Interplay between Randers metrics and the causal geometry of stationary spacetimes

Miguel Ángel Javaloyes (Universidad de Murcia)

Partially supported by MINECO project MTM2012-34037, Regional J. Andalucía Grant P09-FQM-4496 and Fundación Séneca project 04540/GERM/06, Spain

IPMU (Tokyo), Februrary 10th, 2014

Outline

First part: preliminaries on Finsler metrics and stationary spacetimes

- Introduction to Finsler metrics
- Introduction to stationary spacetimes

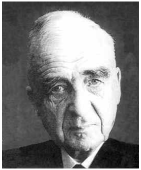
Second part: causality of stationary spacetimes via the Fermat metric

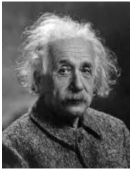
- Fermat Principle in stationary spacetime and Fermat metrics
- Introduction to Causality of a spacetime
- Characterization of causality of a statonary spacetime in terms of the Fermat metric

Third part: almost isometries of Finsler metrics and K-isometries of a stationary spacetime

- Almost isometries of quasi-metrics
- Almost isometries of Finsler metrics
- Applications to the study of K-isometries of a stationary spacetime

First part: preliminaries on Finsler metrics and stationary spacetimes

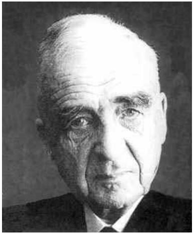




Finsler metrics

DEFINITION: $F: TM \to [0, +\infty)$ continuous and

1. smooth in $TM \setminus \{0\}$
2. Positively homogeneous of degree one $F(\lambda v) = \lambda F(v)$ for all $\lambda > 0$
3. Fiberwise strongly convex square:



PAUL FINSLER (1894-1970)

$$g_v(w,z) = \frac{1}{2}\frac{\partial^2}{\partial t \partial s}F(v+tw+sz)^2|_{t=s=0} = \frac{1}{2}\mathrm{Hess}(F^2)_v(w,z)$$

for every $w, z \in T_{\pi(v)}M$. Then $g_v(w,w) > 0$ for every $0 \neq w \in T_{\pi(v)}M$.

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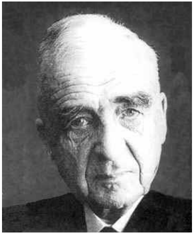

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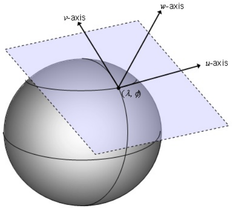
for every $w, z \in T_{\pi(v)}M$. Then $g_v(w,w) > 0$ for every $0 \neq w \in T_{\pi(v)}M$.

It can be showed that this implies:

- F is positive in $TM \setminus \{0\}$ and F^2 is C^1 on TM.
- Triangle inequality holds in the fibers

Interpretation of the fundamental tensor

- S = Indicatrix of $F = \{v \in A : F(v) = 1\}$ (the unit sphere of F).
- The fundamental tensor g_v of F in $v \in S$ coincides with the second fundamental form of S in the hyperplane tangent to S
- $g_v(v, v) = F(v)^2$
- v and $T_v S$ are g_v-orthogonal



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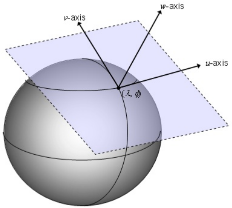
The indicatrix contains all the geometric information of F



Non-symmetric "distance"

- We can define the length of a curve: $\ell_F(\gamma) = \int_a^b F(\dot{\gamma})\mathrm{d}s$
- and then the distance between two points: $\mathrm{dist}(p,q) = \inf_{\gamma\in C^\infty(p,q)} \ell_F(\gamma)$
- dist is non-symmetric because F is non-reversible
- the length of a curve $t \to \gamma(t)$ is different from the length of its reverse $t \to \gamma(-t)$!!

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We have to distinguish between forward and backward:

- balls $B^+(p, r) = \{x \in M : \mathrm{dist}(p, x) < r\}$ and

$$B^-(p, r) = \{x \in M : \mathrm{dist}(x, p) < r\}$$

- Cauchy sequence
- topological completeness
- geodesical completeness

Randers metrics

- Randers metrics in a manifold M is a function $R : TM \to \mathbb{R}$ defined as:

$$R(v) = \sqrt{h(v,v)} + \omega(v)$$

where h is Riemannian and ω a 1-form with $\|\omega\|_h < 1 \ \forall x \in M$,

- are basic examples of non-reversible Finsler metrics: $R(-v) \neq R(v)$.
- Named after the norwegian physicist Gunnar Randers (1914-1992):

G. Randers and A. Einstein

Randers, G.: On an asymmetrical metric in the fourspace of General Relativity. Phys. Rev. (2) **59**, 195–199 (1941)

Zermelo metrics

Given a Riemannian metric g,
Zermelo metric:

$$Z(v) = \sqrt{\frac{1}{\alpha}g(v,v) + \frac{1}{\alpha^2}g(W,v)^2} - \frac{1}{\alpha}g(W,v),$$

where $\alpha = 1 - g(W,W)$.

It is of Randers type

Geodesics minimize time in the presence of a wind or current W.

MEETING OF WATERS

Matsumoto metrics

Given a Riemannian metric g, and a one-form β

$$M(v) = \frac{g(v,v)}{\sqrt{g(v,v)} - \beta(v)}$$

defined in

$$A = \{v \in TM : \sqrt{g(v,v)} > 2\beta(v)\}$$

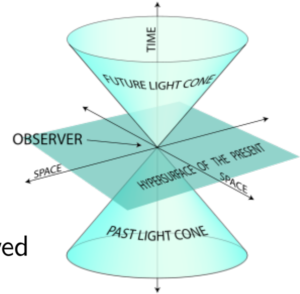
Geodesics minimize time in the presence of a slope

M. Matsumoto. A slope of a mountain is a Finsler surface with respect to a time measure, J. Math. Kyoto Univ., 29 (1989), pp. 17–25

Mount Fuji (near Tokyo)

Makoto Matsumoto (19?? –2005)

- A *Lorentzian manifold* (M, g) with index 1 $(+, \cdots, +, -)$



- $v \in TM$ is $\begin{cases} \text{timelike if } g(v,v) < 0 \\ \text{lightlike if } g(v,v) = 0 \\ \text{causal if } g(v,v) \leq 0 \\ \text{spacelike if } g(v,v) > 0 \end{cases}$
- A spacetime is a Lorentzian manifold endowed with a time-orientation
- The time-orientation is determined by a timelike vector field T
- A causal vector $v \in TM$ is future-pointing if $g(v, T) < 0$ (if $g(v, T) > 0$ is past-pointing)

Stationary spacetimes

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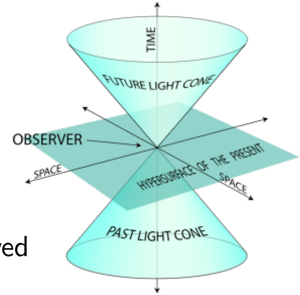

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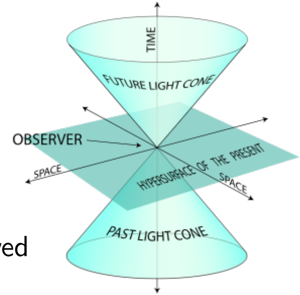

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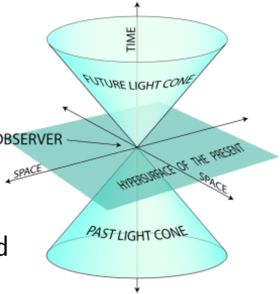

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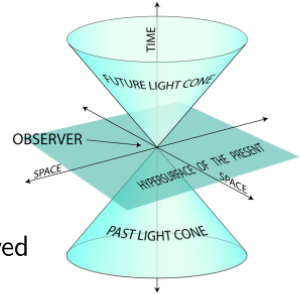
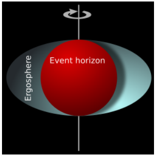
- A causal vector $v \in TM$ is future-pointing if $g(v, T) < 0$ (if $g(v, T) > 0$ is past-pointing)
- A stationary spacetime (M, g) is a Lorentzian manifold endowed with a timelike Killing vector field

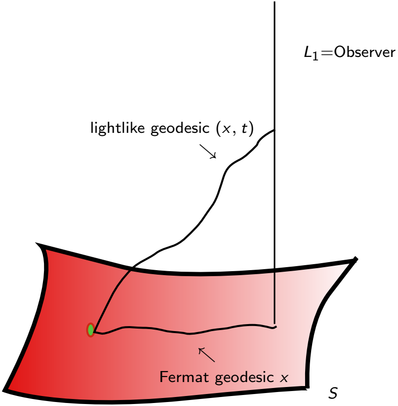




Kerr spacetime

Second part: causality of stationary spacetimes via Finsler geometry

$(S \times \mathbb{R}, l)$ is a standard stationary spacetime



S is naturally endowed with a Randers metric F called the Fermat metric

Conformally Standard Stationary Spacetimes

- A spacetime (M, g) is Conformastationary if it admits a timelike Conformal field K, that is, a timelike vector field satisfying

$$\mathcal{L}_K g = \lambda g,$$

for some function $\lambda : M \to \mathbb{R}$

- Standard Conformastationary means that $M = S \times \mathbb{R}$ and

$$g((v,\tau),(v,\tau)) = \varphi(g_0(v,v) + 2\omega(v)\tau - \tau^2),$$

in $(x,t) \in S \times \mathbb{R}$, where $(v,\tau) \in T_xS \times \mathbb{R}$, $\varphi : S \times \mathbb{R} \to (0,+\infty)$

- and g_0 is a Riemannian metric on S and ω a 1-form on S.
- In this case, ∂_t is a timelike conformal field.

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A conformastationary spacetime is standard whenever it is distinguishing and the timelike conformal vector field is complete:

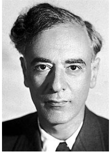
M. A. J. and M. Sánchez, *A note on the existence of standard splittings for conformally stationary spacetimes*, Classical Quantum Gravity, 25 (2008), pp. 168001, 7.

Miguel Sánchez

Fermat principle in General Relativity

- First established by Herman Weyl in 1917 for static spacetimes
- The stationary case is considered by Tulio Levi-Civita in 1927
- It appears in classical books as Landau-Lifshitz "The classical theory of fields" 1962
- I. Kovner gave a version of Fermat principle for an arbitrary spacetime in 1990
- Volker Perlick gave a rigorous proof of this general principle in the same year (1990)

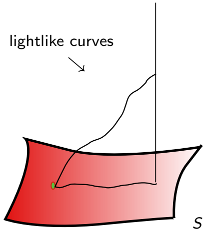





H. Weyl (1885-1955) → T. Levi-Civita (1873-1941) → Lev Landau (1908-1968) → E. Lifshitz (1915-1985) → I. Kovner → V. Perlick (Born in 1956)

Fermat principle in standard stationary spacetimes

- Relativistic Fermat Principle: lightlike pregeodesics are critical points of the arrival time function corresponding to an *observer* in a suitable class of lightlike curves



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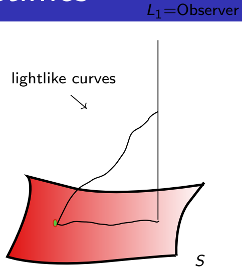

- Relativistic Fermat Principle: lightlike pregeodesics are critical points of the arrival time function corresponding to an *observer* in a suitable class of lightlike curves
- If you consider as observer $s \to L_1(s) = (x_1, s)$ in $(S \times \mathbb{R}, g)$, given a lightlike curve $\gamma = (x, t)$, the arrival time $\mathrm{AT}(\gamma)$ is

$$t(b) = t(a) + \int_a^b \left(\omega(\dot{x}) + \sqrt{g_0(\dot{x},\dot{x}) + \omega(\dot{x})^2}\right) \mathrm{d}s.$$


Pierre de Fermat (1601-1665)

- because $g_0(\dot{x}, \dot{x}) + 2\omega(\dot{x})\dot{t} - \dot{t}^2 = 0$ $(g(\dot{\gamma}, \dot{\gamma}) = 0)$
- Let us define the Fermat (Finslerian) metric in S as

$$F(v) = \omega(v) + \sqrt{g_0(v,v) + \omega(v)^2},$$

Fermat metric and lightlike geodesics

Theorem

A curve $s \to \gamma(s) = (x(s), s)$ *is a lightlike pregeodesic of* $(S \times \mathbb{R}, g)$ *iff* $s \to x(s)$ *is a Fermat geodesic with unit speed.*

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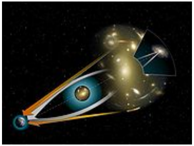
Consequences:

- Gravitational lensing can be studied from geodesic connectedness in Fermat metric
- Existence of t-periodic lightlike geodesics is equivalent to existence of Fermat closed geodesics

EINSTEIN RING



GRAVITATIONAL LENSING

My Collaborators

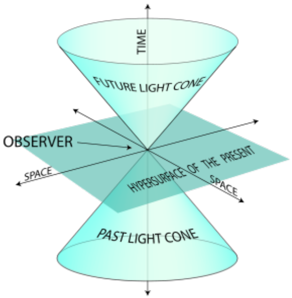
Erasmo Caponio (Bari)

Antonio Masiello (Bari)

Miguel Sánchez (Granada)

Lorentzian Causality

- Causality studies if given two points $p, q \in M$ they are joined by a causal curve
- $p, q \in M$ are chronologically related, and write $p \ll q$ if there exists a future-pointing timelike curve γ from p to q
- $p, q \in M$ are causally related $p < q$) if there exists a future-pointing causal curve γ from p to q
- The chronological future of $p \in M$ is defined as $I^+(p) = \{q \in M : p \ll q\}$
- The causal future of $p \in M$ is defined as $J^+(p) = \{q \in M : p \leq q\}$
- Analogously we define the chronological past $I^-(p)$ and the causal past $J^-(p)$.



The causal ladder

Causal properties classify spacetimes depending on the behaviour of causal cones. A spacetime is:

- Chronological if $p \notin I^+(p)$ for every $p \in M$.
- Distinguishing if $I^+(p) = I^+(q)$ or $I^-(p) = I^-(q)$ implies $p = q$
- Causally continuous if it is distinguishing and the Chronological cones $I^\pm(p)$ are continuous in $p \in M$
- Causally simple if the causal cones $J^\pm(p)$ are closed for every $p \in M$
- Globally hyperbolic if it admits a Cauchy hypersurface (a subset S that meets exactly once every inextendible timelike curve)

Globally hyperbolic

$\Downarrow$

Causally simple

$\Downarrow$

Causally continuous

$\Downarrow$

Stably causal

$\Downarrow$

Strongly causal

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Causality via the Fermat metric

- Let d the non-symmetric distance in S associated to the Fermat metric
- $B^+(x_0, s) = \{p \in S : \mathrm{d}(x_0, p) < s\}$ forward balls
- $B^-(x_0, s) = \{p \in S : \mathrm{d}(p, x_0) < s\}$ backward balls

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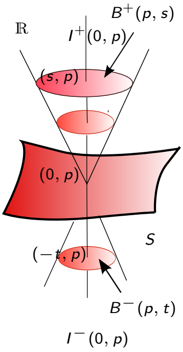
- Let $(\mathbb{R} \times S, g)$ be a standard stationary spacetime. Then

$$I^{\pm}(t_0, x_0) = \cup_{s>0}\{t_0 \pm s\} \times B^{\pm}(x_0, s),$$



Causality through the Fermat metric

Theorem

Let $(S \times \mathbb{R}, g)$ be a standard stationary spacetime. Then $(S \times \mathbb{R}, g)$ is causally continuous and

(a) *the following assertions become equivalent:*
  (i) $(S \times \mathbb{R}, g)$ *is causally simple,*
  (ii) *the associated Finsler manifold (S, F) is convex,*

(b) *it is globally hyperbolic if and only if $\bar{B}^+(x,r) \cap \bar{B}^-(x,r)$ is compact for every $x \in S$ and $r > 0$.*

(c) *a slice $S \times \{t_0\}$, $t_0 \in \mathbb{R}$, is a Cauchy hypersurface if and only if the Fermat metric F on S is forward and backward complete.*

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$\Downarrow$
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Generalized Hopf-Rinow theorem

Theorem (Accurate Hopf-Rinow for Randers metrics)

Let (S, R) a Randers manifold and given a function $f : S \to \mathbb{R}$ such that $df \leq R$ define $R_f(x, v) = R(x, v) - \mathrm{d}f_x(v)$. The following conditions are equivalent:

(A) *the intersection $\bar{B}^+(x, r) \cap \bar{B}^-(x, r)$ of (S, R) is compact for every $r > 0$ and $x \in S$*

(B) *there exists f such that R_f is geodesically complete*

(C) *there exists f and $p \in S$ such that the forward and the backward exponentials of R_f are defined in T_pS*

(E) *there exists f such that the quasi-metric d_f associated to R_f is forward and backward complete*

In such a case, (S, R) is convex.

Heinz Hopf (1894-1971)

References

E. Caponio, M. A. J., and A. Masiello, *On the energy functional on Finsler manifolds and applications to stationary spacetimes*, Math. Ann., 351 (2011), pp. 365–392.

E. Caponio, M. A. J., and M. Sánchez, *On the interplay between Lorentzian Causality and Finsler metrics of Randers type*, Rev. Mat. Iberoamericana, 27 (2011), pp. 919–952.

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For a review see:

M. A. J., *Conformally standard stationary spacetimes and Fermat metrics*, Proceedings of GeLoGra 2011.

Last part: Almost Isometries. My collaborators

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Almost isometries of quasi-metrics

Definition

Given a set X, we say that a function $d : X \times X \to \mathbb{R}$ is a *quasi-metric* if

(i) $d(x,y) \geq 0$ for every $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) + d(y,z) \geq d(x,z)$ (triangle inequality).

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As a consequence of the lack of symmetry, there are two kinds of balls:

- $B_d^+(x, r) = \{y \in X : d(x, y) < r\}$ (forward balls)
- $B_d^-(x, r) = \{y \in X : d(y, x) < r\}$ (backward balls)

respectively, for $x \in X$ and $r > 0$.

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respectively, for $x \in X$ and $r > 0$.

Definition

A pair (X, d) will be called a *quasi-metric space* endowed with the topology induced by the family $B_d^+(x, r) \cap B_d^-(x, r)$, $x \in M$ and $r > 0$.

Let us observe that this topology coincides with the topology generated by (the balls of) the symmetrized metric $\widetilde{d}(x, y) = \frac{1}{2}\big(d(x, y) + d(y, x)\big)$.

Quasi-metrics

Quasi-metrics spaces have been studied by many mathematicians:

- Fréchet 1909, Hausdorff 1914, Mazurkiewicz 1930, Wilson 1931, Busemann 1944
- and also by a spanish mathematician: Julio Rey Pastor 1940

Rey Pastor (1888-1962)

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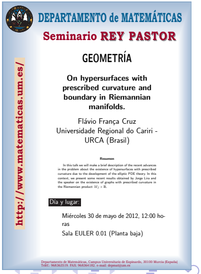
Our seminar in the university of Murcia is called "Rey Pastor" after him

Rey Pastor (1888-1962)



Quasi-metrics and the triangular function

In a quasi-metric space we can define the length of a continuous curve $\alpha : [a, b] \subseteq \mathbb{R} \to X$ as

$$\ell(\alpha) = \sup_{\mathcal{P}} \sum_{1=1}^{r} d(\alpha(s_i), \alpha(s_{i+1})),$$

where $\mathcal{P}$ is the set of partitions $a = s_1 < s_2 < \ldots < s_{r+1} = b$, $r \in \mathbb{N}$.

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- We say that α is *rectifiable* when $\ell(\alpha)$ is finite.
- Moreover, we say that a curve γ in X from p to q is a *minimizing geodesic* if $\ell(\gamma) = d(p, q)$.

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Definition

Let us define the *triangular function* $T : X \times X \times X \to [0, +\infty[$ of a quasi-metric space (X, d) as $T(x, y, z) = d(x, y) + d(y, z) - d(x, z)$ for every $x, y, z \in X$.

Evidently, T is continuous.

Almost isometries

Proposition

A curve $\alpha : [a, b] \subseteq \mathbb{R} \to X$ *is a minimizing geodesic of a quasi-metric space* (X, d) *iff* $T(\alpha(s_1), \alpha(s_2), \alpha(s_3)) = 0$ *for every* $a \leq s_1 < s_2 < s_3 \leq b$.

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Definition

Let (X_1, d_1) and (X_2, d_2) be two quasi-metric spaces. A bijection $\varphi : X_1 \to X_2$ is an *almost isometry* if it preserves the triangular function, that is,

$$T_2(\varphi(x), \varphi(y), \varphi(z)) = T_1(x, y, z)$$

for every $x, y, z \in X_1$, where T_1 and T_2 are the triangular functions associated respectively to (X_1, d_1) and (X_2, d_2).

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Corollary

Almost isometries preserve minimizing geodesics.

Almost isometries

Proposition

Given quasi-metric spaces (X_1, d_1) and (X_2, d_2), a bijection $\varphi : X_1 \to X_2$ is an almost isometry iff $\exists\ f : X_2 \to \mathbb{R}$ such that for every $x, y \in X_1$:

$$d_2\big(\varphi(x), \varphi(y)\big) = d_1(x, y) + f\big(\varphi(x)\big) - f\big(\varphi(y)\big) \qquad (1)$$

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Given quasi-metric spaces (X_1, d_1) *and* (X_2, d_2)*, a bijection* $\varphi : X_1 \to X_2$ *is an almost isometry iff* $\exists\ f : X_2 \to \mathbb{R}$ *such that for every* $x, y \in X_1$*:*

$$d_2(\varphi(x), \varphi(y)) = d_1(x, y) + f(\varphi(x)) - f(\varphi(y)) \quad (1)$$

Proof.

$\Rightarrow$ (the converse is straightforward)

- Fix a point $x_0 \in X_1$ and define $f : X_2 \to \mathbb{R}$ as $f(z) = d_2(z, \varphi(x_0)) - d_1(\varphi^{-1}(z), x_0)$ for every $z \in X_2$.

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- Given $x, y \in X_1$, as φ preserves the triangular function, we have

$$d_1(x, y) + d_1(y, x_0) - d_1(x, x_0)$$
$$= d_2(\varphi(x), \varphi(y)) + d_2(\varphi(y), \varphi(x_0)) - d_2(\varphi(x), \varphi(x_0)),$$

  which is equivalent to (1).

Almost isometries

Some observations:

- In metric spaces, almost isometries are always isometries
- If $\varphi : (X_1, d_1) \to (X_2, d_2)$ is an almost isometry, then

$$\varphi : (X_1, \widetilde{d}_1) \to (X_2, \widetilde{d}_2)$$

  is an isometry, where

$$\widetilde{d}_1(x, y) = \tfrac{1}{2}\big(d_1(x, y) + d_1(y, x)\big),$$
$$\widetilde{d}_2(x, y) = \tfrac{1}{2}\big(d_2(x, y) + d_2(y, x)\big).$$

- Moreover, φ is a homeomorphism and the functions $f : X_2 \to \mathbb{R}$ are continuous

Almost isometries

Notation:

- $\mathrm{Iso}(X, d)$ is the group of isometries of (X, d)
- $\widetilde{\mathrm{Iso}}(X, d)$ is the group of almost isometries of (X, d). It will be called the *extended isometry group* of (X, d).

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Proposition

- *With the above notation, $\widetilde{\mathrm{Iso}}(X, d)$ and $\mathrm{Iso}(X, d)$ are topological groups endowed with the compact-open topology.*
- *If the topology induced by d is locally compact, then $\widetilde{\mathrm{Iso}}(X, d)$ and $\mathrm{Iso}(X, d)$ are locally compact.*

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- *If the topology induced by* d *is locally compact, then* $\widetilde{\mathrm{Iso}}(X,d)$ *and* $\mathrm{Iso}(X,d)$ *are locally compact.*

Proof.

The proof follows from the inclusions:

$$\mathrm{Iso}(X,d) \subseteq \widetilde{\mathrm{Iso}}(X,d) \subseteq \mathrm{Iso}(X,\tilde{d}).$$

□

Local almost isometries

Definition

Let (X_1, d_1) and (X_2, d_2) be two quasi-metric spaces. A map $\varphi : X_1 \to X_2$ is a *local almost isometry* if $\forall x \in X_1$, $\exists$ $U \subseteq X_1$, $V \subseteq X_2$ open subsets, with $x \in U$, such that $\varphi|_U : U \to V$ is an almost isometry.

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- define d_l as the infimum of the lengths of curves between two points. We say that (X, d) is a *length space* when $d_l = d$.
- We say that a quasi-metric space is weakly finitely compact if $B^+(x, r) \cap B^-(x, r)$ are precompact $\forall x \in X$ and $r > 0$.

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Theorem

Let $\varphi : (X_1, d_1) \to (X_2, d_2)$ be a local almost isometry. Assume that (X_1, d_1) and (X_2, d_2) are length spaces, d_1 is weakly finitely compact and X_2 is locally arc-connected and simply connected. Then φ is an almost isometry.

Almost isometries of Finsler metrics

Let us define the symmetrized Finsler metric of F as

$$\hat{F}(v) = \frac{1}{2}\big[F(v) + F(-v)\big]$$

for every $v \in TM$. The sum of Finsler metrics is a Finsler metric:

M. A. J. and M. Sánchez, *On the definition and examples of Finsler metrics*, to appear in Ann. Sc. Nor. Pisa

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Lemma

If $\varphi : (M_1, F_1) \to (M_2, F_2)$ *is an almost isometry then*

$$\varphi : (M_1, \hat{F}_1) \to (M_2, \hat{F}_2)$$

is an isometry and φ *is smooth.*

Proof.

- To see that φ is an isometry prove that preserves the length of curves
- φ is smooth because it is an isometry of a Riemannian average metric

Finsler metrics

Proposition

- *If* $\exists$ *an almost isometry* $\varphi : (M_1, F_1) \to (M_2, F_2)$, *then there exists a smooth* $f : M_2 \to \mathbb{R}$ *such that* $\varphi^*(F_1) = F_2 + \mathrm{d}f$.
- *Conversely, if* $\varphi^*(F_1) = F_2 + \mathrm{d}f$, *the map* φ *is an almost isometry.*

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Proposition

Let (M, F) *be a Finsler manifold. Then the extended isometry group* $\widetilde{\mathrm{Iso}}(M, F)$ *is a closed subgroup of* $\mathrm{Iso}(M, \hat{F})$. *In particular,* $\widetilde{\mathrm{Iso}}(M, F)$ *is a Lie group.*

Finsler metrics

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Proof.

Use that $\widetilde{\mathrm{Iso}}(M, F) \subset \mathrm{Iso}(M, \hat{F})$ □

Corollary

Let (M, R) be a Randers manifold and $\varphi : M \to M$ an almost isometry for R. Then φ is an isometry for h.

Randers metrics

Corollary

Let (M, R) be a Randers manifold and $\varphi : M \to M$ an almost isometry for R. Then φ is an isometry for h.

Proof.

Just observe that the symmetrized Finsler metric of R is given by $\hat{R}(v) = \sqrt{h(v, v)}$ for $v \in TM$. □

K-conformal maps

- Let (M, g) be a spacetime and K a Killing field
- We say that a diffeomorphism $\psi : M \to M$ is K-conformal if
  - It is conformal, $\psi_*(g) = \lambda g$, $\lambda \neq 0$, (ψ_* is the pushforward) and
  - preserves K, $\psi_*(K) = K$
- Now consider a normalized standard stationary spacetime $(S \times \mathbb{R}, g)$ with

$$g((v, \tau), (v, \tau)) = g_0(v, v) + 2\omega(v)\tau - \tau^2$$

$v \in TS$ and $\tau \in \mathbb{R}$.

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$v \in TS$ and $\tau \in \mathbb{R}$.

Theorem

If $\psi : (S \times \mathbb{R}, g) \to (S \times \mathbb{R}, g)$ is a K-conformal map, then

$$\psi(x, t) = (\varphi(x), t + f(x)),$$

and $\varphi_*(F) = F + df$ and $\varphi : (S, h) \to (S, h)$ is an isometry, where

$$h(v, v) = g_0(v, v) + \omega(v)^2.$$

K-conformal maps

Theorem

If $\psi : (S \times \mathbb{R}, g) \to (S \times \mathbb{R}, g)$ *is a* K*-conformal map, then* $\psi(x,t) = (\varphi(x), t + f(x))$ *and* $\varphi_*(F) = F + df$ *and* $\varphi : (S, h) \to (S, h)$ *is an isometry, where*

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Proof.

□

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$$h(v,v) = g_0(v,v) + \omega(v)^2.$$

Proof.

- K-conformal implies that ψ maps orbits of ∂_t to orbits of ∂_t, that is, $\psi(x,t) = (\varphi(x), t + f(x))$
- As ψ is conformal, maps lightlike pregeodesics to lightlike pregeodesics
- Then Fermat metric maps Fermat pregeodesics to Fermat pregeodesics and $\ell_{\varphi_*(F)}(\gamma) = \ell_F(\gamma) + f(\gamma(1)) - f(\gamma(0))$
- This means that $\varphi_*(F)$ and $F + df$ have the same geodesics and therefore they are equal

□

K-conformal maps

Lemma

$\mathrm{Conf}_K(M,g)$ (here $M=\mathbb{R}\times S$) is a closed subgroup of $\mathrm{Conf}(M,g)$. Moreover the one-parameter subgroup $\mathcal{K}$ generated by K is closed and normal in $\mathrm{Conf}_K(M,g)$.

Proof.

□

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Proof.

- First part is obvious in the C^1 topology.
- If $\psi \in \mathrm{Conf}_K(M,g)$ then $\psi(x,t) = (\varphi(x), t + f(x))$ with $\varphi \in \widetilde{\mathrm{Iso}}(S,F)$
- Moreover, $\psi^{-1}(x,t) = (\varphi^{-1}(x), t - f(\varphi^{-1}(x)))$
- Then if $K^T : M \to M$ is given by $K^T(x,t) = (x, t+T)$, it follows that $\psi \circ K^T \circ \psi^{-1} = K^T$ ($\mathcal{K}$ is normal)

□

K-conformal maps

Proposition

The map $\pi : \mathrm{Conf}_K(M, g) \to \widetilde{\mathrm{Iso}}(S, F)$ *defined as* $\pi(\psi) = \varphi$ *is a Lie group homomorphism and* $\bar{\pi} : \mathrm{Conf}_K(M, g)/\mathcal{K} \to \widetilde{\mathrm{Iso}}(S, F)$ *is an isomorphism.*

Proof.

K-conformal maps

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Proof.

- We just have to prove that $\bar{\pi}$ is one-to-one.
- Injective: if ψ_1 and ψ_2 project on the same almost isometry map φ, then by last Prop. $\psi_1(x,t) = (\varphi(x), t + f(x) + c_1)$ and $\psi_2(x,t) = (\varphi(x), t + f(x) + c_2)$, $\psi_2 \circ \psi_1^{-1} = K^{c_2 - c_1}$ and $[\psi_1] = [\psi_2]$
- Surjective: given an almost isometry φ, we construct the map

$$\psi(x,t) = (\varphi(x), t + f(x))$$

Clearly, it preserves ∂_t. By Fermat principle, it maps lightlike pregeodesics to lightlike pregeodesics, then it preserves the lightcone and it must be conformal (by Dajcker-Nomizu [83]).

Applications

Corollary

Given a manifold S, for a generic set of data (g_0, ω), the stationary metric $g = g(g_0, \omega)$ on $M = S \times \mathbb{R}$ has discrete K-conformal group $\mathrm{Conf}_K(M, g)/\mathcal{K}$.

Applications

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Corollary

If S *is compact, then* $\mathrm{Conf}_K(S \times \mathbb{R}, g)/\mathcal{K}$ *and* $\widetilde{\mathrm{Iso}}(S, F)$ *are compact Lie groups.*

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Open problems and work in progress

- Compute explicitly some extended isometry group
- to study the case in that K is not timelike

Thanks a lot for your attention