogradski's theorem, they are:

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II. In the literature there are several HD theories claimed to be stable, they DO NOT violate the Ostrogradski's theorem, they are:

1. Constrained: ghost in f(R) gravity is removed by gauge symmetry.

2. Healthy non-higher-derivative theories: Galileon, Lovelock gravity.

Toy model: HD scalar field theory

- **The instability can be found in the Hamiltonian by two different procedures**

Toy model: HD scalar field theory

- The instability can be found in the Hamiltonian by two different procedures

1. Auxiliary field method

$$S=\int d^4x-\frac{1}{2}\nabla^{\mu}\phi\nabla_{\mu}\phi-\frac{1}{2M^2}(\nabla^{\mu}\nabla_{\mu}\phi)^2+\frac{1}{2M^2}\left[\nabla^{\mu}\nabla_{\mu}\phi-\frac{M^2(\lambda-\phi)}{2}\right]^2,$$

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which can be diagonalized as

$$S=\int d^4x-\frac{1}{2}\nabla^\mu\Phi\nabla_\mu\Phi+\frac{1}{2}\nabla^\mu\Psi\nabla_\mu\Psi+\frac{M^2}{2}\Psi^2,$$

where $\phi=\Phi-\Psi$ and $\lambda=\Phi+\Psi$.

Toy model: HD scalar field theory

- **The instability can be found in the Hamiltonian by two different procedures**

2. Ostrogradski's method

$$S=\int d^4x-\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi-\frac{1}{2M^2}(\nabla^\mu\nabla_\mu\phi)^2.$$

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2. Ostrogradski's method

$$S = \int d^4x - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - \frac{1}{2M^2}(\nabla^\mu\nabla_\mu\phi)^2.$$

The canonical coordinates are defined by

$$q_1 \equiv \phi, \qquad q_2 \equiv \dot{\phi},$$

$$p_1 \equiv \frac{\delta S}{\delta\dot{\phi}}, \qquad p_2 \equiv \frac{\delta S}{\delta\ddot{\phi}}$$

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$$p_1 \equiv \frac{\delta S}{\delta\dot{\phi}}, \qquad \boxed{p_2 \equiv \frac{\delta S}{\delta\ddot{\phi}} = F(\ddot{\phi},\ldots)} \Longrightarrow \textbf{Nondegeneracy}$$

Toy model: HD scalar field theory

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$$S = \int d^4x - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - \frac{1}{2M^2}(\nabla^\mu\nabla_\mu\phi)^2 .$$

The Hamiltonian is

$$H = \int d^3x p_1 q_2 + q_1\left(-\frac{1}{2}\vec{\nabla}^2 + \frac{1}{2M^2}\vec{\nabla}^2\vec{\nabla}^2\right)q_1 - \frac{M^2 p_2^2}{2} + q_2\left(-\frac{1}{2} + \frac{\vec{\nabla}^2}{M^2}\right)q_2 .$$

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$$S = \int d^4x - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - \frac{1}{2M^2}(\nabla^\mu\nabla_\mu\phi)^2 .$$

The Hamiltonian can be diagonalized by

$$H = \int d^3x \left(\frac{p_\Phi^2}{2} + \frac{(\partial\Phi)^2}{2} \right) - \left(\frac{p_\Psi^2}{2} + \frac{(\partial\Psi)^2}{2} + \frac{M^2\Psi^2}{2} \right)$$

Massless normal mode **Massive ghost**

Linear higher derivative gravity

- **Linearize the action around constant curvature background**

$$S = \frac{M_P^2}{2}\int d^4x\sqrt{g}\,(R - 2\Lambda + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}),$$

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- **8 degrees of freedom.**

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- **2 → massless graviton → GR**

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- **2** → **massless graviton** → **GR**
- **5** → **massive graviton** → $\beta R_{\mu\nu}R^{\mu\nu}$

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- 8 degrees of freedom.
- 2 → massless graviton → GR
- 5 → massive graviton → $\beta R_{\mu\nu}R^{\mu\nu}$
- 1 → massive scalar → vanishes if $\beta+3\alpha=0$

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- $\beta=0$, $\alpha>0\ (\alpha<0)$ $\Rightarrow$ **healthy (tachyonic) f(R) gravity.**

Linear higher derivative gravity

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- $\beta+3\alpha=0$ ⇒ bigravity theory with ghostlike massive graviton.

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- $\beta = 0, \alpha > 0 \ (\alpha < 0)$ ⟹ **healthy (tachyonic) f(R) gravity.**
- $\beta + 3\alpha = 0$ ⟹ **bigravity theory with ghostlike massive graviton.**
- $\beta + 3\alpha = 0$ **and total minus sign** ⟹ **Fierz-Pauli + ghostlike GR.**

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- **Ghosts can be turned off at linear level by choosing boundary conditions, but in general they will be turned on at nonlinear level.**

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- E.g. Fierz-Pauli + ghostlike GR, at nonlinear level the Boulware–Deser ghost will be turned on, one need dRGT theory to avoid it.

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- E.g. Fierz-Pauli + ghostlike GR, at nonlinear level the Boulware–Deser ghost will be turned on, one need dRGT theory to avoid it.
- By entropic argument, the empty state will decay into some collection of positive and negative energy particles.

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- **In 3D,** $3\beta + 8\alpha = 0$ **with total minus sign** ⇒ **New Massive Gravity**

Linear higher derivative gravity

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- In 3D, $3\beta+8\alpha=0$ with total minus sign ⇒ New Massive Gravity
- The only physical degree of freedom is healthy massive graviton, the ghostlike massless graviton is a gauge degree of freedom.

Parameterization of metric fluctuation

- **The metric fluctuation around Minkowski background can be parameterized as**

$$ds^2 = -(1+2\phi)dt^2 + 2B_i dx^i dt + \left[(1-2\psi)\delta_{ij} + 2E_{ij}\right]dx^i dx^j,$$

where B_i and E_{ij} can be decomposed into helicity-0,1,2 modes,

$$B_i = \partial_i B + B_i^T,$$

$$E_{ij} = \partial_{\langle i}\partial_{j\rangle}E + \partial_{(i}E_{i)}^T + E_{ij}^{TT}.$$

Helicity-2 sector

- **The action of helicity-2 sector is**

$$S=\frac{M_P^2}{2}\int d^4x\left\{\beta\left[\ddot{E}_{ij}^2+2\dot{E}_{ij}\nabla\dot{E}^{ij}+(\nabla^2E_{ij})^2\right]+\dot{E}_{ij}^2+E_{ij}\nabla^2E^{ij}\right\}$$

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Higher derivative terms **GR**

- **Using Ostrogradski's method, the canonical coordinates are**

$$E_{ij}\equiv E_{ij},\qquad q_{ij}\equiv\dot{E}_{ij},$$

$$\pi_{ij}\equiv\frac{\delta S}{\delta\dot{E}^{ij}},\qquad p_{ij}\equiv\frac{\delta S}{\delta\ddot{E}^{ij}}.$$

Helicity-2 sector

- **The Hamiltonian is**

$$H = \frac{M_P^2}{2} \int d^3x \pi_{ij} q^{ij} + \frac{p_{ij} p^{ij}}{4\beta} - q_{ij} \left(1 + 2\beta \nabla^2\right) q^{ij} - E_{ij} \left(\nabla^2 + \beta \nabla^2 \nabla^2\right) E^{ij},$$

which is unstable because if the first term,

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$$H = \frac{M_P^2}{2}\int d^3x\left(\frac{p_{\phi ij}p_{\phi}^{ij}}{2} - \frac{\phi_{ij}\nabla^2\phi^{ij}}{2}\right) - \left(\frac{p_{\psi ij}p_{\psi}^{ij}}{2} - \frac{\psi_{ij}\nabla^2\psi^{ij}}{2} - \frac{\psi_{ij}\psi^{ij}}{2\beta}\right),$$

which contain one healthy mode and one massive ghost.

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which contain one healthy mode and one massive ghost.

- **The mass of the ghost is** $m^2=\left(-\beta\right)^{-1}$**, the choice** $\beta=0$ **makes the ghost infinitely massive, and hence become non-dynamical.**

Helicity-2 sector (constrained)

- To stabilize the helicity-2 sector, we introduce the auxiliary field λ_{ij},

$$S=\frac{M_P^2}{2}\int d^4x\left\{\beta\left[(\ddot{E}_{ij}-\lambda_{ij})^2+2\dot{E}_{ij}\nabla\dot{E}^{ij}+(\nabla^2E_{ij})^2+4\lambda_{ij}\nabla^2E_{ij}\right]+\dot{E}_{ij}^2+E_{ij}\nabla^2E^{ij}\right\}$$

where the Lorentz invariance is explicitly broken.

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- Consider introducing constraints without breaking Lorentz invariance

1. No full theory → Don't know how to couple λ_{ij} with other fields.

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where the Lorentz invariance is explicitly broken.

- Consider introducing constraints without breaking Lorentz invariance

1. No full theory → Don’t know how to couple λ_{ij} with other fields.
2. Auxiliary field only couples to $(\ddot{E}_{ij} - \nabla^2 E_{ij})$ has no improved renormalization properties.

Helicity-2 sector (constrained)

- **Ostrogradski's coordinates in the constrained theory are**

$$E_{ij} \equiv E_{ij}, \qquad q_{ij} \equiv \dot{E}_{ij}, \qquad \lambda_{ij} \equiv \lambda_{ij},$$

$$\pi_{ij} \equiv \frac{\delta S}{\delta \dot{E}^{ij}}, \qquad p_{ij} \equiv \frac{\delta S}{\delta \ddot{E}^{ij}}, \qquad p_{\lambda_{ij}} \equiv \frac{\delta S}{\delta \dot{\lambda}^{ij}} = 0,$$

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while the Hamiltonian reads

$$H = \frac{M_P^2}{2} \int d^3x \left\{ \pi_{ij} q^{ij} + \frac{p_{ij} p^{ij}}{4\beta} - q_{ij} \left(1 + 2\beta \nabla^2\right) q^{ij} - E_{ij} \left(\nabla^2 + \beta \nabla^2 \nabla^2\right) E^{ij} \right.$$

$$\left. + \lambda_{ij} \left(p^{ij} - 4\beta \nabla^2 E^{ij}\right) \right\}.$$

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Primary constraint φ_1

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$$\left. + \lambda_{ij}\left(p^{ij} - 4\beta\nabla^2 E^{ij}\right) \right\}.$$

Helicity-2 sector (constrained)

- **The secondary constraints can be generated by Dirac's consistency relation**

$$\varphi_2 : p_{ij} - 4\beta\nabla^2 E_{ij} \approx 0, \qquad \varphi_3 : \pi_{ij} - 2q_{ij} \approx 0, \qquad \varphi_4 : F(\lambda_{ij},\ldots) \approx 0,$$

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while the reduced Hamiltonian is

$$H_R = \frac{M_P^2}{2}\int d^3x \frac{1}{4}\pi_{ij}\left(1 - 2\beta\nabla^2\right)\pi^{ij} + E_{ij}\left(-\nabla^2 + 3\beta\nabla^2\nabla^2\right)E^{ij},$$

which is bounded below if $\beta > 0$**.**

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which is bounded below if $\beta > 0$.

- **Propagator**

$$\hat{P} = \left[\left(1 + 2\beta k^2\right)\left(-w_k^2\right) + \left(k^2 + 3\beta k^4\right)\right]^{-1} \propto k^{-4}.$$

Helicity-1 sector

- The action of helicity-1 modes is

$$S = \frac{M_P^2}{2}\int d^4x\,\frac{\beta}{2}\left(\dot{v}_i\dot{v}^i + v_i\nabla^2 v^i + \frac{1}{\beta}v_i v^i\right),$$

where $v_i = \sqrt{-\nabla^2}\left(B_i^T - \dot{E}_i^T\right)$, and the Hamiltonian is

$$H = \frac{M_P^2}{2}\int d^3x\,\frac{p_{vi}p_v^i}{2\beta} - \frac{\beta}{2}v_i\nabla^2 v^i - \frac{1}{2}v_i v^i.$$

The helicity-1 sector is either tachyonic if $\beta > 0$ or a massive ghost with mass $m^2 = (-\beta)^{-1}$ if $\beta < 0.$

Helicity-1 sector (constrained)

- **To stabilize helicity-1 sector, we need to constrain all the degrees of freedom by introducing auxiliary field λ_i**

$$S=\frac{M_P^2}{2}\int d^4x\,\frac{\beta}{2}\left[\left(\dot{v}_i-\lambda_i\right)^2+v_i\nabla^2v^i+\frac{1}{\beta}v_iv^i\right].$$

Helicity-1 sector (constrained)

- To stabilize helicity-1 sector, we need to constrain all the degrees of freedom by introducing auxiliary field λ_i

$$S = \frac{M_P^2}{2}\int d^4x\,\frac{\beta}{2}\left[\left(\dot{v}_i - \lambda_i\right)^2 + v_i\nabla^2 v^i + \frac{1}{\beta}v_i v^i\right].$$

- The theory admits four constraints which eliminate all the degrees of freedom, and make the reduced

$$\varphi_1 : p_\lambda^i = 0, \qquad \varphi_4 : \left(\frac{1}{\beta} + \nabla^2\right)\left(p_v^i + \beta\lambda^i\right) \approx 0,$$

$$\varphi_2 : p_v^i \approx 0, \qquad \varphi_3 : v^i + \beta\nabla^2 v^i \approx 0.$$

Helicity-0 sector

- **The action of helicity-0 modes is**

$$S=\frac{M_P^2}{2}\int d^4x\Big[\left(-6\dot{\Psi}^2-2\Psi\nabla^2\Psi+4\Psi\nabla^2\Phi\right)$$

$$+4\left(\beta+3\alpha\right)\left(3\ddot{\Psi}^2+4\dot{\Psi}\nabla^2\Psi+2\ddot{\Psi}\nabla^2\Phi\right)$$

$$+2\left(3\beta+8\alpha\right)\left(\nabla^2\Psi\right)^2+2\left(\beta+2\alpha\right)\left(\nabla^2\Phi\right)^2-4\left(\beta+4\alpha\right)\nabla^2\Psi\nabla^2\Phi\Big]$$

where $\Phi\equiv\phi+\dot{B}-\ddot{E},\ \Psi\equiv\psi+\frac{1}{3}\nabla^2E$ **are gauge invariant variables.**

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- Φ **is an auxiliary field generating 2 constraints (4 if** $\beta = 0$**).**
- $\beta + 3\alpha = 0$ → **no longer higher derivative theory.**

Helicity-0 sector

- **The constraints of helicity-0 modes are**

$$\varphi_1 : p_\Phi = 0, \qquad \varphi_2 : \nabla^2\left[\frac{p_\chi}{3} + 4\Psi - 4(\beta + 4\alpha)\nabla^2\Psi + \frac{4\beta}{3}\nabla^2\Phi\right] \approx 0,$$

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with the diagonalized reduced Hamiltonian

$$H_R = \frac{M_P^2}{2}\int d^3x \left[\left(\frac{P_1^2}{2} - \frac{1}{2} Q_1 \nabla^2 Q_1 + \frac{1}{2} m_1^2 Q_1^2 \right) - \left(\frac{P_2^2}{2} - \frac{1}{2} Q_2 \nabla^2 Q_2 + \frac{1}{2} m_2^2 Q_2^2 \right) \right],$$

where

$$m_1^2 = \frac{1}{2(\beta + 3\alpha)}, \qquad m_2^2 = \frac{1}{(-\beta)}.$$

Helicity-0 sector (constrained)

- **Similarly, stabilize helicity-0 sector by modifying the action by**

$$
\begin{aligned}
S = \frac{M_P^2}{2}\int d^4x\Big[&\Big(-6\dot{\Psi}^2 - 2\Psi\nabla^2\Psi + 4\Psi\nabla^2\Phi\Big)\\
&+4\left(\beta+3\alpha\right)\left(3\ddot{\Psi}^2 + 4\dot{\Psi}\nabla^2\Psi + 2\ddot{\Psi}\nabla^2\Phi\right)\\
&+2\left(3\beta+8\alpha\right)\left(\nabla^2\Psi\right)^2 + 2\left(\beta+2\alpha\right)\left(\nabla^2\Phi\right)^2 - 4\left(\beta+4\alpha\right)\nabla^2\Psi\nabla^2\Phi\\
&+4\left(\beta+3\alpha\right)\lambda\left(8\nabla^2\Psi + 3\lambda - 6\ddot{\Psi} - 2\nabla^2\Phi\right) - 8\lambda\Psi\Big].
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- **The constrained theory admits six constraints which can be used to remove three degrees of freedom λ, Φ, $\dot{\Psi}$.**

Helicity-0 sector (constrained)

$$H_R = \frac{M_P^2}{2}\int d^3x\left\{\frac{1}{8}p_\Psi\left[1-8\left(\beta+3\alpha\right)\nabla^2\right]p_\Psi\right.$$

$$\left.+\Psi\left[\frac{2\left(\beta+\alpha\right)}{\beta\left(\beta+3\alpha\right)}+\left(-2+\frac{16\alpha}{\beta}\right)\nabla^2+\frac{32\left(\beta+\alpha\right)\left(\beta+3\alpha\right)}{\beta}\nabla^2\nabla^2\right]\Psi\right\}.$$

Helicity-0 sector (constrained)

- **The reduced Hamiltonian is**

$$H_R = \frac{M_P^2}{2}\int d^3x \left\{ \frac{1}{8} p_\Psi \left[1 - 8(\beta + 3\alpha)\nabla^2\right] p_\Psi \right.$$

$$\left. + \Psi \left[\frac{2(\beta+\alpha)}{\beta(\beta+3\alpha)} + \left(-2 + \frac{16\alpha}{\beta}\right)\nabla^2 + \frac{32(\beta+\alpha)(\beta+3\alpha)}{\beta}\nabla^2\nabla^2 \right]\Psi \right\}.$$

- **It I bounded from below if**

1. $\alpha \le 0, \quad \beta + 3\alpha > 0.$

2. $\alpha > 0, \quad -3\alpha < \beta < -\alpha$ **or** $\beta > 8\alpha.$

de Sitter background: helicity-2 example

- **The action around dS space is**

$$S = \frac{M_P^2}{2}\int d^4x\left\{\beta\left[\ddot{E}_{ij}^2 + 2\dot{E}_{ij}\nabla\dot{E}^{ij} + (\nabla^2 E_{ij})^2\right] + ca^2(t)\left[\dot{E}_{ij}^2 + E_{ij}\nabla^2 E^{ij}\right]\right\},$$

where $c \equiv 1 + 8H^2\left(\beta + 3\alpha\right).$

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where $c \equiv 1 + 8H^2\left(\beta + 3\alpha\right).$

- **Could have two stable modes if** $c < 0.$
- **However,** $c < 0$ **violates the assumption that higher derivative terms are corrections of GR. In dS background,** $R \sim H^2$**, we thus expect**

$$\left|H^2\left(\beta + 3\alpha\right)\right| << 1.$$

Summary

- Higher derivative gravity with quadratic curvature invariant $\alpha R^2 + \beta R^{\mu\nu} R_{\mu\nu}$ is a renormalizable theory but suffers from Ostrogradski's instability.
- Ostrogradski's instability in higher derivative theory can be saved by additional constraints only if the dimensionality of phase space is reduced.
- Applying the same method to linearized HD gravity around Minkowski background, the instabilities in different helicities can be consistently removed by suitable auxiliary fields while preserving the improved renormalizable properties if the Lorentz invariance is explicitly broken.
- Similar result applies to the same theory around de Sitter background, with the reasonable assumption that the higher derivative terms are correction to usual General Relativity.

Thank you!