Response theory of relativistic quantum Hall: a new topological current

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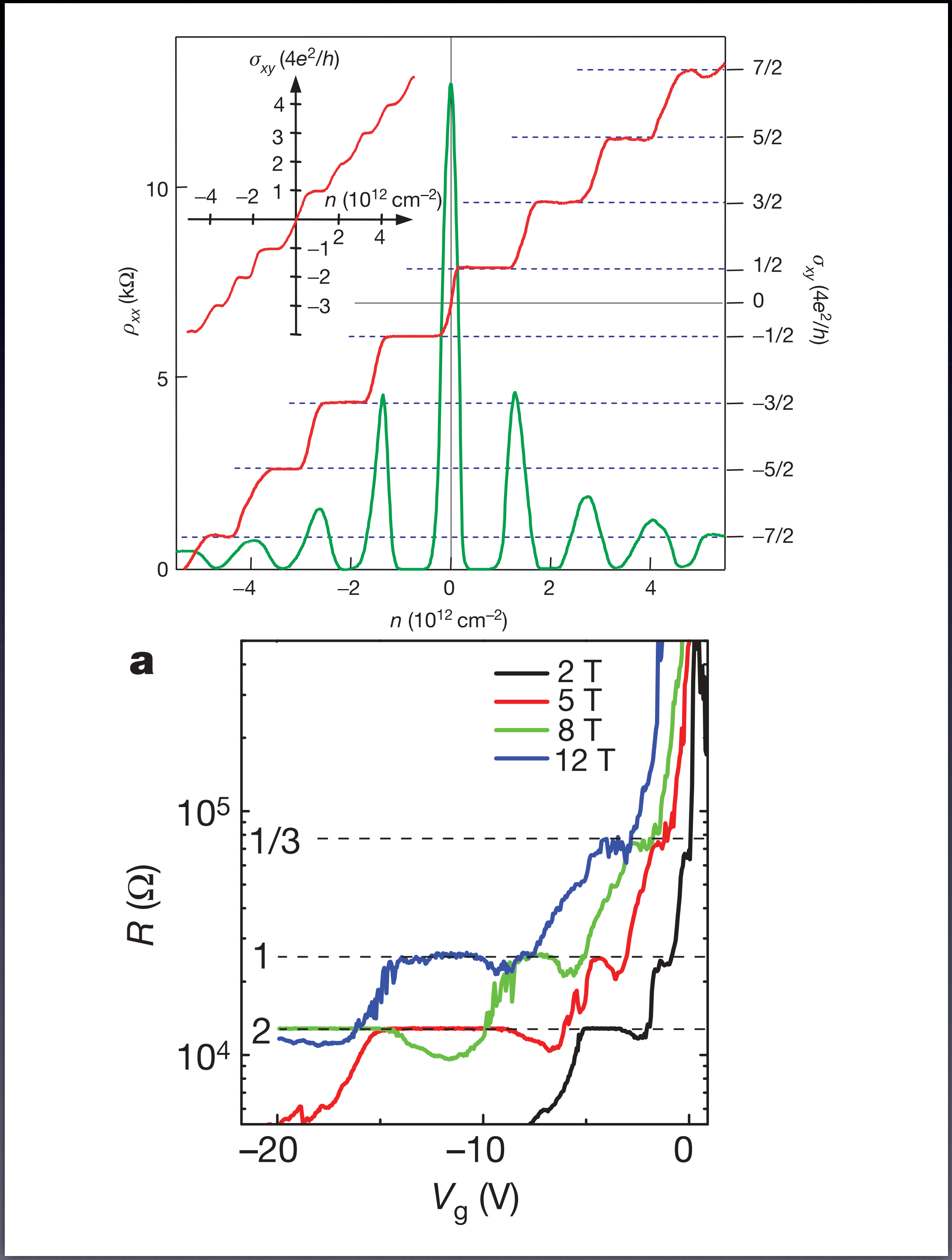
Outline

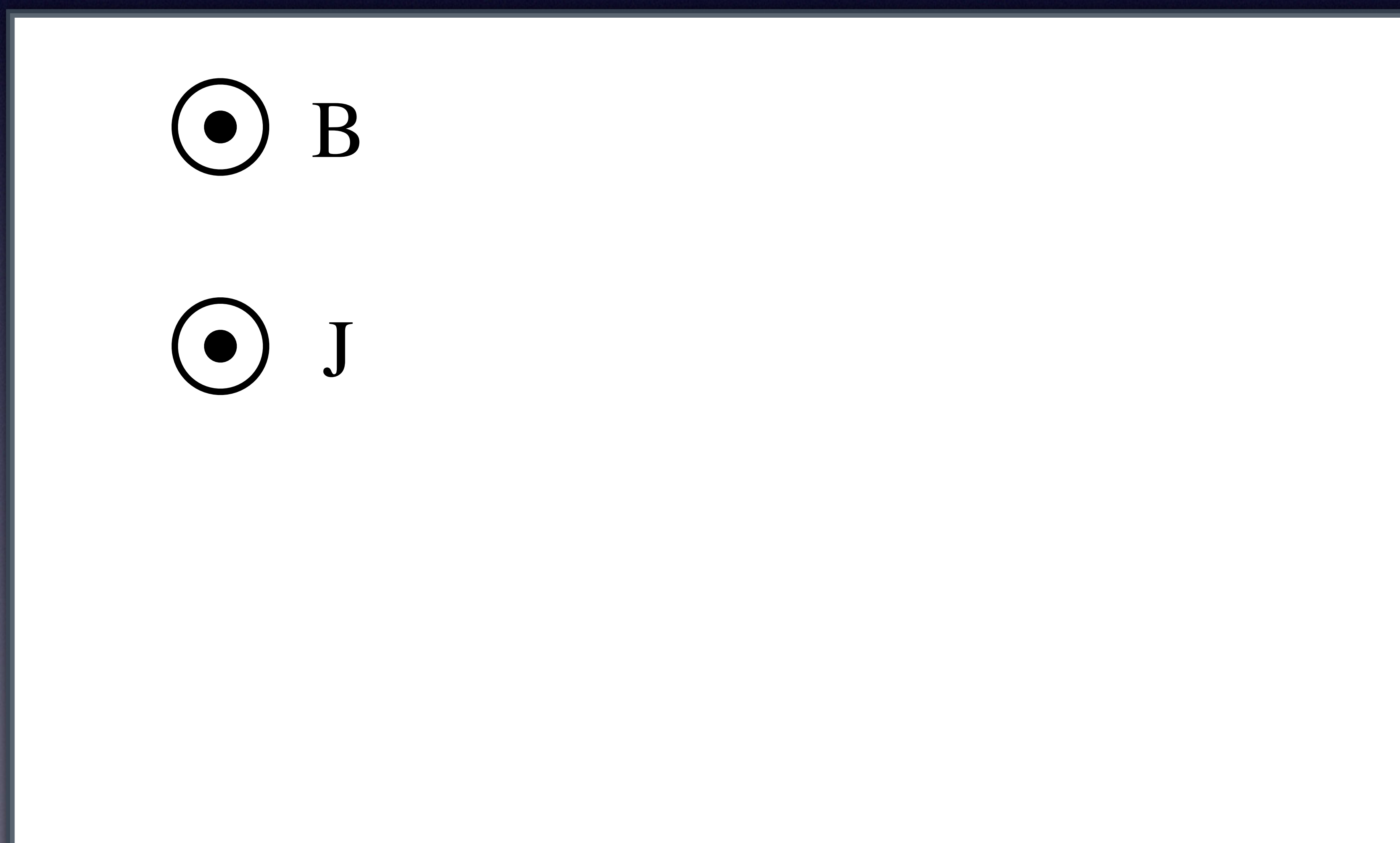
- Part I:
  - What is the QHE?
  - Response theory
  - Shift & Hall transports
- Part II:
  - Response theory pt. 2
  - A 2+1 topological current
  - Shift and Hall transports
  - Simplifications at the “LLL”
  - Conformal QH systems



Quantum Hall Effect

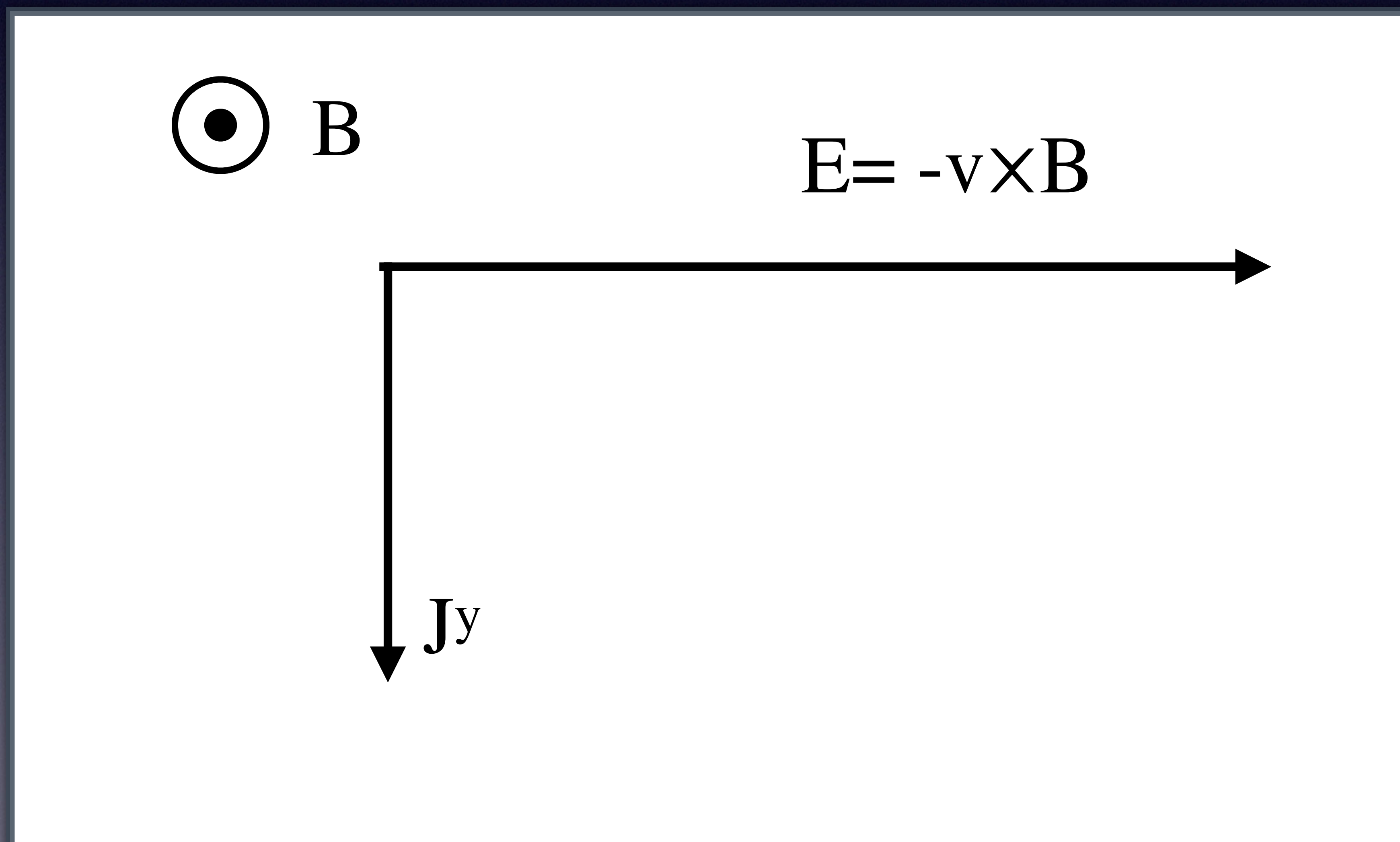
- 2D Electron gas in magnetic field
  - Classical Hall (Easy)
  - Integer Quantum Hall (Medium)
  - Fractional Quantum Hall (Hard)

- Classical Hall Effect: A perfect 2D e^- gas
  - Const. B_z, electron density $J^0 = -ne$



- Boost down: $E = -\frac{1}{c} v \times B = \frac{1}{nec} J \times B,$

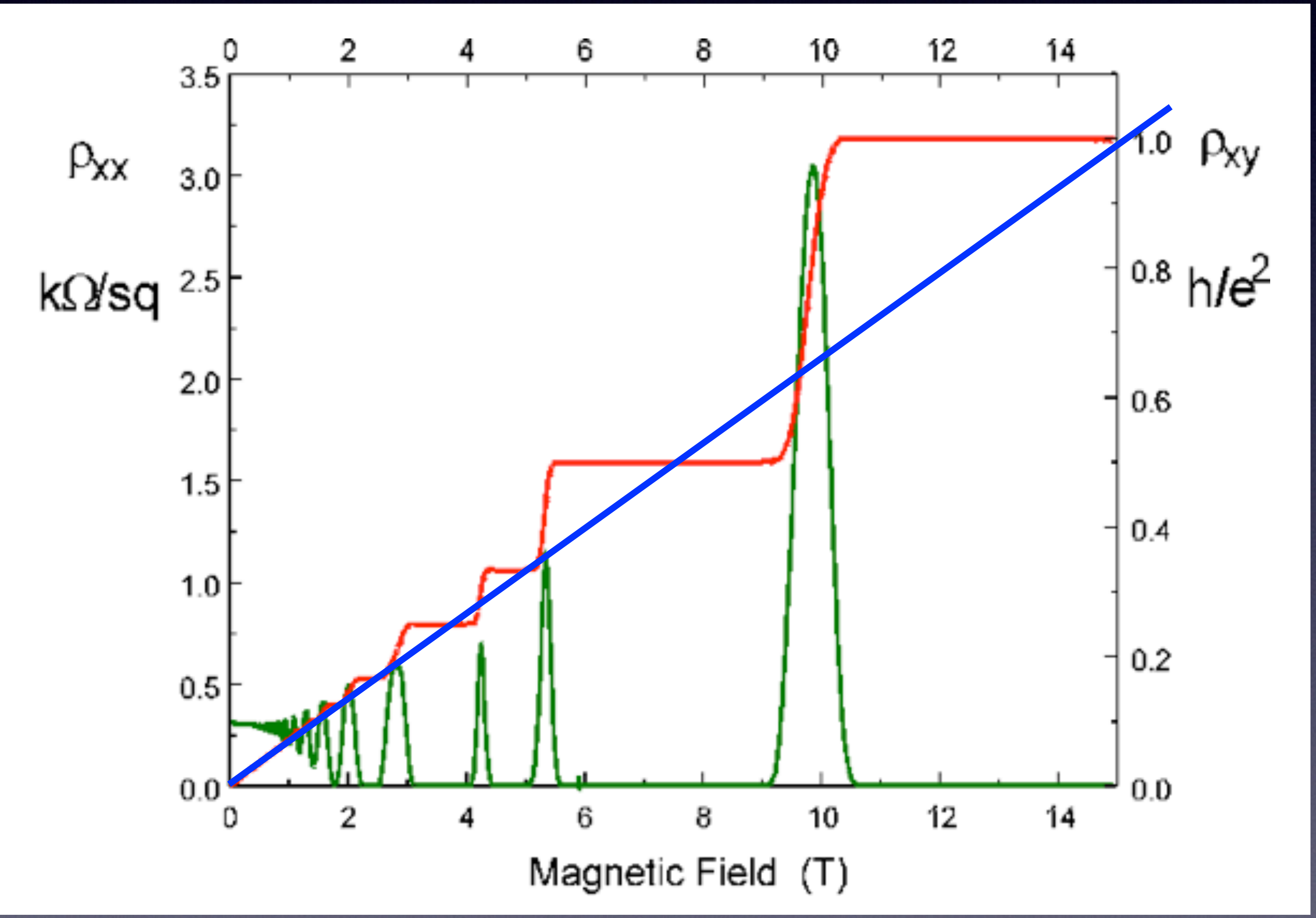
$$E = \rho \cdot J, \rho_{xy} = \frac{B}{nec}$$



- Integer Quantum Hall Effect: Schrödinger w/ B_z

$$\sigma_{xy} = 1/\rho_{xy} = n\frac{e^2}{2\pi\hbar}$$

$$\sigma_{ij} = \begin{pmatrix} 0 & \sigma_{xy} \\ \sigma_{xy} & 0 \end{pmatrix}$$



- Note $\rho_{xx} = 0 = \sigma_{xx}$

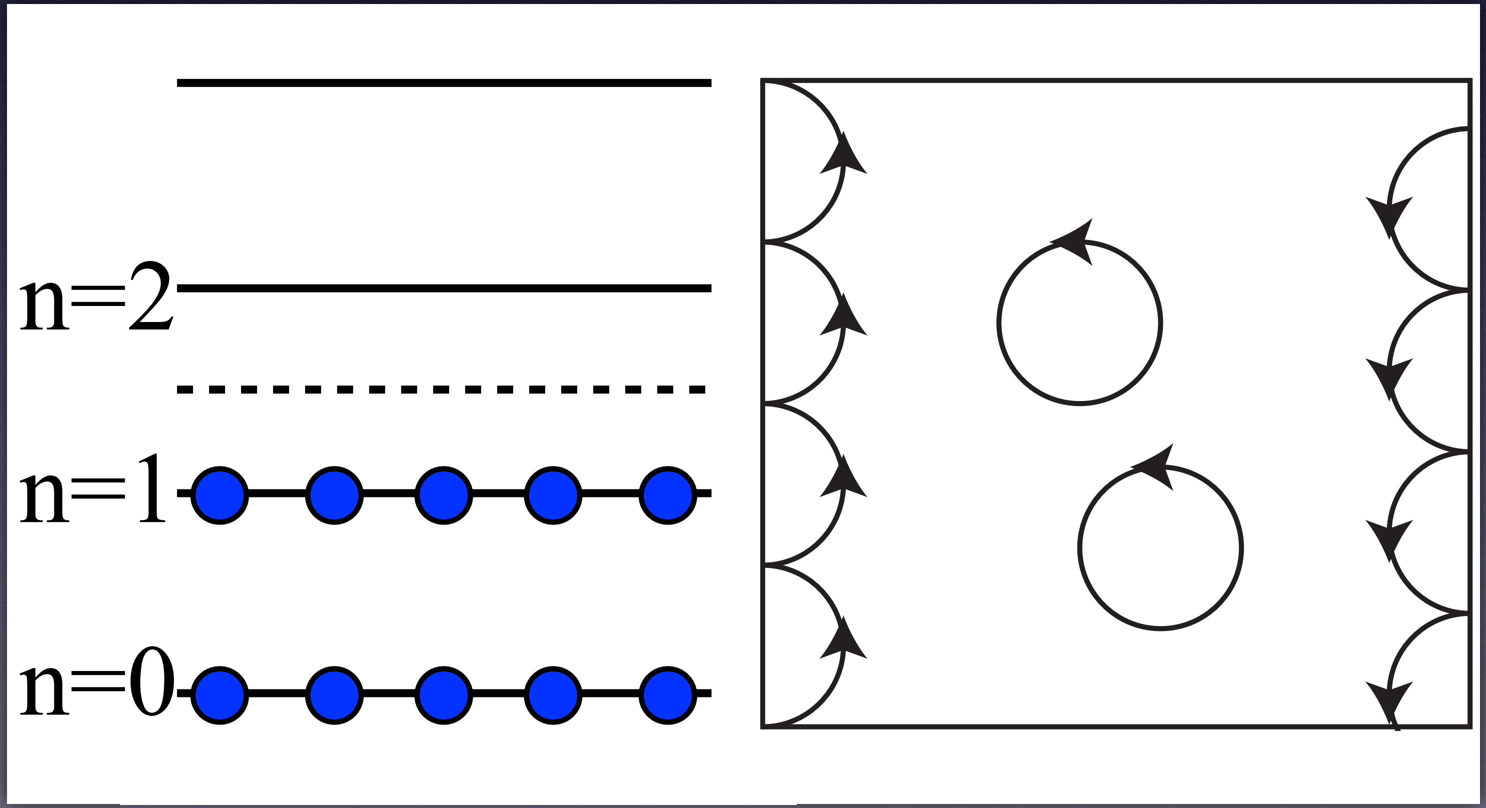
- Integer Quantum Hall Effect: Schrödinger w/ B_z

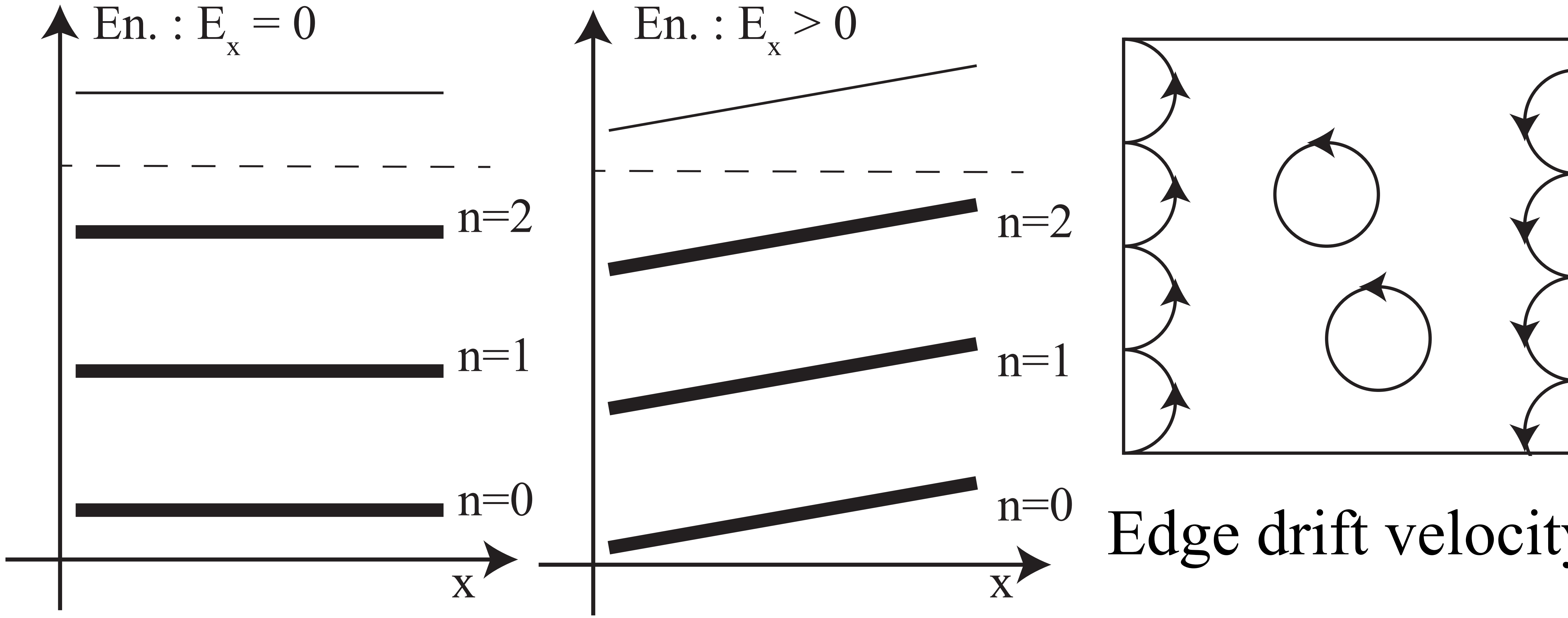
$$H = \frac{1}{2m}(p - eA/c)^2,\ A = \frac{B}{2}(xdy - ydx)$$

$$E_n = \left(n + \frac{1}{2}\right)\frac{eB}{mc},\ \psi_{n=0}(z,\bar{z}) = f(z)e^{-\frac{Be}{4\hbar c}|z|^2}$$

1 state per *hc*/*eB* area

- Huge degeneracy!
- filled levels: no free e^-s
  - localized in orbits
  - transport in edge





Edge drift velocity

$$v = E \times B/B^2$$

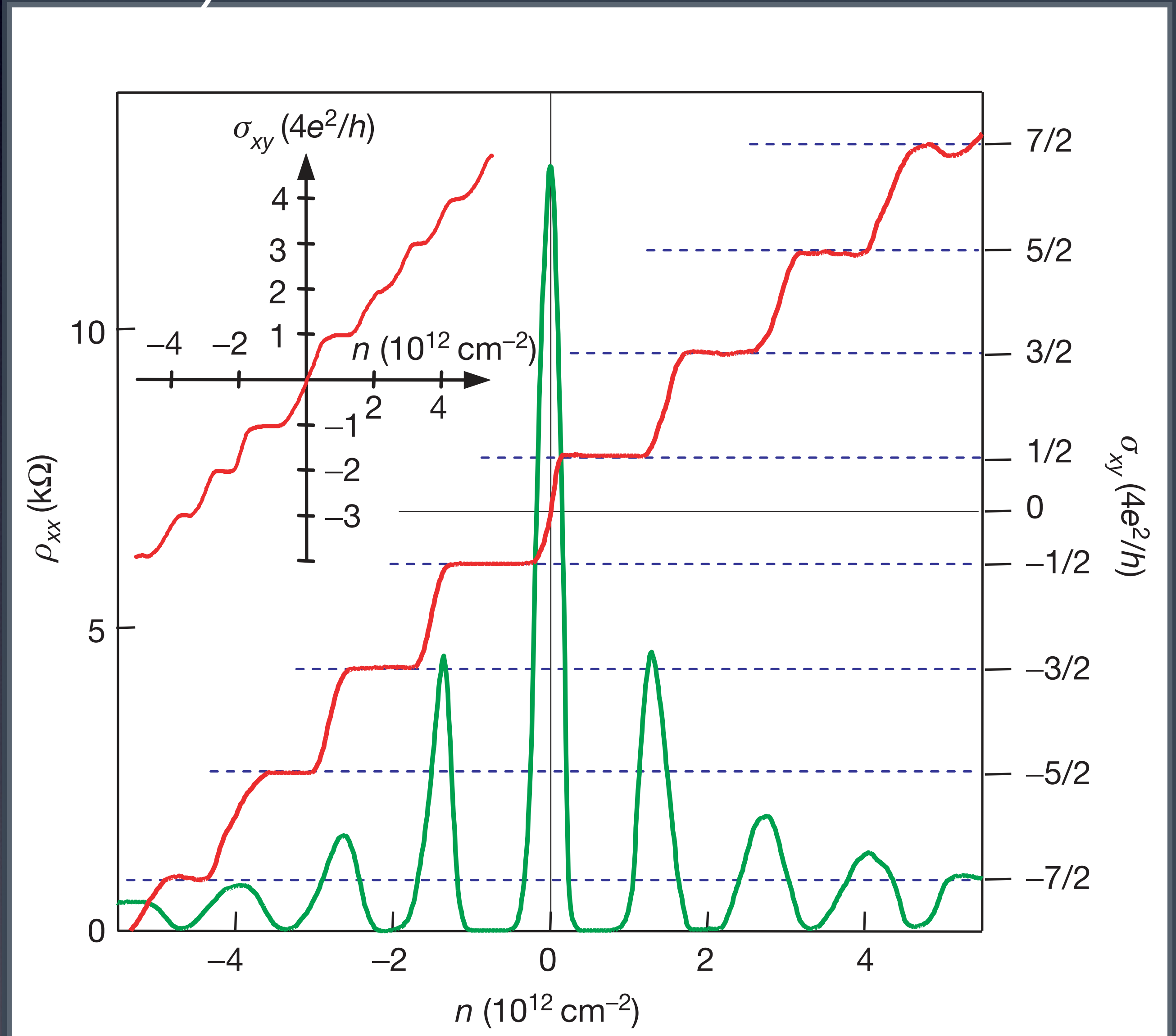
- Relativistic IQHE: Dirac w/ B_z: $(\text{Dirac})^2$ = Schr.

$$\sigma_{xy} = 1/\rho_{xy} = N_f \left(n + \frac{1}{2} \right) \frac{e^2}{2\pi\hbar}$$

$$\sigma_{ij} = \begin{pmatrix} 0 & \sigma_{xy} \\ \sigma_{xy} & 0 \end{pmatrix}$$

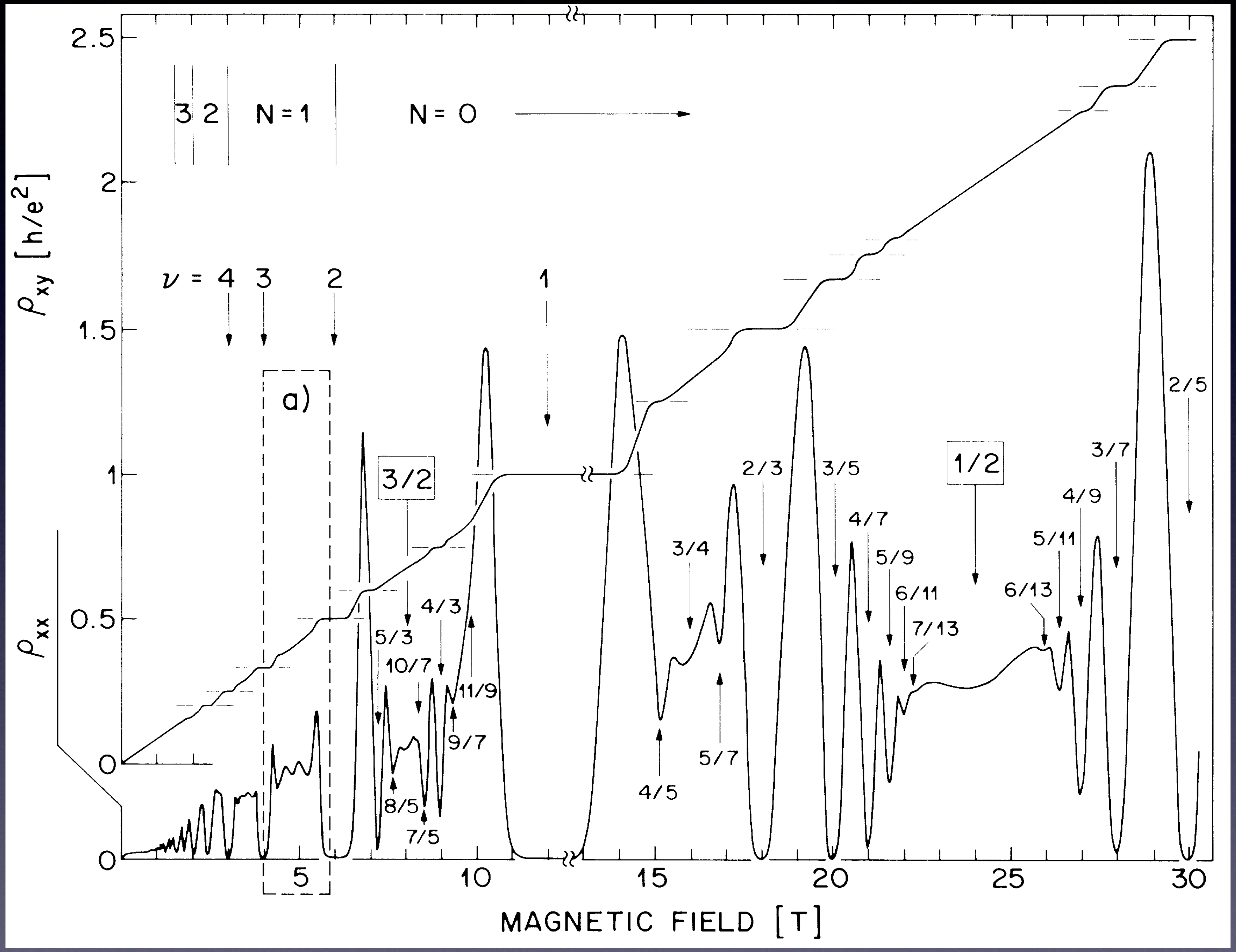
- Again,

$$\rho_{xx} = 0 = \sigma_{xx}$$



Novoselov et al Nature 438, 197 (2005)

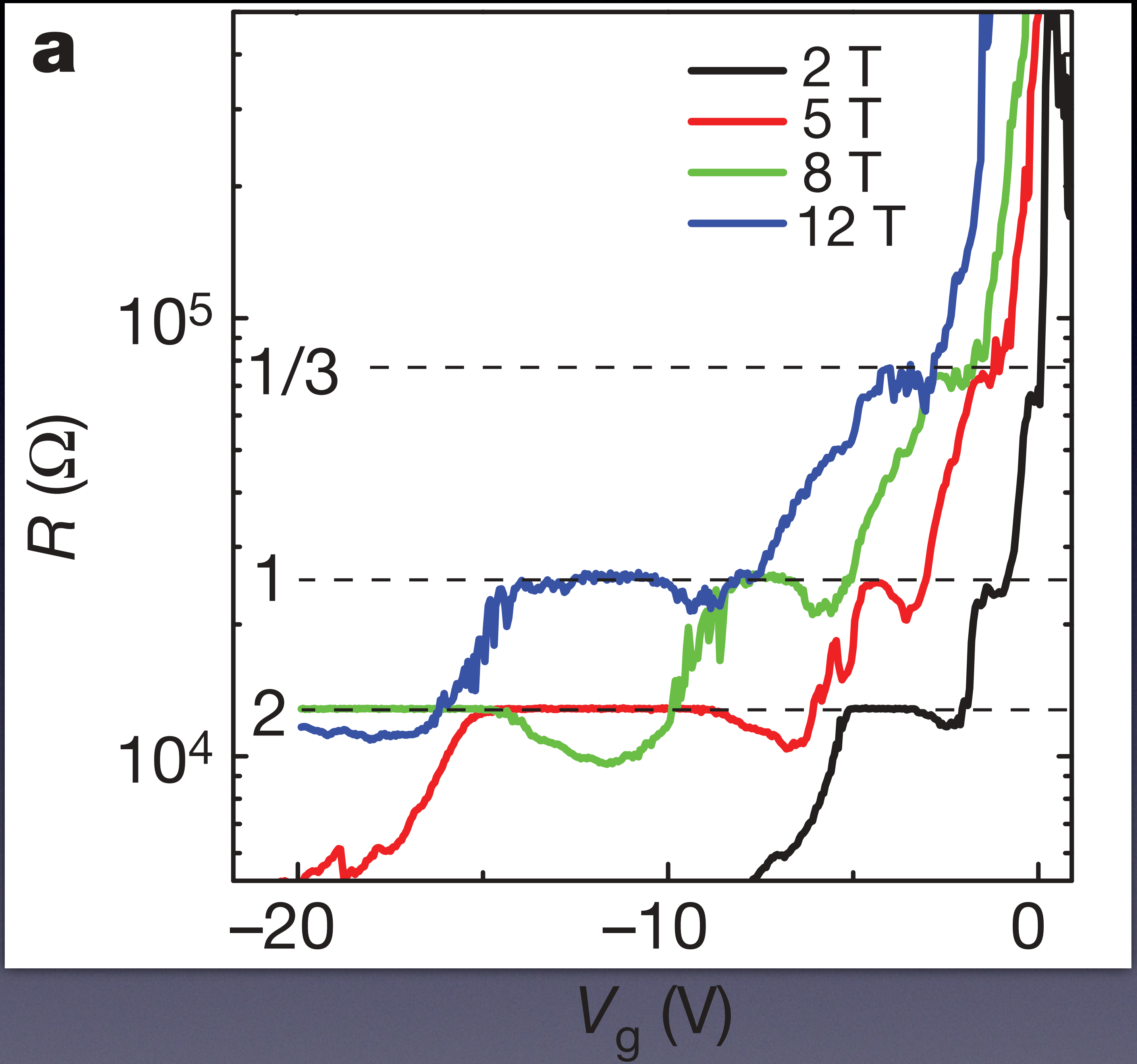
- Fractional Quantum Hall Effect:



Willet et al Phys. Rev. Lett. 59, 1776 (1987)

- $$\sigma_{xy} = \frac{p}{q}\frac{e^2}{2\pi\hbar}$$

- Fractional Relativistic Quantum Hall Effect:



- $$\sigma_{xy} = \frac{p}{q}\frac{e^2}{2\pi\hbar}$$

Du et al Nature 462, 192 (2009)

- Fractional Quantum Hall Effect:
- Naive: partial filled LL has huge degeneracy
- How do interactions lift degen. & gap system?
- How do we describe the system? Response physics?

Response Theory

- Chern-Simons: does it give all "universal" response?

$$S[A_{EM}] = \frac{\nu}{4\pi} \int d^3x \epsilon^{abc} A_a \partial_b A_c$$

$$j^a = \frac{\delta S}{\delta A_a} = \frac{\nu}{4\pi} \epsilon^{abc} F_{bc}, \; Q = \frac{\nu}{2\pi} B$$

$$j_y = \sigma_{xy} E_x, \; \sigma_{xy} = \frac{\nu}{2\pi}$$

- Is there more?

Shift & Wen-Zee

- QH on plane: $Q = \nu N_\phi$ (def'n of filling fraction!)
- IQH on sphere: $\nu=1$ w/ filled 1st LL
  - $Q = \nu(N_\phi + 1)$
- IQH on sphere: $\nu=1$ w/ filled 2nd LL
  - $Q = \nu(N_\phi + 3)$
- Topological parameter in EFT: Shift!
  - $Q = \nu(N_\phi + S\,\chi)$

- In NR case, we can add a term that looks like
  - $\Delta S = \# \, \eta_H \int A \wedge d\omega$ (ω: U(1) connection for 2D)
  - $\Delta Q \propto \eta_H \, \chi$
- We can write in NR case due to 2+1 break
  - This term gives a non-dissipative transport
  - $\langle T_{xx}(\omega) \, T_{xy}(\omega) \rangle = 2i\omega \, \eta_H$ (Hall viscosity)
- How do we get this in Lorentz Inv't EFT?

- A new 3D topological current:
- With $B >> E$ everywhere, $F^2 > 0$,

$$b = \left(\frac{1}{2}F^2\right)^{1/2}, \; u^a = \frac{1}{2b}\epsilon^{abc}F_{bc}$$

$$J^a = \frac{1}{8\pi}\epsilon^{abc}\epsilon^{def}u_d\left(\nabla_b u_e \nabla_c u_f - \frac{1}{2}R_{bcef}\right),$$

$$\nabla \cdot J = 0, \; \Delta S = \kappa \int A \cdot J$$

- CS + this: $Q = \nu N_\phi + \kappa\chi/2, \; \eta_H = \dfrac{\kappa B}{8\pi}$

Outline of Part II

- Response theory in gapped systems (generalities)
- What is that 3D current?
- Shift and Hall transports
- Simplifications at the “LLL”
- Conformal QH systems

~ 5 minute break ~

Response theory

- Free massive scalar, $Z[J] =$

$$\int \mathcal{D}\phi e^{i\int -\frac{1}{2}(\nabla\phi)^2 - \frac{m^2}{2}\phi^2 + J\phi} = e^{\frac{i}{2}\int \frac{d^dk}{(2\pi)^d}\frac{J(k)J(-k)}{k^2+m^2}}$$

- When studying $|k| \ll m$ resp.,

$$\ln Z = \frac{i}{2}\int \frac{d^dk}{(2\pi)^d} J(k)\left[\frac{1}{m^2} - \frac{k^2}{m^4} + \dots\right] J(-k)$$

- For QHE, if $|k| \ll m_{LL} \propto eB$, CS+… resp. thy. works

- Power counting: background of ~ const. B, so
- $F = O(p^0)$, $A = O(p^{-1})$. $g_{ab} = O(p^0)$, Riem. $= O(p^2)$.
    - $A \wedge F = O(p^{-1})$.
        - This gives Hall cond. and filling fraction
- We will work out all gauge inv't terms to $O(p^1)$
    - (where the Hall visc. & shift live)

- Recall $b = \left(\frac{1}{2}F^2\right)^{1/2}, \ u^a = \frac{1}{2b}\epsilon^{abc}F_{bc}$

$$b\ u_a = (*F)_a, \ \nabla_a(bu^a) = 0$$

- Bianchi identity gives conserved Hall fluid current
- gauge invariant and O(p^0) by our counting. We will work with these in what follows.

- O(p[0]) term: $S_0 = \int \varepsilon(b)$
  - Depends on metric, gives stress tensor

$$T^{ab} = (\varepsilon + P)u^a u^b + P g^{ab},\ P = b\varepsilon'(b) - \varepsilon(b)$$

  - Ideal fluid with velocity from B field
    - EOS comes from ε(b)

- O(p[1]) terms: $f_1(b)\epsilon^{abc}u_a\partial_b u_c,\ \ f_2(b)u^a\partial_a b$
  - But if $f_2 = bf_3',\ \ f_2(b)u^a\partial_a b = \nabla_a(bu^a f_3(b))$
  - Total derivative! So we only need this first term

- This isn't quite it - as advertised, there is another term
  - New term, like A∧F, is topological

Topological current

- Consider the following current:

$$J^a = \frac{1}{8\pi}\epsilon^{abc}\epsilon^{def}u_d\left(\nabla_b u_e \nabla_c u_f - \frac{1}{2}R_{bcef}\right),$$

- This current is locally conserved when $u^2=-1$
- Proof: (apologies for the next slide!)

$$
\begin{aligned}
&8\pi\nabla_\mu J^\mu \\
=&\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\nabla_\mu u_\alpha\nabla_\nu u_\beta\nabla_\rho u_\gamma + 2\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}u_\alpha\nabla_\mu\nabla_\nu u_\beta\nabla_\rho u_\gamma \\
&\frac{1}{2}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\nabla_\mu u_\alpha R_{\nu\rho\beta\gamma} - \frac{1}{2}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}u_\alpha\nabla_\mu R_{\nu\rho\beta\gamma} \\
=&\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\nabla_\rho u_\gamma\left(R_{\mu\nu\beta\lambda}u^\lambda u_\alpha + \frac{1}{2}R_{\mu\nu\beta\alpha}\right) \\
=&\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\nabla_\rho u_\gamma\Bigg(-R_{\mu\nu\delta\lambda}({P_\parallel}^\delta_\beta + {P_\perp}^\delta_\beta){P_\parallel}^\lambda_\alpha \\
&\quad+\frac{1}{2}R_{\mu\nu\delta\lambda}({P_\parallel}^\delta_\beta + {P_\perp}^\delta_\beta)({P_\parallel}^\lambda_\alpha + {P_\perp}^\lambda_\alpha)\Bigg) \\
=&+\frac{1}{2}\varepsilon^{\mu\nu\rho}\varepsilon^{\alpha\beta\gamma}\nabla_\rho u_\gamma R_{\mu\nu\delta\lambda}{P_\perp}^\delta_\beta{P_\perp}^\lambda_\alpha = 0.
\end{aligned}
$$

- What is this current?
  - Consider M_3 = (Time) × (Riemann Surface), $u = \partial_t$

$$J^0 = \frac{^{(2)}R}{8\pi}, \; Q = \frac{\chi}{2} = 1 - g$$

  - Since J is conserved & locally defined by u, interpolate between metric & u factorization,

$$u^a = (\partial_t)^a + \mathcal{O}(t^3),$$

$$ds^2 = -dt^2 + ds_\Sigma + \mathcal{O}(t^3)$$

  - $Q = (1\text{-}g)$ for any (smooth) u^a, g_{ab}!

- We may now add a new term, $\kappa \int A_a J^a$
- Our final action, to O(p[1]), is

$$\mathcal{L} = \frac{\nu}{4\pi}\epsilon^{abc}A_a\partial_b A_c - \varepsilon(b) + f(b)\epsilon^{abc}u_a\partial_b u_c + \frac{\kappa}{8\pi}A_a\epsilon^{abc}\epsilon^{def}u_d\left(\nabla_b u_e \nabla_e u_f - \frac{1}{2}R_{bcef}\right)$$

Relativistic Shift

- From that previous action, one can quickly check

$$Q = \int d^2x \left(\frac{\nu}{2\pi} F_{12} + \frac{\kappa}{8\pi} J^0 \right) = \nu N_\phi + \kappa \chi / 2$$

Hall viscosity

- Linear response to strain:

$$\delta\langle T_{ij}\rangle = \lambda_{ijk\ell}u^{k\ell} - \eta_{ijk\ell}\dot{u}^{k\ell}$$

$$\eta_{ijk\ell} = \eta^S_{ijk\ell} + \eta^A_{ijk\ell},$$
$$\eta^S_{ijk\ell} = +\eta^S_{k\ell ij},$$
$$\eta^A_{ijk\ell} = -\eta^A_{k\ell ij}$$

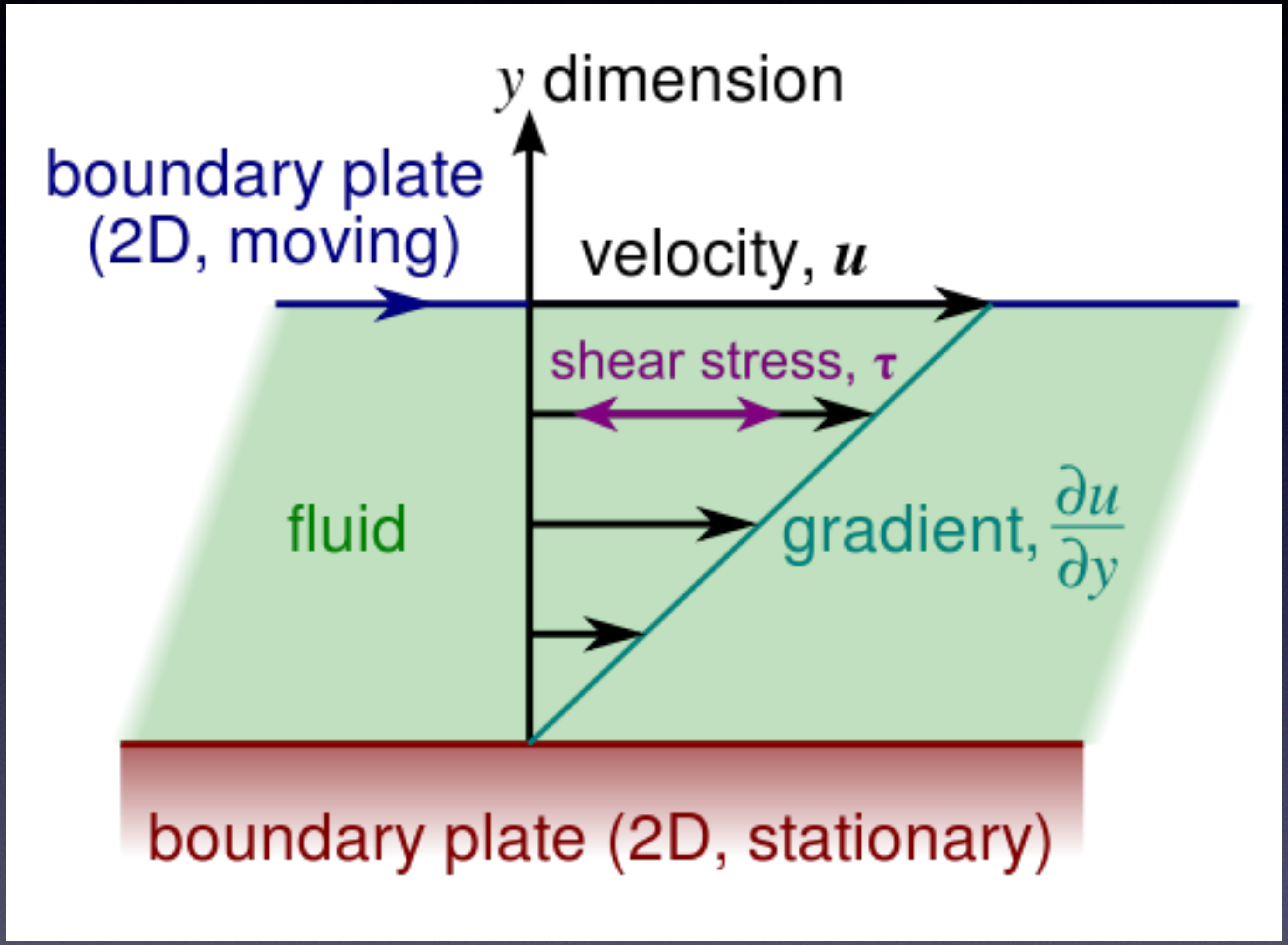
- For rot. inv't d+1,

$$\eta^S_{ijk\ell} = \zeta\delta_{ij}\delta_{k\ell} + \eta^{\mathrm{sh}}\left(\frac{1}{2}\delta_{i[k|}\delta_{j|\ell]} - \frac{2}{d}\delta_{ij}\delta_{k\ell}\right)$$

- P (or Lorentz Inv't T) broken 2+1:

$$\eta^A_{ijk\ell} = \eta^H(\delta_{jk}\epsilon_{i\ell} - \delta_{i\ell}\epsilon_{kj})$$

- Shearing a system:



$$\delta T_{xy} = -\eta^{sh}\dot{u}_{xy},$$
$$\delta T_{xx} = -\delta T_{yy} = \eta^{H}\dot{u}_{xy}$$

- Only contribution from $\kappa \int AJ: \ \delta g_{ij} = h_{ij}(t)$

$$\kappa \int AJ = -\frac{\kappa B}{32\pi} \int \epsilon^{jk} h^i{}_j \dot{h}_{ik}, \ \eta_H = \frac{\kappa B}{8\pi}$$

- Shift and Hall viscosity are simply related!

Response to static E

$$T_{ij} = P\delta_{ij} - \left(\frac{\kappa}{4\pi} + f'(B)\right)(\nabla \cdot E)\delta_{ij} + \frac{\kappa}{8\pi}(\partial_i E_j + \partial_j E_i)$$

- Rewrite in fluid language: drift vel., shear rate

$$v^i = \epsilon^{ij} E_j / B, \; V_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i - \delta_{ij} \partial \cdot v)$$

- Stress is $$T_{ij} = P\delta_{ij} + (\eta_H + Bf'(B))(\nabla \times v)\delta_{ij} + \eta_H \left(\epsilon_i{}^k V_{kj} + \epsilon_j{}^k V_{ki}\right)$$

Trace-free part fixed by η_H

- A few other transport results:
  - Corrections to Hall conductance:

$$\sigma_{xy}(\omega, q) = \frac{\nu}{2\pi} + \left(\frac{\kappa}{4\pi} + \frac{2f(B)}{B}\right)\frac{\omega^2}{B} - \frac{f'(B)}{B}\frac{q^2}{B} + \ldots$$

  - Thermal Hall:

$$\langle T^{tx}\rangle = \left(\frac{\kappa}{4\pi}B + f(B)\right)\partial_y h_{tt}$$

- (Gen'l $\langle TT\rangle$, $\langle Tj\rangle$, $\langle jj\rangle$ responses calculable…)

Time permitting…

- Simplifications at the zeroth Landau level
- Conformal quantum Hall
- Boundary terms
- 2+1 superfluid couplings
- General odd dimensions

Simplifications at "LLL"

- Relativistic system: zeroth level = Dirac zero modes
- With a representation $\gamma^0 = \sigma_3$, $\gamma^1 = i\sigma_2$, $\gamma^2 = -i\sigma_1$,

$$H = -i\gamma^0\gamma^i D_i - A_0 = -2i \begin{pmatrix} 0 & D \\ \overline{D} & 0 \end{pmatrix} - A_0.$$

- zeroth level: $\psi = \begin{pmatrix} \varphi \\ 0 \end{pmatrix}, \ \overline{D}\varphi = 0$

- Calculating the momentum density,

$$T^{0i} = -\frac{i}{4}\varphi^* \overleftrightarrow{D}^i \varphi = -\frac{1}{4}\epsilon^{ij}\partial_j n, \;\; n = \varphi^*\varphi$$

- From our response theory, we have

$$T^{0i} = -\epsilon^{ij}\partial_j \left(\frac{\kappa}{8\pi} b + f(b) \right)$$

- Therefore partially filled zeroth LL

$$f(b) = \frac{1}{8\pi}(\nu - \kappa) b$$

$$\sigma_{xy}(\omega, q) = \frac{\nu}{2\pi} + \frac{\nu}{\pi}\frac{\omega^2}{B} + \frac{\kappa - \nu}{8\pi}\frac{q^2}{B}$$

Conformal QH

- We said J was "topological"
- Define $\theta_{ab} = -\epsilon_{abc} J^c$, charge density 2-form
- Weyl trans. $g_{ab} \to e^{2\Omega} g_{ab},\ A_a \to A_a,\ u_a \to e^{\Omega} u_a,$

$$\Delta_\Omega \theta_{ab} = 4\epsilon_{abc} \nabla_d u^{[c} \partial^{d]} \Omega$$

- So charge density is Weyl invariant!

- A.J is therefore also Weyl invariant:

$$A_a J^a = *(A \wedge \theta), \ \Delta_\Omega(*A \wedge \theta) = A_a \nabla_b (u^{[c} \partial^{d]} \Omega)$$

- Integrating by parts,

$$\Delta_\Omega \left(\int A_a J^a \right) = \int F_{ab} u^a \partial^b \Omega$$

- This vanishes,

$$u \propto *F, \ F_{ab} u^a \propto \epsilon_{abc} u^c u^b = 0$$

- Can we make the entire EFT Weyl invariant?

$$b \to e^{-2\Omega} b,\ \Delta_\Omega \left(\sqrt{-g}\varepsilon\right) \propto e^{3\Omega}\varepsilon(e^{-2\Omega}b) - \varepsilon(b)$$

- Chosing $\varepsilon \propto b^{3/2}$ this cancels!

- From O(p^0) stress tensor:

$$T^{a}_{\varepsilon\ a} = 2P - \varepsilon = 2b\varepsilon' - 3\varepsilon : \varepsilon \propto b^{3/2},\ T^{a}{}_{a} = 0.$$

- $u \wedge du$ term:

$$\Delta_\Omega \left(f u \wedge du \right) \propto e^{2\Omega} f(e^{-2\Omega} b) - f(b)$$

- With $f(b) \propto b,$ entire action is Weyl invariant

- Response theory for CFT deformed to QH state!

Boundary term

- On manifold w/ bdy., Euler dens. has bdy. term.
  - bdy. current $k^a = \frac{1}{4\pi}\epsilon^{abc} n_c n^d \nabla_b t_a, \; t \propto u \wedge n$

$$u \cdot k = \frac{1}{4\pi}\int_\partial ds \; k, \;\; \int J^0 + \int_\partial k^0 = \chi/2$$

- Careful treatment requires study of edge modes…

2+1 superfluids

- for 2+1 superfluids, dualize phase to gauge field

$$\partial_a \psi \sim (\partial_t)_a, \;\; f = *d\psi$$

- Fluid conservation is Bianchi, so $f = da$
- Away from vortices, $f^2>0$: define u, b…

$$S = \ldots + \kappa \int a \cdot J, \;\; \eta_H = \frac{\kappa n_0}{16\pi}, \ldots$$

Higher dimensions

- When d odd, iterate:

$$J_d^\mu = \varepsilon^{\mu\mu_1\cdots\mu_{d-1}}\varepsilon^{\nu_1\cdots\nu_d}u_{\nu_1}$$

$$\Bigg(\nabla_{\mu_1}u_{\nu_2}\cdots\nabla_{\mu_{d-1}}u_{\nu_d} + \frac{\sigma}{2}\frac{d-1}{2!!}R_{\mu_1\mu_2\nu_2\nu_3}\nabla_{\mu_3}u_{\nu_4}\cdots\nabla_{\mu_{d-1}}u_{\nu_d}$$

$$+\left(\tfrac{\sigma}{2}\right)^2\frac{(d-1)(d-3)}{4!!}R_{\mu_1\mu_2\nu_2\nu_3}R_{\mu_3\mu_4\nu_4\nu_5}\nabla_{\mu_5}u_{\nu_6}\cdots\nabla_{\mu_{d-1}}u_{\nu_d}$$

$$+\left(\tfrac{\sigma}{2}\right)^3\frac{(d-1)(d-3)(d-5)}{6!!}R_{\mu_1\mu_2\nu_2\nu_3}R_{\mu_3\mu_4\nu_4\nu_5}R_{\mu_5\mu_6\nu_6\nu_7}$$

$$\times\nabla_{\mu_7}u_{\nu_8}\cdots\nabla_{\mu_{d-1}}u_{\nu_d}+\cdots+\left(\tfrac{\sigma}{2}\right)^{\frac{d-1}{2}}R_{\mu_1\mu_2\nu_2\nu_3}\cdots R_{\mu_{d-2}\mu_{d-1}\nu_{d-1}\nu_d}\Bigg).$$

- Consider again when u is normal to hypersurface Σ:

$$u \cdot J = (-\tfrac{1}{2})^{\frac{d-1}{2}} \epsilon^{a_1 \ldots a_d} \epsilon^{b_1 \ldots b_d} u_{a_1} u_{b_1}$$
$$\times\, {}^{(d-1)}R_{a_2 a_3 b_2 b_3} \cdots {}^{(d-1)}R_{a_{d-1} a_d b_{d-1} b_d}$$

$$\int u \cdot J = (-4\pi)^{\frac{d-1}{2}} (\tfrac{d-1}{2})! \chi(\Sigma)$$

- This motivates the odd-dimensionality of our current.
- In even dimensions, we get $\nabla \cdot J \propto (\mathrm{Euler\ Density})$

Conclusions

- Found analog of Wen-Zee term for relativistic QH
  - New current of topological nature
  - Reproduces shift without breaking Lorentz
  - Relates shift and η^H

- New interesting current to couple to odd-d fluids
- Lots more to investigate in this current!