GTM 2

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X closed, oriented, Riem 4-manifold

I. Set up :

Spin^c structure

$$\mathrm{Spin}(4) \xrightarrow{2:1} SO(4) \quad \text{universal cover}$$

$$\mathrm{Spin}^c(4) = (\mathrm{Spin}(4) \times U(1)) / \{\pm 1\}$$

$$0 \to S^1 \to \mathrm{Spin}^c(4) \to SO(4) \to 0$$

Def A spin^c structure on X is a principal $\mathrm{Spin}^c(4)$-bundle

$$\begin{tikzcd} P \arrow[rr] \arrow[dr] & & P_{SO(4)}(TX) \arrow[dl] \\ & X & \end{tikzcd}$$

$$\mathrm{Spin}(4) = SU(2) \times SU(2)$$

A spin^c structure gives rise to a rank 4 hermitian bundle

$$S_X = S^+ \oplus S^- \qquad \text{spinor bundles}$$

Alternatively, a spin^c structure is an isometry

$$\rho : TX \to \mathrm{Hom}(S_X, S_X) \qquad \text{Clifford multiplication}$$

locally $\{e_0, e_1, e_2, e_3\}$ ONB, $\rho(e_i) = \begin{bmatrix} 0 & -\sigma_i^* \\ \sigma_i & 0 \end{bmatrix}$

$$\sigma_0 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} i & \\ & -i \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} & i \\ i & \end{bmatrix}$$

ρ extends to $\Lambda^* T^* X$

Fact A spin^c structure for an oriented 4-mfd always exists.

Furthermore, $\{\text{spin}^c \text{ str.}\} \xleftrightarrow{1:1} \left\{ \begin{matrix} \text{complex} \\ \text{line bundle} \end{matrix} \right\} \simeq H^2(X; \mathbb{Z})$

II. SW equations : Fix X and a spin^c str. $\mathfrak{s}$

- A unitary connection on S_X is a spin^c connection if ρ is parallel

$$\nabla_A (X \cdot \varphi) = \nabla_{LC} X \cdot \varphi + X \cdot \nabla_A \varphi$$

$\mathcal{U} := \{\text{spin}^c \text{ connection}\} \longleftrightarrow \Omega^1(X; i\mathbb{R})$ affine space

- Given a spin^c connection, we define the Dirac operator D_A as a composite

$$\Gamma(S_X) \xrightarrow{\nabla_A} \Gamma(T^*X \otimes S_X) \xrightarrow{\rho} \Gamma(S_X)$$

There is a decomposition $\quad D_A = D_A^+ + D_A^-$

$$D_A^{\pm} : \Gamma(S^{\pm}) \to \Gamma(S^{\mp})$$

- Hodge star $\quad * : \Omega^2(X) \to \Omega^2(X)$

self-dual $\quad \Omega^+(X) = \{ \omega \in \Omega^2(X) \ ; \ *\omega = \omega \}$

projection $\quad \Omega^2(X) \to \Omega^+(X)$

$$\omega \mapsto \omega^+ = \frac{\omega + *\omega}{2}$$

Note $\quad \rho : \Omega^+(X) \to \mathfrak{su}(S^+)$

- For a pair $(A, \Phi) \in \mathcal{A} \times \Gamma(S^+)$

$$\left.\begin{aligned} \rho(F_A^+) - (\Phi\Phi^*)_0 &= 0 \\ D_A^+ \Phi &= 0 \end{aligned}\right\} \quad \text{Seiberg-Witten equations}$$

F_A curvature 2-form, $(\Phi\Phi^*)_0$ traceless endomorphism

The gauge group $\quad \mathcal{G} = Map(X, S^1)$

$$u \cdot (A, \Phi) = (A - u^{-1}du, u\Phi)$$

Def. $\quad \mathcal{M}(X, s) = \{ (A, \Phi) \text{ satisfies SW-eqn} \} / \mathcal{G}$

The monopole moduli space

Fact $\quad \mathcal{M}(X, s) \subset \mathcal{B}(X, s) = \mathcal{A} \times \Gamma(S^+) / \mathcal{G}$ is compact

A pair (A, Φ) s.t. $\Phi \equiv 0$ is called reducible

Otherwise, it is called irreducible

Fact When $b^+(X) \geq 1$, one can perturb the SW equation so that $\mathcal{M}(X, s_X)$ is regular and contains no reducibles.

$\Rightarrow \mathcal{M}(X, s_X)$ is a compact, smooth manifold of dimension

$$d = \frac{1}{4}\left(c_1(S')[X] - 2\chi(X) - 3\sigma(X) \right)$$
$$\quad \text{ind } d^+ + 2\,\text{ind } D_A^+$$

In this case,

$\mathcal{M}(X,s) \subset \mathcal{B}^*(X,s)$ irreducible part

$\mathcal{G}_0 = \text{Map}_*(X, S^1)$ with a basepoint $x_0 \in X$

$$\mathcal{A} \times (\Gamma(S^+) \setminus \{0\}) / \mathcal{G}_0 \longrightarrow \mathcal{B}^*(X,s) \quad \text{is an } S^1\text{-bundle}$$

$$\rightsquigarrow \quad u \in H^2(\mathcal{B}^*(X,s);\mathbb{Z})$$

<u>Def.</u> $SW(X,s) = \langle [\mathcal{M}(X,s)], u^{d/2} \rangle \in \mathbb{Z}$

if $b^+(X) \geq 2$ this is independent of the metric

When $d=0$, this is the sign count.

<u>III Calculation</u>

1. Positive scalar curvature

Weitzenböck formula

$$\int_X |D_A^+ \Phi|^2 = \int_X |\nabla_A \Phi|^2 + \int_X \langle \Phi, \rho(F_A^+)\Phi \rangle + \frac{1}{4}\int_X s|\Phi|^2$$

If (A,Φ) is a SW-solution,

$$0 = \int_X |\nabla_A \Phi|^2 + \frac{1}{4}\int_X |\Phi|^4 + \frac{1}{4}\int_X s|\Phi|^2$$

<u>Prop</u> If X has a metric of positive scalar curvature, then $SW(X) = 0$

Ex. S^4, $\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$, $k\,\mathbb{C}P^2 \# \ell\,\overline{\mathbb{C}P}^2$

2. Kahler <u>surfaces</u>

There is a canonical spin^c structure $K^* = \Lambda^{0,2}$

$$S^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}, \quad S^- = \Lambda^{0,1}$$

Thm. $SW(X, K^*) = \pm 1$

Taubes generalized to symplectic manifolds

Idea of proof: Show that $\left(A_0, \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}\right)$ is the only solution canonical connection

Ex. · K3

- A quintic surface of $\mathbb{C}P^3$ with odd intersection form
  $b^+(X) = 9$, $b^-(X) = 44$

Freedman $\Rightarrow$ X is homeomorphic to $9\,\mathbb{C}P^2 \# 44\,\overline{\mathbb{C}P^2}$

Above $\Rightarrow$ X is not diffeomorphic to ↑

3. Surgery

SW series $SW(X) = \sum_{s \in H^2(X;\mathbb{Z})} SW(X, s)\, e^s$

Suppose X is simply-connected and contains an embedded torus T s.t. (near cusp)

- T is homologically nontrivial
- zero self-intersection
- a basis of $H_1(T;\mathbb{Z})$ each bounds a disk of self-intersection -1

Assume that $X \setminus T$ is simply-connected

A tubular nbhd of T is $T \times D^2$ with boundary $T \times S^1$

Pick a knot $K \subset S^3$

Note that $S^3 \setminus K \simeq S^1 \times D^2$ homologically

$\partial((S^3 \setminus K) \times S^1) \simeq \partial(T \times D^2)$ "

Identify by sending the meridian $c_m \times 1$ to $1 \times S^1$

Define $X_K = (X \setminus T) \cup_{T^3} ((S^3 \setminus K) \times S^1)$

This preserves the intersection form, so $X_K \overset{\text{homeo}}{\simeq} X$

Thm. (Fintushel - Stern) $SW(X_K) = SW(X) \cdot \Delta_K(e^{2T})$

where $\Delta_K(\ )$ is the Alexander polynomial of K

(Idea of proof: Use a sequence of generalized log transform & gluing formula)

Ex. $X = K3$, $SW(X_K) = \Delta_K(e^{2F})$

F generic torus fiber

$\Rightarrow$ K3 has infinitely many smooth structures