Derived categories, noncommutative motives and commutative theorems

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X: smooth proj var$/k$

$$Z_i(X) := \{\text{i-dim'l alg cycles on } X\}$$

$$Z_i(X)_R := Z_i(X) \otimes_{\mathbb{Z}} R$$

R: a ring

Adequate equivalent relation on $Z_*(X)$

$$Z_*(X) := \bigoplus_{i \geq 0} Z_i(X)$$

adequate respects dim filtration
compatible with base change

$\sim_{RAT}$: rational equivalence

$\sim_{Alg}$ algebraic . . . . . $CH^i(X) := Z^i(X) / \sim_{Alg}$

$\sim_{\otimes\text{-NIL}}$ $\alpha \sim 0 \iff \exists N \gg 0$

s.t $\underbrace{\alpha \times \cdots \times \alpha}_{N\text{-times}} \sim_{rat} 0$

in $Z_*(\underbrace{X \times \cdots \times X}_{N\text{-times}})$

homological equiv

$\sim_{Hom}$ $\gamma^*(\alpha) = 0$ for any any γ^* a Weil cohomology

$\Updownarrow$

$\alpha \sim_{Hom} 0$

numerical equiv

$\alpha \sim_{NUM} 0$ if $\forall \beta \in Z_{d-i}(X)$

$\langle \beta, \alpha \rangle = 0$

intersection of cycles

(1) Algebraically equiv cycles and rationality

$k = \mathbb{C}$

$$J^i(X) := \frac{H^{2i}(X,\mathbb{C})}{\underbrace{F^i H^{2i}(X,\mathbb{C})}_{\text{Hodge filtration}} \oplus H^{2i}(X,\mathbb{Z})}$$ complex torus

[Griffiths]

$$\begin{array}{ccc} AJ^i : CH^i(X)_{\mathbb{Q}} & \longrightarrow & J^i(X)_{\mathbb{Q}} \\ \cup & & \cup \\ A^i(X)_{\mathbb{Q}} & \longrightarrow & J^i_a(X)_{\mathbb{Q}} \\ \| & & \text{abelian var} \\ \{\alpha \mid \alpha \sim_{Alg} 0\}_{\mathbb{Q}} & & \end{array}$$

<u>Example</u>

$J^1(X) = \mathrm{Pic}^0(X)$, $J^d(X) = \mathrm{Alb}(X)$

[Clemens–Griffiths]

$X \subset \mathbb{P}^4$ cubic 3-fold $\quad J^i(X) = \begin{cases} 0 & i \neq 2 \\ J(X) & \text{ppav.} \end{cases}$

Y is a Fano 3-fold
$\Rightarrow J(Y) = \bigoplus_{i=1}^{h} J(C_i)$
C_i: smooth proj curves

Apply X cubic 3-fold is not rational!

X is representable if
$A^i(X) = 0$ if $i \neq \lfloor \frac{\dim X}{2} \rfloor =: k$
$\exists$
$A^k(X) \simeq J(X)$, principally polarized

Example
X a curve
X a cubic 3-fold, quartic double solid
$X = Q_1 \cap Q_2$ $\dim Q_i = 2n$
$X = Q_1 \cap Q_2 \cap Q_3$ $\dim Q_i = 2n+1$
Q_i: quadric
in these cases Torelli holds; $X \simeq X' \Leftrightarrow J(X) = J(X')$

(2) Chow group of low degree varieties

$CH_i(\mathbb{P}^n) = \mathbb{Z}$

if $X \subset \mathbb{P}^n$ is a complete intersection of type $(d_1, \cdots, d_k)$ $(d_i \geq d_{i+1} \neq 1)$

<u>Conjecture</u> [Paranjape] $\ell := \left\lfloor \dfrac{n - \sum_{j=1}^{k} d_j}{d_1} \right\rfloor$

$CH_i(X)_{\mathbb{Q}} = \mathbb{Q} \quad \forall i < \ell$

round down

known for "low values of i"

[Esnault-Levine-Viehweg]

(3) Voevodsky's smash nilpotence conjecture

<u>Conj</u> X: smooth proj var$/k$

$\sim_{num} = \sim_{\otimes\text{-Nil}}$ on $Z_*(X)$.

(In this case we say $C_{voev}(X)$ holds) known for

$\dim X \leq 2$, X abelian 3-fold (char $k=0$)

X : Fano 3-folds

(for motivic reason)

$$M(X)_{\mathbb{Q}} = \mathbb{1} \oplus \mathbb{L}^{\oplus q} \oplus h^1(J) \oplus \underbrace{h^1(J)}_{\text{abelian}} \otimes (\mathbb{L}^{\otimes 2})^{\oplus q} \oplus \mathbb{L}^{\otimes 3}$$

$$h^1(J) \hookrightarrow h^1(\text{curve})$$

Chow motives

$$\mathrm{Sm\,proj}(k) \xrightarrow[\text{Weil cohomology}]{H^*} \mathrm{Vect}(L)$$

(category of sm proj/k) (the cat of v.s./L)

$\mathrm{Chow}(k)$: objects (X, p, m)

$X \in \mathrm{Sm\,proj}\ k$, $m \in \mathbb{Z}$

$p \in CH_{\dim X - m}(X \times X)$

$p : CH_i(X) \longrightarrow CH_{i-m}(X) \qquad p^2 = p$

<u>Morphism</u>

$(X, p, m) \to (Y, q, n)$

are element in $CH_*(X \times Y) \ni \alpha$

composed with p and q $\quad q \circ \alpha \circ p$

$$\begin{array}{ccc} M : \text{sm proj } k & \longrightarrow & \text{Chow } k \\ \cup & & \cup \\ X & \longmapsto & (X, \Delta_X, 0) \end{array}$$

Chow(k) is idempotent complete

$M(\mathbb{P}^1) = \mathbb{1} \oplus \mathbb{L}$

$\mathbb{1} = M(\operatorname{Spec} k)$

$\mathbb{L} = (\operatorname{Spec} k, \Delta, 1)$ Lefschetz motive

$\mathbb{L}^{\otimes n} = (\operatorname{Spec} k, \Delta, n)$

C : curve

$$M(C)_{\mathbb{Q}} = \mathbb{1} \oplus h^1(C) \oplus \mathbb{L}$$

correspond to $J(C)_{\mathbb{Q}}$

$: \mathrm{Mor}(h^1(C)_{\mathbb{Q}}, h^1(C')_{\mathbb{Q}})$

$= \mathrm{Isog}(J(C), J(C'))$

in fact

$$Ab(k)_{\mathbb{Q}} \hookrightarrow Chow(k)_{\mathbb{Q}}$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (A)_{\mathbb{Q}} & \longmapsto & h^1(C)_{\mathbb{Q}} \text{ s.t. } J(C) \underset{\text{isog}}{\sim} A \end{array}$$

(Part II)

ERRATA

(1) $\mathrm{Smproj}\,k \xrightarrow[\text{Weil}]{J^*} \mathrm{Vect}\,L$

(2) $(X, p, m) \in \mathrm{Chow}(k)$

$p \in CH_{d_X}(X \times X) \qquad p^2 = p$

$\mathrm{Hom}((X,p,m),(Y,q,n)) =$

$p\,CH_{d_Y - n + m}(X, Y)\,q$

(3) AJ Map $\quad A^{i+1}(X)_{\mathbb{Q}} \to J^i(X)_{\mathbb{Q}}$

$$\begin{array}{ccc} A^{i+1}(X)_{\mathbb{Q}} & & \\ \cap & & = \dfrac{H^{2i+1}(X, \mathbb{C})}{F^{i+1}H^{2i+1}(X,\mathbb{C}) \oplus H^{2i+1}(X,\mathbb{Z})} \\ CH^{i+1}(X)_{\mathbb{Q}} & & \end{array}$$

$\left(\begin{array}{l} \alpha \in A^{i+1} \quad [\alpha] = 0 \text{ in } H^{2i+2}(X, \mathbb{C}) \\ \therefore\ \exists Z \in H^{2i+1}(X, \mathbb{C}) \text{ s.t. } \partial Z = \alpha \end{array} \right)$

Noncommutative motives

$\mathcal{A} \in dgcat(k)$ dg-categories$/k$

$\mathcal{A}$: pretriangulated if $H_0(\mathcal{A})$: same obj is triangulated

$Mor_{H_0(\mathcal{A})}(A, B) = H^0(Mor(A, B))$

[Lunts-Orlov]: X/k smooth proj

$\Rightarrow \mathrm{Perf}\,X$ has a natural (canonical) dg-structure (pretriangulated)

$\overset{\shortparallel}{D^b(X)}$ $\qquad \mathrm{Perf}^{dg}(X)$

$H_0(\mathrm{Perf}^{dg}(X)) = \mathrm{Perf}(X)$

<u>Fact</u> $F: \mathrm{Perf}(X) \to \mathrm{Perf}(Y)$

exact functors, k-linear

(Δ-structure)

$\exists F^{dg}: \mathrm{Perf}^{dg}(X) \to \mathrm{Perf}^{dg}(Y)$

s.t. $H_0(F^{dg}) = F$

$\Longleftrightarrow$ F is Fourier-Mukai

<u>Conjecture</u> [Kawatani]

if $A \hookrightarrow \mathrm{Perf}(X)$ admissible

$\mathrm{Perf}(X) = \langle A, A^{\perp}_{X} \rangle$

and $A \hookrightarrow \mathrm{Perf}(Y)$ admissible, $\langle A, A^{\perp}_{Y} \rangle$

$\Phi : \mathrm{Perf}\, X \to A \simeq A \hookrightarrow \mathrm{Perf}(Y)$

is FM

<u>Note</u> This conj is equiv to

$\exists \Phi^{dg}$ s.t. $H_0(\Phi^{dg}) = \Phi$. //

I will assume this conjecture

[Tabuada]

$A, B \in \mathrm{dgCat}(k)$.

Morita equivalence: $F : A \to B$ is Morita equiv

the induced functor $D(A) \to D(B)$ equiv

Via a model structure

$Hmo(k)$: Homotopy cat of dg cat/k,

has Morita eq as weak equiv.

Fact

$\mathrm{Hom}_{Hmo(k)}(A, B) = A\text{-}B^{op}$ perfect bimodule

$\rightsquigarrow$ pre-triangulated cat

Define Hmo_0 by

$\mathrm{Hom}_{Hmo_0(k)}(A, B) = K_0(\mathrm{Hom}_{Hmo(k)}(A, B))$

obj same as $Hmo(A, B)$

Ex. $X, Y \in \mathrm{Smproj}\, k$

$\mathrm{Hom}_{Hmo_0(k)}(\mathrm{Perf}^{dg}(X), \mathrm{Perf}^{dg}(Y)) = K_0(X \times Y)$

$\mathrm{Hom}_{Hmo(k)}(\mathrm{Perf}^{dg}(X), \mathrm{Perf}^{dg}(Y)) = \mathrm{Perf}^{dg}(X \times Y)$

[Toen]

Last step

Take ps-abelian envelop of $\mathrm{Hmo}_0(k)$.

$\mathrm{NChow}(k)$: the cat of NC Chow motieves$/k$

[Tabuada] (1) if $\mathcal{A} = \langle \mathcal{B}, \mathcal{C} \rangle$, $NC(\mathcal{A}) = NC(\mathcal{B}) \oplus NC(\mathcal{C})$

(2) if $U : \mathrm{dgcat}(k) \longrightarrow \mathrm{Vect}(L)$

is additive, factors via $\mathrm{NChow}(k)$

Semiorth $\longrightarrow \oplus$, ...

$$\begin{tikzcd}
\mathrm{SmProj}(k) \arrow[rr, "\mathrm{Perf}^{dg}(-)"] \arrow[d, "M"'] & & \mathrm{dgcat}(k) \arrow[d, "\mathrm{NChow}"] \arrow[dddd, bend left, "N"] \\
\mathrm{Chow}(k) \arrow[r] \arrow[d, "\mathrm{Ab}(k)"'] & \mathrm{Chow}(k)/_{\otimes \mathbb{L}} \arrow[r, "R"', "\text{Full Faithful}"] & \mathrm{NChow}(k) \arrow[d, "\text{quotient by the } \otimes\mathrm{Nil} \text{ ideal}"] \\
\mathrm{Voev}(k) \arrow[r] \arrow[d] & \mathrm{Voev}(k)/_{\otimes \mathbb{L}} \arrow[r, "R_V^{FF}"] & \mathrm{NVoev}(k) \arrow[d] \\
\mathrm{Hom}(k) \arrow[r] \arrow[d] & \mathrm{Hom}(k)/_{\otimes \mathbb{L}} \arrow[r, "R_H"', "\text{Full}"] & \mathrm{NHom}(k) \arrow[d] \\
\mathrm{Num}(k) \arrow[r] & \mathrm{Num}(k)/_{\otimes \mathbb{L}} \arrow[r, "R_N^{FF}"] & \mathrm{NNum}(k)
\end{tikzcd}$$

すべて $\otimes \mathbb{Q}$

Grothendieck-Riemann-Roch

<u>Example</u> $X = \mathbb{P}^2$

$$M(X) = \mathbb{1} \oplus \mathbb{L} \oplus \mathbb{L}^{\otimes 2}$$

$$D^b(\mathbb{P}^2) = \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle = \langle D^b(\mathrm{Spec}\,k), D^b(\mathrm{Spec}\,k), D^b(\mathrm{Spec}\,k) \rangle$$

$NC(\mathrm{Spec}\,k) =: \mathbb{1}$ "unit"

$$\Rightarrow NC(X) = \mathbb{1}^{\oplus 3}$$

Orbit category

$\mathcal{C}$: additive, $\otimes$-category, $\mathbb{L} \in \mathcal{C}$ invertible

$\mathcal{C}/_{\otimes \mathbb{L}}$ is the cat with same obj as $\mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}/_{\otimes\mathbb{L}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}\left(X, \bigoplus_{i \in \mathbb{Z}} (Y \otimes \mathbb{L}^{\otimes i})\right)$$

R cannot be defined

if $R = \mathbb{Z}$

$\exists$ Quadric Q s.t. $D^b(Q)$ generated by exc obj

$\longrightarrow N(Q)_{\mathbb{Z}} = 1^{\oplus l}$

But $M(Q)_{\mathbb{Z}} \neq \oplus \mathbb{L}^i$ in the orbit cat.

[Marcolli-Tabuada]

$$\mathrm{Ab}(k)_{\mathbb{Q}} \hookrightarrow \underbrace{\mathrm{NNum}(k)}_{\text{ps-abel}}{}_{\mathbb{Q}} \text{ is fully faithful}$$

$A \in \mathrm{dgcat}(k)$ $\quad J(A) :=$ the comp of $N(A)$ in the image of $\mathrm{Ab}(k)_{\mathbb{Q}}$

Thm. If $\mathcal{A} = \mathrm{Perf}^{dg}(X)$
$$\mathbb{J}(X)_{\mathbb{Q}} \simeq \prod_{i=1}^{d-1} J_a^i(X)_{\mathbb{Q}}$$

Suppose: $X, Y \in \mathrm{Smproj}(\mathbb{C})$
$$\Phi : \mathrm{Perf}^{dg}(X) \to \mathcal{A}_X \subset \mathcal{A}_Y \hookrightarrow \mathrm{Perf}^{dg}(Y)$$
$$\mathrm{Perf}^{dg}(X) = \langle \mathcal{A}_X, \mathcal{A}_X^{\perp} \rangle \qquad \mathrm{Perf}^{dg}(Y) = \langle \mathcal{A}_Y, \mathcal{A}_Y^{\perp} \rangle$$

Thm [- , Tabuada]

(1) $\exists$ well-defined $\tau : \prod_{i=0}^{d_X-1} J_a^i(X)_{\mathbb{Q}} \to \prod_{i=1}^{d_Y-1} J_a^i(Y)_{\mathbb{Q}}$.

(2) if $\mathbb{J}(\mathcal{A}) = 0$, τ is injective, and if $\mathbb{J}(\mathcal{A}_Y) = 0$ too, then τ isogeny

(3) X, Y are representable and ω_X ample, antiample or trivial

and $\mathbb{J}(\mathcal{A}) = 0$, then $\tau : J(X) \hookrightarrow J(Y)$ preserves the principal polarization

Ex. $J(\mathcal{A}_X) = 0$ if $\mathcal{A}_X$ is generated by exc obj.

Application categorical Torelli
X satisfies classical Torelli

Thm [–, Tabuada]
Conjecture [Paranjape] is true
if X is of type (2,2) on X
or (2,2,2) dim X odd

Thm [–, Marcolli, Tabuada]
$NCvoev(X) = NNum(X)$
is equiv to $Cvoev(X)$.