FQH/CFT and q-CFT

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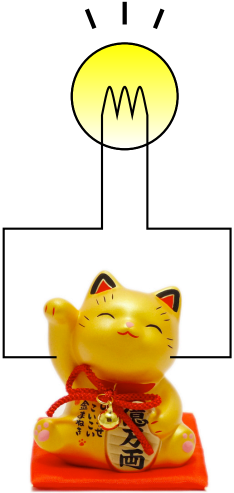
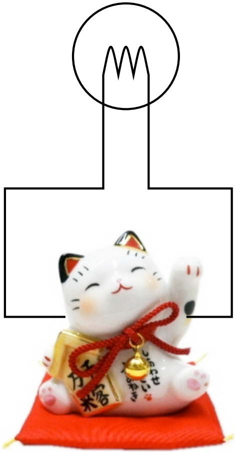




- How to distinguish metal and insulator?

Electric conductivity

$$J \quad = \quad \sigma \quad \times \quad E$$

(current = **conductivity** × voltage)

$\sigma = 0$: **insulator**, $\sigma \neq 0$: **metal**

- 2 dimensions
  - σ : conductivity matrix

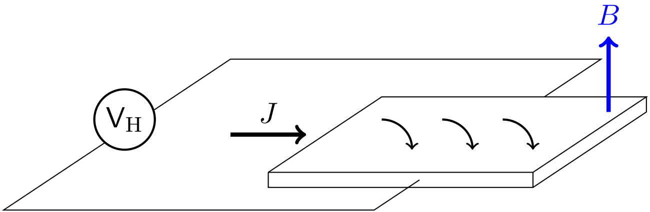
$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \to \begin{pmatrix} \sigma_{\mathrm{L}} & \sigma_{\mathrm{H}} \\ -\sigma_{\mathrm{H}} & \sigma_{\mathrm{L}} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

  - $\sigma_{xx} = \sigma_{yy} = \sigma_{\mathrm{L}}, \quad \sigma_{xy} = -\sigma_{yx} = \sigma_{\mathrm{H}}$

$\sigma_{\mathrm{L}} = 0$: **insulator**, $\sigma_{\mathrm{L}} \neq 0$: **metal**

  - What's the role of σ_{H}?

Hall effect

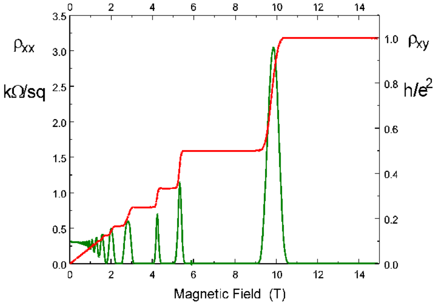


(Classical) Hall conductivity: $\sigma_{\mathrm{H}} \sim B$

- cf. resistivity matrix: $E_\mu = \rho_{\mu\nu} J^\nu$

$$\begin{pmatrix} \rho_{\mathrm{L}} & \rho_{\mathrm{H}} \\ -\rho_{\mathrm{H}} & \rho_{\mathrm{L}} \end{pmatrix} \equiv \begin{pmatrix} \sigma_{\mathrm{L}} & \sigma_{\mathrm{H}} \\ -\sigma_{\mathrm{H}} & \sigma_{\mathrm{L}} \end{pmatrix}^{-1} = \frac{1}{\sigma_{\mathrm{L}}^2 + \sigma_{\mathrm{H}}^2} \begin{pmatrix} \sigma_{\mathrm{L}} & -\sigma_{\mathrm{H}} \\ \sigma_{\mathrm{H}} & \sigma_{\mathrm{L}} \end{pmatrix}$$

- Longitudinal and transverse resistivity



- Longitudinal mode

$$\rho_{\rm L} = 0 \ \rightarrow \ \underline{\sigma_{\rm L} = 0}$$

- Transverse mode

$$\sigma_{\rm H} = \nu \frac{e^2}{h} \quad \text{w/} \quad \underline{\nu \in \mathbb{Z}}$$

Quantum Hall effect

1. Hall conductivity is quantized with high-precision
2. $\sigma_{\rm L} = 0$, but $\sigma_{\rm H} \neq 0$: **insulator with non-trivial response**

- What's a field theory for QHE?
  1. $d = 2 + 1$
  2. Parity-broken (due to B)

Chern–Simons theory

$$S_{\rm CS} = \frac{k}{4\pi} \int \mathcal{A}\, d\mathcal{A}\,, \quad k \in \mathbb{Z}$$

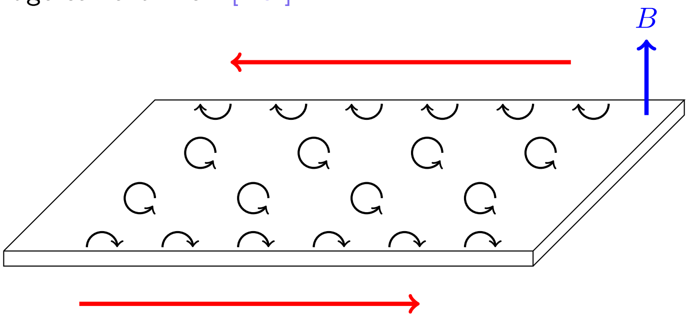
- Electric current

$$J_\mu = \frac{\delta S_{\rm CS}}{\delta A^\mu} = \frac{k}{2\pi} \epsilon_{\mu\nu\rho}\, \partial^\nu A^\rho \xrightarrow{\nu = t} \frac{k}{2\pi} \epsilon_{\mu\nu} E^\nu \equiv \sigma_{\mu\nu} E^\nu$$

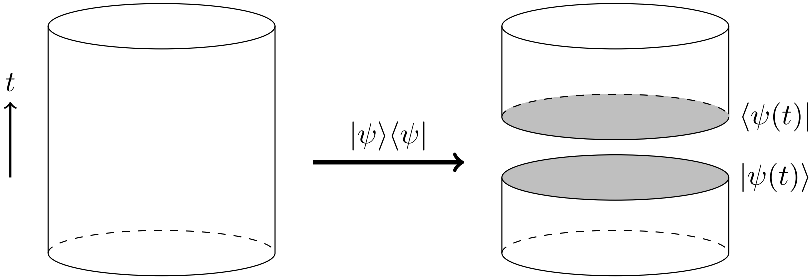
$$\sigma_{\rm H} = \frac{k}{2\pi} \;\to\; \frac{k}{2\pi}\frac{e^2}{\hbar} = k\,\frac{e^2}{h}\,, \quad \sigma_{\rm L} = 0$$

- Chern–Simons is topological:
  - non-dynamical in **the bulk**
  - unless **the boundary** $\rightarrow$ CFT on the boundary

1. Edge current   cf. [Wen]



2. Bulk wavefunction [Moore–Read]



bulk wavefunction = conformal block

- Ex.: Laughlin wavefunction

$$\Phi_{\mathrm{L}}(\{z_i\}) = \prod_{i<j}^{N} (z_i - z_j)^m$$

  - Vertex operator: $V(z) = e^{i\alpha\varphi(z)}$ with $\alpha = \sqrt{m}$

$$\Phi_{\mathrm{L}}(\{z_i\}) = \Big\langle V(z_1) V(z_2) \cdots V(z_N) \Big\rangle$$

- Bulk wavefunction = conformal block
  - Particle statistics $\longleftrightarrow$ CFT fusion rule (monodromy)
  - Non-Abelian & q-deformed CFT

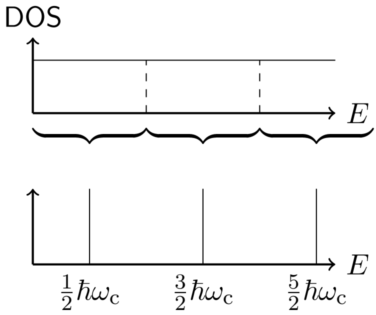
Summary

- Quantum Hall effect
  - quantized Hall conductivity $\sigma_{\mathrm{H}} = \nu e^2/h$
  - insulator with non-trivial response ($\sigma_{\mathrm{L}} = 0$ but $\sigma_{\mathrm{H}} \neq 0$)
- Chern–Simons effective theory
  - level quantization $=$ quantization of σ_{H}
  - non-dynamical bulk & CFT on the boundary
    1. edge state
    2. bulk wavefunction

- Quantum mechanics in magnetic field ($d = 2$)

$$\mathcal{H} = \frac{1}{2m}\left(\vec{p} - \vec{A}\right)^2 = \hbar\omega_{\mathrm{c}}\left(a^\dagger a + \frac{1}{2}\right)$$

  - cyclotron freq.: $\omega_{\mathrm{c}} \sim B$



$$\text{DOS} = \sum_n N_\phi \, \delta(E - E_n)$$

degeneracy of Landau level

$$N_\phi = \frac{1}{2\pi}\int d^2x \, B$$

- Lowest Landau level (LLL)
  - annihilation op.: $a = \partial_{\bar{z}} + \frac{z}{4\ell_B^2}$
  - magnetic length: $\ell_B = \sqrt{\frac{\hbar}{eB}}$

$$a\,\phi_m(z,\bar{z}) = 0 \;\to\; \phi_m(z,\bar{z}) = z^m\, e^{-\frac{1}{4\ell_B^2}|z|^2}$$

  - LLL state is labeled by angular momentnum index:

$$m = 0,\, 1,\, \cdots,\, N_\phi - 1$$

- Many-body wavefunction in LLL

$$\Psi(\{z_i\}) = \Phi(\{z_i\}) \prod_{i=1}^{N} e^{-\frac{1}{4\ell_B^2}|z_i|^2}$$

  - (anti-)symmetric polynomial: $\Phi(\{z_i\}) = z_1^{m_1} \cdots z_N^{m_N} + \cdots$
  - Highest power in $\Phi(\{z_i\})$ is $N_\phi - 1$

- How to read σ_{H} from $\Phi(\{z_i\})$?

$$J_\mu = \frac{\nu}{2\pi} \epsilon_{\mu\lambda\rho} \partial^\lambda A^\rho \quad \xrightarrow{\mu = t} \quad \nu = \frac{2\pi J_t}{B}$$

- Filling fraction:

$$\nu = \frac{\#\text{particles}}{\#\text{magnetic fluxes}} = \frac{N}{N_\phi}$$

- Hall conductivity:

$$\sigma_{\mathrm{H}} = \nu \frac{e^2}{h}$$

- Ex. 1: Slater determinant

$$\Phi(\{z_i\}) = \det\left(z_i^{j-1}\right) = \prod_{i<j}^{N}(z_i - z_j)$$

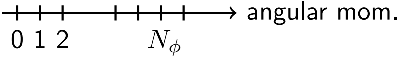
  - Highest power:

$$\Phi(\{z_i\}) = z_1^0 z_2^1 \cdots z_N^{N-1} + \cdots \;\rightarrow\; N_\phi = N$$

  - Occupation number representation:

"Fermi sea"

$$|\Phi\rangle = |1\;1\;1\cdots 1\;1\,\vdots\,0\;0\cdots\rangle$$



  - Filling fraction: $\nu = \dfrac{N}{N_\phi} = 1$

$\nu = 1$ integer QH state

- Ex. 2: Laughlin state

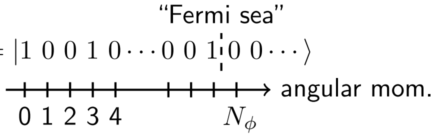
$$\Phi_{\mathrm{L}}(\{z_i\}) = \prod_{i<j}^{N} (z_i - z_j)^m$$

  - Highest power:

$$\Phi_{\mathrm{L}}(\{z_i\}) = z_1^0 z_2^m \cdots z_N^{m(N-1)} + \cdots \ \rightarrow \ N_\phi - 1 = m(N-1)$$

  - Occupation number representation:

$$|\Phi_{\mathrm{L}}\rangle = |1\ 0\ 0\ 1\ 0 \cdots 0\ 0\ 1 \,\vdots\, 0\ 0 \cdots\rangle$$



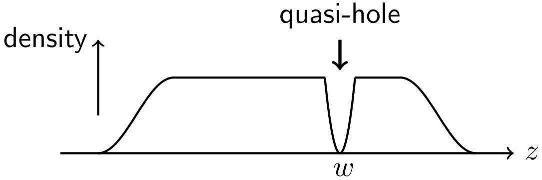
  - Filling fraction: $\nu = \dfrac{N}{m(N-1)+1} \overset{N\to\infty}{\longrightarrow} \dfrac{1}{m}$

$\nu = 1/m$ **fractional QH state**

- Excitation state: quasi-hole state

$$\Phi_{\mathrm{L}}(\{z_i\}|w) = \prod_{i<j}^{N}(z_i - z_j)^m \prod_{i=1}^{N}(z_i - w)$$

  - Φ_{L} has zeros at $z_i = w$



- Multi-quasi-hole state

$$\Phi_{\mathrm{L}}(\{z_i\}|\{w_i\}) = \prod_{i<j}^{N}(z_i - z_j)^m \prod_{i=1}^{N}\prod_{j=1}^{M}(z_i - w_j) \prod_{i<j}^{M}(w_i - w_j)^{1/m}$$

- Connection to CFT
  - Electron operator: $V_{\mathrm{e}}(z) = e^{i\sqrt{m}\varphi(z)}$

$$\Phi_{\mathrm{L}}(\{z_i\}) = \Big\langle V_{\mathrm{e}}(z_1) V_{\mathrm{e}}(z_2) \cdots V_{\mathrm{e}}(z_N) \Big\rangle$$

  - $\mathrm{U}(1)$ free boson: $\varphi(z)\varphi(w) = -\log(z-w)$

- Quasi-hole state
  - Quasi-hole operator: $V_{\mathrm{qh}}(z) = e^{i\sqrt{1/m}\,\varphi(z)}$

$$\Phi_{\mathrm{L}}(\{z_i\}|\{w_i\}) = \Big\langle V_{\mathrm{e}}(z_1) \cdots V_{\mathrm{e}}(z_N) V_{\mathrm{qh}}(w_1) \cdots V_{\mathrm{qh}}(w_M) \Big\rangle$$

Summary

- LLL many-body wavefunction

$$\Psi(\{z_i\}) = \Phi(\{z_i\}) \prod_{i=1}^{N} e^{-\frac{1}{4\ell_B^2}|z_i|^2}$$

  - (anti-)symmetric polynomial: $\Phi(\{z_i\}) = z_1^{m_1} \cdots z_N^{m_N} + \cdots$

- Filling fraction: $\nu = \dfrac{\#\text{particles}}{\#\text{magnetic fluxes}} = \dfrac{N}{N_\phi}$

- Conformal block as a wavefunction

$$\Phi(\{z_i\}) = \Big\langle V_{\mathrm{e}}(z_1) V_{\mathrm{e}}(z_2) \cdots V_{\mathrm{e}}(z_N) \Big\rangle$$

- Electron operator
  - Laughlin: $\mathrm{U}(1)$ free boson

$$V_\mathrm{e}(z) = e^{i\sqrt{m}\,\varphi(z)} \longrightarrow \Phi_\mathrm{L}(\{z_i\}) = \prod_{i<j}^{N} (z_i - z_j)^m$$

  - Moore–Read: $\mathrm{SU}(2)_{k=2}$ (Ising; Majorana) [Moore–Read]

$$V_\mathrm{e}(z) = \psi(z)\, e^{i\varphi(z)} \longrightarrow \Phi_\mathrm{MR}(\{z_i\}) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j}^{N} (z_i - z_j)^2$$

  - Read–Rezayi: $\mathrm{SU}(2)_k$ ($\mathbb{Z}_k$-parafermion) [Read–Rezayi]

$$V_\mathrm{e}(z) = \psi_1(z)\, e^{i\sqrt{2/k}\,\varphi(z)}$$

Non-Abelian particle statistics (NA anyon)

- Moore–Read state

$$\Phi_{\mathrm{MR}}(\{z_i\}) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}^{N}(z_i - z_j)^2$$

  - Filling fraction: $\nu = 1/2$ **(even denominator!)**
  - Ising CFT: $\mathrm{SU}(2)_{k=2}/\mathrm{U}(1)$
  - Electron & quasi-hole operators

$$V_{\mathrm{e}}(z) = \psi(z)\, e^{i\varphi(z)}\,, \quad V_{\mathrm{qh}}(w) = \sigma(w)\, e^{i\varphi(w)/2}$$

  - Quasi-hole state

$$\Phi_{\mathrm{MR}}(\{z_i\}|\{w_i\}) \sim \left\langle \psi(z_1)\cdots\psi(z_N)\sigma(w_1)\cdots\sigma(w_M)\right\rangle$$

Fusion rules

$$\psi \times \psi = \mathbb{1}, \quad \psi \times \sigma = \sigma, \quad \sigma \times \sigma = \mathbb{1} + \psi$$

- Bosonic Moore–Read state

$$\Phi_{\mathrm{MR}}(\{z_i\}) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}^{N}(z_i - z_j)$$

  - Filling fraction: $\nu = 1$ **(odd denominator!)**
  - Occupation number rep.:

$$|\Phi_{\mathrm{MR}}\rangle = |2\ 0\ 2\ 0 \cdots 0\ 2\ 0 \cdots\rangle$$

- Bosonic Read–Rezayi state: $\mathrm{SU}(2)_k$

$$|\Phi_{\mathrm{RR}_k}\rangle = |k\ 0\ k\ 0 \cdots 0\ k\ 0 \cdots\rangle$$

- More generic NA state: **Jack state** [Bernevig–Haldane]

Jack polynomial

$$\Phi(\{z_i\}) = J^{(\alpha)}(\{z_i\})$$

- 1-parameter generalization of Schur polynomial
- Laplace–Beltrami op. (Calogero–Sutherland model)

$$\mathcal{L}_{\mathrm{LB}} = \sum_{i=1}^{N} \left(z_i \frac{\partial}{\partial z_i} \right)^2 + \frac{1}{\alpha} \sum_{i<j}^{N} \frac{z_i + z_j}{z_i - z_j} \left(z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right)$$

- (k,r)-admissible state at $\alpha = -\dfrac{k+1}{r-1}$ [Feigin et al.]

$$|\Phi_{(k,r)}\rangle = |k \ \underbrace{0\,0\cdots 0}_{r-1} \ k \ 0 \cdots \rangle$$

- Examples:

Laughlin $= (1,r)$, Moore–Read $= (2,2)$, Read–Rezayi $= (k,2)$

- Underlying CFT: $\mathrm{WA}_{k-1}(k+1, k+r)$ [Bernevig–Gurarie–Simon]

Summary

- Non-Abelian FQH state

$$\Phi(\{z_i\}|\{w_i\}) = \Big\langle V_{\mathrm{e}}(z_1)\cdots V_{\mathrm{e}}(z_N) V_{\mathrm{qh}}(w_1)\cdots V_{\mathrm{qh}}(w_M)\Big\rangle$$

  - Electron & quasi-hole operators

$$V_{\mathrm{e}}(z) = \psi(z)\, e^{i\varphi(z)}\,, \quad V_{\mathrm{qh}}(w) = \sigma(w)\, e^{i\varphi(w)/2}$$

  - Fusion rules $\rightarrow$ NA statistics

$$\psi\times\psi = \mathbb{1}, \quad \psi\times\sigma = \sigma, \quad \sigma\times\sigma = \mathbb{1}+\psi$$

- (k,r)-admissible state: Jack polynomial at $\alpha = -\dfrac{k+1}{r-1}$

- Spinful FQH state [Halperin]

Halperin state

$$\Phi_{\mathrm{H}}(\{z_i\},\{w_i\}) = \prod_{i=1}^{N^\uparrow}(z_i - z_j)^r \prod_{i=1}^{N^\downarrow}(w_i - w_j)^r \prod_{i,j}^{N^{\uparrow,\downarrow}}(z_i - w_j)^s$$

z_i: up-spin particle, w_i: down-spin particle

- Spin operator

$$S_i^+ : w_i \longrightarrow z_{N^\uparrow+1}, \quad S_i^- : z_i \longrightarrow w_{N^\downarrow+1}, \quad S_z = \frac{N^\uparrow - N^\downarrow}{2}$$

- Spin-singlet condition: $\mathrm{SU}(2)$ invariance

$$S^{\pm}\Phi_{\mathrm{H}} = 0, \; S_z \Phi_{\mathrm{H}} = 0 \; \longrightarrow \; r = s + 1, \; N^{\uparrow} = N^{\downarrow} = N$$

$\mathrm{SU}(2)$ spin-singlet Halperin state

$$\Phi_{\mathrm{H}}(\{z_i\}, \{w_i\}) = \prod_{i=1}^{N} (z_i - z_j)^r \prod_{i=1}^{N} (w_i - w_j)^r \prod_{i,j=1}^{N} (z_i - w_j)^{r-1}$$

- $\mathrm{SU}(M)$ generalization: $E_a \Phi_{\mathrm{H}} = 0, \; F_a \Phi_{\mathrm{H}} = 0, \; H_a \Phi_{\mathrm{H}} = 0$

$\mathrm{SU}(M)$-singlet Halperin state

$$\Phi_{\mathrm{H}}(\{z_i^{(u)}\}) = \prod_{u=1}^{M} \prod_{i=1}^{N} (z_i^{(u)} - z_j^{(u)})^r \prod_{u<v}^{M} \prod_{i,j=1}^{N} (z_i^{(u)} - z_j^{(v)})^{r-1}$$

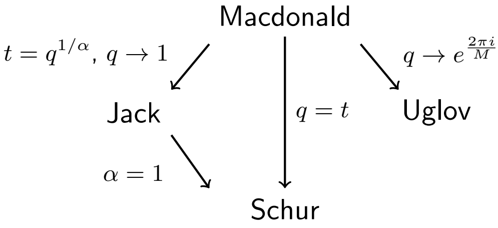
- Non-Abelian spin-singlet state
  - Generalized parafermion: $\mathrm{SU}(3)_k/\mathrm{U}(1)^2$ [Ardonne–Schoutens]
- Jack state
  - non-symmetric Jack [Ardonne–Regnault] [Estienne–Bernevig]
  - Uglov polynomial [TK]

- Non-symmetric Jack: spin Laplace–Beltrami operator

$$\mathcal{L}_{\mathrm{sLB}} = \mathcal{L}_{\mathrm{LB}} - \frac{1}{\alpha} \sum_{i \neq j} (1 - K_{ij}) \frac{z_i z_j}{(z_i - z_j)^2}$$

  - K_{ij}: particle exchange operator $\sim \vec{S}_i \cdot \vec{S}_j$
  - Spectrum degeneracy due to the spin symmetry

- Uglov polynomial is [Uglov]
  - Yangian basis for spin Calogero–Sutherland model
  - given by **the root of unity limit** of **Macdonald polynomial**



Let's consider q-CFT, and then take the limit $q \to e^{\frac{2\pi i}{M}}$

- Ex.: q-Laughlin state
  - q-boson with $t = q^r$ [Shiraishi–Kubo–Awata–Odake]

$$[a_n, a_m] = n \frac{1-q^{|n|}}{1-t^{|n|}} \delta_{n+m,0}\,, \qquad [a_n, Q] = \frac{1}{r}\delta_{n,0}$$

  - OPE at the root of unity limit

$$\begin{aligned} \varphi(z)\varphi(w) \quad &\sim \quad -\log\left[\frac{(w/z;q)_\infty}{(tw/z;q)_\infty} z^r\right] \\ &\to \quad -\log\left[\left(z^M - w^M\right)^{(r-1)/M}(z-w)\right] \end{aligned}$$

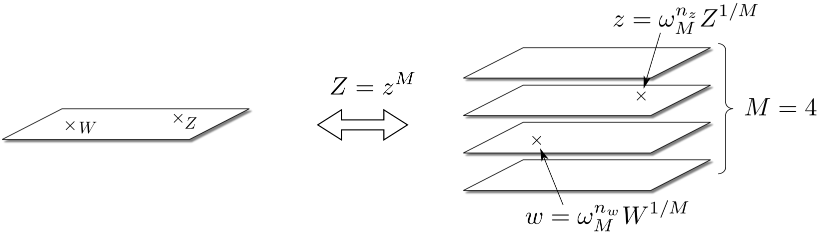
  - $Z = z^M$, $W = w^M \ \rightarrow\ z = \omega_M^{n_z} Z^{1/M}$, $w = \omega_M^{n_w} W^{1/M}$

$$\varphi(Z)\varphi(W) \sim \begin{cases} -\log(Z-W)^{(r-1)/M+1} & (n_z = n_w) \\ -\log(Z-W)^{(r-1)/M} & (n_z \neq n_w) \end{cases}$$

- Conformal block at the root of unity of q-CFT

$$\prod_{u=1}^{M}\prod_{I<J}^{N^{(u)}}\left(z_I^{(u)}-z_J^{(u)}\right)^{(r-1)/M+1}\prod_{u<v}^{M}\prod_{I,J}^{N^{(u,v)}}\left(z_I^{(u)}-z_J^{(v)}\right)^{(r-1)/M}$$

This is the $\mathrm{SU}(M)$ spin-singlet Halperin state!



Branch of $Z = z^M$ distinguishes the particle state

Summary

- Spin-singlet FQH states:
  - Halperin state
  - Generalized parafermion state: $\dfrac{\mathrm{SU}(3)_k}{\mathrm{U}(1)^2} \;\to\; \dfrac{\mathrm{SU}(M+1)_k}{\mathrm{U}(1)^M}$
- Spin-singlet Jack state
  - Non-symmetric Jack
  - Uglov polynomial (Macdonald at the root of unity limit)

Summary

- Conformal block = FQH wavefunction
  - Laughlin: $\mathrm{U}(1)$
  - Moore–Read: $\mathrm{SU}(2)_{k=2}/\mathrm{U}(1)$
  - Read–Rezayi: $\mathrm{SU}(2)_k/\mathrm{U}(1)$
  - NA spin-singlet: $\mathrm{SU}(3)_k/\mathrm{U}(1)^2$

- FQH wavefunction is characterized by symmetric polynomial
  - Generically described by Jack polynomial

$$\Psi(\{z_i\}) = \Phi(\{z_i\}) \prod_{i=1}^{N} e^{-\frac{1}{4\ell_B^2}|z_i|^2}$$

- Spin-singlet Jack state
  - Uglov polynomial obtained from Macdonald polynomial

Discussion

- FQH/q-CFT
  - Refined Chern–Simons theory [Aganagic–Shakirov]
- Holography of the non-Abelian FQH state
  - Minimal model holography [Gaberdiel–Gopakumar]
- q-CFT and gauge theory at $q \to e^{\frac{2\pi i}{M}}$
  - Instanton counting on $\mathbb{C}^2/\mathbb{Z}_M$ [TK] [Itoyama–Oota–Yoshioka]