Dirac graphs: Local Lorentz symmetry from the graph

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Outline

1. Introductory part (ca. 20min.)
    - Efective vs. exact Lorentz symmetry
    - Dirac lattices
    - Dirac graphs
    - (Quasi)Conclusion
2. Technical part (ca. 50min.)
    - Pseudo Fermi points
    - Existence and properties of pFps
    - Continuum limit: Dirac fermion
    - Graph deformations: coupling to external gauge and gravity fields
    - Conclusion&Outlook

☞ Based on: work in progress... see also **1112.5937**, and **1012.5354**

Lorentz symmetry: exact?...

- QFT models of HEP are relativistic theories, based on Lorentz symmetry group
- In gravity it is the local symmetry group
- In the observable range of energies the Lorentz symmetry is an exact symmetry, no violation observed so far
- But must it be indeed an exact symmetry?

. . . or effective?

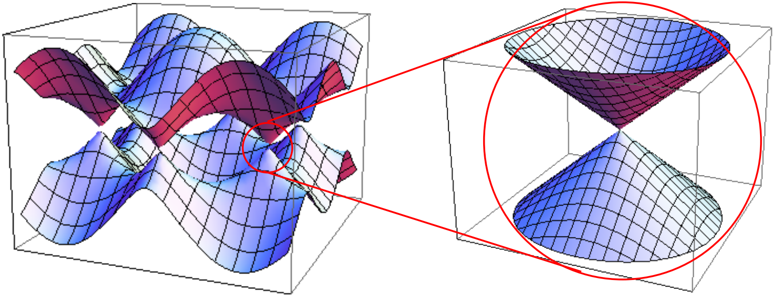
- [Penrose'70s]: *"The concept based on continuous space and Lorentz symmetry is faulty and should be replaced in the theory including quantum gravity. There are other models where the space-time and Lorentz symmetry emerge in an effective description e.g. in low energy theory". . .*
- A possibility: The Lorentz symmetry could be the effective symmetry at IR (Lifshitz) point [Hořava'08-09]

...in Fermi systems

- In some CMT models Lorentz symmetry emerges at the IR fixed point of initially non relativistic dynamics
  - Graphene
  - He^3
  - Topological insulators
  - etc...
- In all these cases the Fermi surface degenerates to a point (or set of disjoint points) and the dispersion relations are linearly non-degenerate
- The degrees of freedom are organised into Dirac fermion(s)

. . . in graphene

The graphene's dispersion relation



. . . accidental?

- In all known examples the Dirac fermion emerges as soon as
  - The Fermi surface degenerates to a point
  - The spectrum of fluctuations around each Fermi point is linearly non-degenerate
- In fact, ABS construction [Atiyah–Bott–Shapiro'64] implies that this spectrum is non-degenerate only when the fluctuations can be organised into a faithful representation of the Clifford algebra ⇒ Dirac fermion ⇒ Lorentz symmetry. . . So it's a universal property!

Stable and non-stable Fermi points

- In some cases one can find non-trivial topological charges related to the Fermi point [Volovik'01]. This implies stability of such systems against small deformations of the system
- More generally, the Fermi surfaces can be naturally classified in terms of K-theory [Hořava'05]
- We can design the dynamical system (e.g. a discrete model) so that they have stable Fermi points. The effective models for fluctuations around this points will be Weyl/Dirac fermions and Lorentz symmetry

Dirac systems

- 2D lattice systems with stable Fermi points in 2D were studied by [Asano-Hotta'11...]
- Arbitrary graphs leading to Dirac fermions (arbitrary dimension) we call Dirac lattices [CS'13]
- Local deformations of the graph from the Dirac lattice appear as gauge field and Yukawa field modes coupled to the Dirac fermion (in $D = 2 + 1$) [CS'12]
- In $D > 2 + 1$ we expect to get coupling to gravity as well

Dirac lattice

- The Model: fermi particle hopping on a graph specified by the adjacency matrix T (see the main part)
  - Non-relativistic
  - Dynamical
- Absolute time is the dynamical parameter which exists *ab initio*
- The structure of the adjacency matrix:
  - $\Rightarrow$ Non-degenerate linear dispersion relations
  - $\Rightarrow$ Dirac fermion

Dirac lattices → Dirac graphs

- Absolute time + Spacial geometry inherited from the graph = Lorentz invariance
- Product geometry: we don't have any control over the time extension
- Deformations of the lattice will not produce a generic geometry
- A recipe to cure: generalise to a model where the time extension is effectively generated from the graph as well
- We call such models Dirac graphs✍
  (better name suggestions accepted)

Dirac graph model

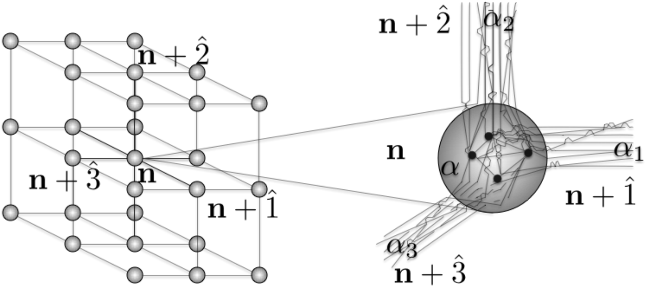
- No dynamics $\Rightarrow$ Statistical model with 'partition function'

$$Z(T) = \int [\mathrm{d}\psi^\dagger][\mathrm{d}\psi]\mathrm{e}^{\mathrm{i}S[\psi,\psi^\dagger]}$$
$$S = \psi^\dagger \cdot T \cdot \psi$$

  T is the adjacency matrix
- We start with 'rigid' T obeying some properties like translational symmetry, etc.
- 'Saddle point integral' well defined as an analytic continuation (Wick rotation) $\Rightarrow$ can define an analogue of Fermi surface: call it pseudo-Fermi surface
- When does it describe a Dirac fermion in low energy (saddle point) limit?

(More) on translational invariance

Translation invariant T corresponds to a D-dimensional lattice generated by translations of the unit cell graph



Our findings

- We considered the simplest nontrivial case: $d_{\text{internal space}} = 2$
- For a region of parameter space a generic adjacency matrix T has a set of linear non degenerate pseudo-Fermi points when $D \leq 4$
- When $D = 4$ the pFp's are stable and the continuum limit is described by $D = 3 + 1$ Dirac fermion
- We considered local deformation of the graph $T \mapsto T + \mathfrak{A}$ by nearest cell operators $\mathfrak{A}$
- In this case the Dirac field is coupled to (non-dynamical) deformation-induced Abelian gauge field and gravity background

Break!

The model

- We consider ‘partition function’

$$Z(T) = \int [\mathrm{d}\psi^\dagger][\mathrm{d}\psi]\mathrm{e}^{\mathrm{i}S[\psi,\psi^\dagger]}$$

  with the ‘action’: $S = \psi^\dagger \cdot T \cdot \psi$

- The Hermitian adjacency matrix T is non-dynamical and possesses a D-dimensional translational symmetry

The Adjacency Matrix

- The translational symmetry $\Rightarrow$ block structure of adjacency matrix

$$T = \alpha + \sum_I \alpha^I T_I + \sum_I \alpha^{I\dagger} T_I^{-1} + \sum_{IJ} \alpha^{IJ} T_I T_J + \dots$$

  where T_I, $I = 1, \dots, D$ are generators of translations; α, α^I etc. commute with all T_I

- Locality: $T = \alpha + \sum_I \left(\alpha^I T_I + \alpha^{I\dagger} T_I^{-1}\right)$

➪ Dimensions of 'internal space' are related to D by ABS construction. Simplest case: $d_{\text{internal space}} = 2 \Rightarrow D = 4$ (our choice)

- So α, α_I, etc. are 2×2 matrices, and $I = 0, 1, 2, 3$

Fourier transform

- The translation symmetry allows Fourier transform in terms of cell numbers

$$\tilde{f}_\alpha(\mathbf{k}) = \sum_{\mathbf{n}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{n}} f_{\alpha\mathbf{n}}$$

- Inverse Fourier transform

$$f_{\alpha\mathbf{n}} = \int \frac{\mathrm{d}^D k}{(2\pi)^D} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{n}} f_\alpha(\mathbf{k})$$

- This allows us to rewrite the 'action' in the 'momentum space'

Momentum space

- The action

$$S = \int \frac{\mathrm{d}^D k}{(2\pi)^D} \psi^\dagger(k) T(k) \psi(k)$$

where

$$T(k) = \alpha + \sum_I (\alpha^I \mathrm{e}^{-\mathrm{i}k_I} + \alpha^{I\dagger} \mathrm{e}^{\mathrm{i}k_I})$$

- The two-dimensional matrices α, α_I and $\alpha_I^\dagger$ can be expanded in terms of the extended set of Pauli matrices σ_A, $A = 0, 1, 2, 3$,

$$\alpha = \alpha_A \sigma^A, \quad \alpha^I = \alpha^I_A \sigma^A, \quad \alpha^{I\dagger} = \bar{\alpha}^I_A \sigma^A$$

- $T(k) = T_A(k)\sigma^A, \qquad T_A(k) = \alpha_A + \sum_I (\alpha^I_A \mathrm{e}^{-\mathrm{i}k_I} + \bar{\alpha}^I_A \mathrm{e}^{\mathrm{i}k_I})$

Stationary points

- The stationary points are given by zero modes of the adjacency matrix
- In the momentum space: $\det T = 0$,

$$T_0^2(k) - T_1^2(k) - T_2^2(k) - T_3^2(k) = 0$$

- Not a single point, but a surface with a conical singularity

Wick rotation

- after the Wick rotation:

$$-T_0^2(k) - T_1^2(k) - T_2^2(k) - T_3^2(k) = 0$$

- the only degenerate point is the tip of the cone K

$$T_A(K) = 0$$

- We call K linearly non-degenerate if $T(K+k) \neq 0$, for any $k \neq 0$ in linear approximation
- Pseudo-Fermi point (pFp)

Pseudo-Fermi points

- A pFp is solution to $T_A(k) \equiv \alpha_A + \sum_I (\alpha_A^I \mathrm{e}^{-\mathrm{i}k_I} + \bar{\alpha}_A^I \mathrm{e}^{\mathrm{i}k_I}) = 0$
- Questions...
  - For which matrices T the pFps exist?
  - How many pFps?
  - Their locations, etc
- Once pFp's are known...
  - What is the fermionic spectrum in the vicinity of pFp
  - How the spectrum combines globally (for all pFp in the Brillouin zone)
  - What happens when the graph is deformed away from diracity?

Existence of pFp's

- Finding conditions that at least one pFp exists is extremely easy
- Given the location K and the parameters α_A^I the remaining parameters α_A are recovered from the consistency:

$$\alpha_A = -\sum_I (\alpha_A^I \mathrm{e}^{-\mathrm{i}K_I} + \bar{\alpha}_A^I \mathrm{e}^{\mathrm{i}K_I})$$

- Slightly less trivial is to show that this is an isolated linearly non-degenerate point, i.e.,

$$\det \mathbf{h}(K) \neq 0, \qquad h_A^I(k) = \frac{\partial T_A(k)}{\partial k_I}$$

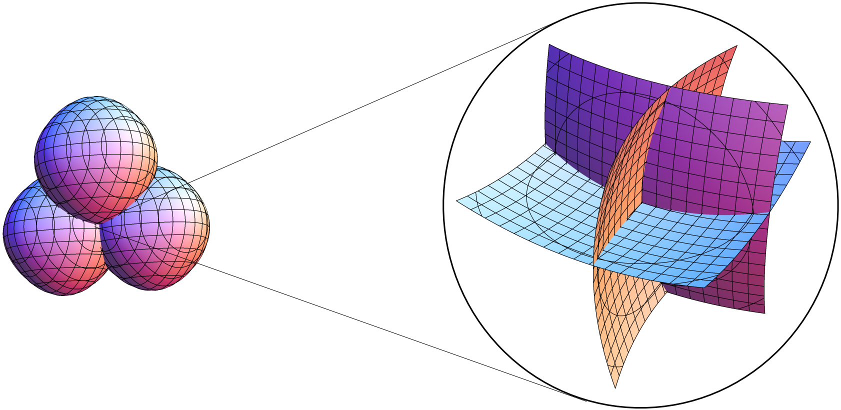
- Even less trivial is to find all other pFp. (They are always there!)

pFp's as intersections of hyper surfaces

- We can regard each equation $T_A(k) = 0$ in the pFp condition as defining a hyper-surface in the momentum space
- Then the pFps appear as points of intersection of all four hyper-surfaces corresponding to each direction A
- 'Most common' 4 convex 3-surfaces in $\mathbb{R}^4$ have two points of intersection carrying opposite indices
- However, other situations are also possible

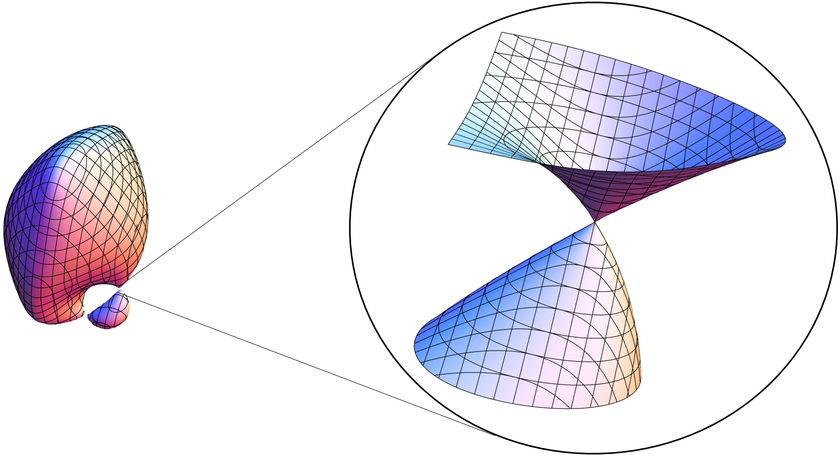
Example: Intersection of three 2-surfaces in $\mathbb{R}^3$

Surfaces $T_A(k) = 0$ and their intersections in $D = 3$



Example: Intersection of three 2-surfaces in $\mathbb{R}^3$

The surface $T_0^2(k) - T_1^2(k) - T_2^2(k) = 0$ in $D = 3$



General properties of pFp

- For a specific range of parameters pFp exist and well-defined. Even more: they are stable against small deformations of parameters
- There is a topological argument that there can not be a single pFp [Volovik'01,Hořava'05]:
    - Associate a topological charge to each pFp $K^{(\alpha)}$,

$$n_\alpha = \frac{1}{24\pi^2} \int_{S^3_\alpha} \mathrm{tr}(\mathrm{d}TT^{-1} \wedge \mathrm{d}TT^{-1} \wedge \mathrm{d}TT^{-1})$$

    - The compactness of the Brillouin zone implies: $\sum_\alpha n_\alpha = 0$ and $n_\alpha \in \pi_3(S^3) = \mathbb{Z}$
    - In lattice gauge theory this fact is known as the Nielson-Ninomiya theorem [Nielson-Ninomiya'81]

Continuum limit

- In the saddle point approximation the main contribution comes from configurations in the vicinity of pFp
- Depending on the value of the topological charge we get fermion modes as follows
  - $n_\alpha = +1$: a positive chirality Weyl fermion
  - $n_\alpha = -1$: a negative chirality Weyl fermion
  - $|n_\alpha| > 1$: a fundamental multiplet of $\mathrm{U}(n_\alpha)$ Weyl fermions
- In the simplest setup (as we have seen) there are exactly two pFps with opposite topological charges: $n_0 = -n_1 = 1$
- This corresponds to one Dirac fermion

The modes near pFp

- Near a pFp K^α, $\alpha = 1, 2$, we have

$$T_A(K^\alpha + k) = h_A^{\alpha I} k_I + O(k^2), \qquad h_A^{\alpha I} \equiv h_A^I(K^\alpha) = \left.\frac{\partial T_A(k)}{\partial k_I}\right|_{k=K^\alpha}$$

- Fermionic fluctuations near pFps can be labelled as

$$\psi_\alpha(k) = \psi(K^\alpha + k)$$

- The contribution of near pFp modes to the action

$$S = \int \frac{\mathrm{d}^D k}{(2\pi)^D} \sum_\alpha \psi_\alpha^\dagger(k) \sigma_A h_A^{\alpha I}(K_\alpha) k_I \psi_\alpha(k) + O(k^2)$$

'Cartesian Momenta'

- It is tempting to introduce the Cartesian momentum q_A defined by

$$q^A = h_A^{\alpha I} k_I$$

- This is a good choice near pFp where $\det \mathbf{h}^\alpha > 0$
- If $\det \mathbf{h}^\alpha < 0 \Rightarrow$ orientation problem. In this case we should replace $\mathbf{h}^\alpha$ by a matrix $\mathbf{h}'^\alpha$, s.t. $\det \mathbf{h}'^\alpha = |\det \mathbf{h}^\alpha|$
- There are many (equivalent?) ways to do it. We choose,

$$h'^{\alpha I}_A = \begin{cases} h_0^{\alpha I} \\ \epsilon_\alpha h_a^{\alpha I}, \ a = 1,2,3 \end{cases} \qquad \epsilon_\alpha = \operatorname{sign} \det \mathbf{h}^\alpha$$

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Dirac fermion, finally!

- With the properly defined cartesian momenta $q = \mathbf{h}'^{\alpha} \cdot k$, the action becomes

$$S = \int \frac{\mathrm{d}^D q}{(2\pi)^D} \Psi^\dagger \left(q_0(\mathbb{I} \otimes \mathbb{I}) + q_a(\sigma^a \otimes \sigma^3) \right) \Psi$$

where $\Psi_\alpha = h_\alpha^{-1/2} \psi_\alpha$, and $h_\alpha = |\det \mathbf{h}_\alpha|$

- 'Luckily' we can identify

$$\gamma^0 = \mathbb{I} \otimes \sigma^1, \quad \gamma^a = \gamma^0 \cdot (\sigma^a \otimes \sigma^3) = -\mathrm{i}(\sigma^a \otimes \sigma^2), \quad \bar{\Psi} = \Psi^\dagger \cdot \gamma^0$$

so that the action is just the momentum space action of the Dirac particle!

$$S = \int \mathrm{d}^D x \bar{\Psi}(-\mathrm{i}\gamma^A)\partial_A \Psi$$

What did we get? What could we...

- In the minimal setup we got: (one positive chirality fermion) + (one negative chirality fermion) = Dirac fermion with global $U(1)$ gauge symmetry
- In general you can get a fermion in the fundamental representation of $U_{\epsilon_1}(n_1) \times U_{\epsilon_2}(n_2) \times \cdots \times U_{\epsilon_k}(n_k)$ with $\sum_{\alpha=1}^{k} \epsilon_k n_k = 0$
- Free Dirac fermion: No gauge fields, gravity or anything like that (so far)...

The graph deformations

- Until now the graph was considered as rigid and the adjacency matrix taking some special values
- In particular, we required $4D$ translation symmetry
- This resulted in Dirac fermion
- Giving up translational symmetry, but keeping locality we get coupling to
  a) external U(1) gauge field
  b) background geometry
- Still work in progress. . .

Conclusion&Outlook...

- We considered the fermi particle on a graph parameterised by the adjacency matrix
- With translational invariance the model describes a Dirac particle or particles in the IR limit at a generic point in parameter space
- The particle spectrum depends on zeroes of the adjacency matrix. At generic point in the parameter space there are two pFps
- Graph deformations lead to gauge and gravity fields coupled to Dirac particles
- It is interesting to study graphs leading to realistic QFT models, like QCD or SM...
- ...but what if it doesn't make any sense as a fundamental theory?

Applications

- The model still makes sense as an economical discretisation of a fermionic gauge theory. . .