

α' adventures

with Jim Liu ([arXiv:1304.3137 + in progress])

with Ilarion Melnikov and Savdeep Sethi ([arXiv:1403.4298])

with A. Coimbra, M. Graña, H. Triendl and D. Waldram([to appear])

“If a person has lived through war, poverty and love, he has lived a full life” (O. Henry)

if not ... computing higher derivative terms is the closest you get

Higher-derivative terms

- Probe the stringy regime (α' and genus expansion)
- Needed for consistency
- Important for applications

▷ Main equation - Heterotic Bianchi Identity :

$$dH_3 = \frac{\alpha'}{4} (\text{tr } R^2(\Omega_+) - \text{tr } F^2) + \mathcal{O}(\alpha')$$

where

$$R(\Omega_+) = R(\Omega^{\text{LC}} + \frac{1}{2}\mathcal{H}) = R(\Omega) + \frac{1}{2}d\mathcal{H} + \frac{1}{4}\mathcal{H} \wedge \mathcal{H}, \quad \mathcal{H}^{ab} = H_\mu{}^{ab} dx^\mu.$$

▷ In this talk:

- dualities: more α'
- supersymmetric heterotic backgrounds
- heterotic generalised geometry

- Choice of connection in heterotic BI

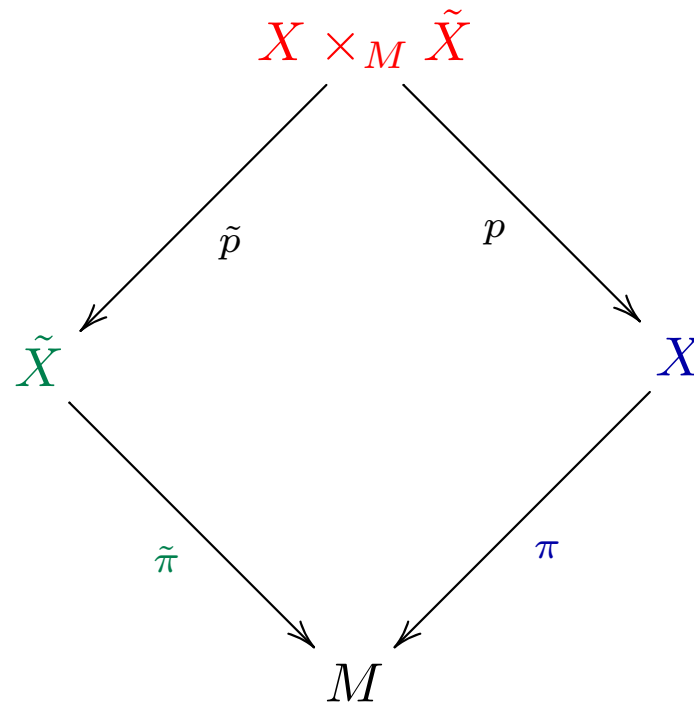
- ▷ tied to a choice of field redefinitions in the higher order curvature corrections to supergravity
- ▷ manifest (0,1) world-sheet supersymmetry: covariant Hermitian space-time metric $G \leftrightarrow \Omega_+ = \Omega^{\text{LC}} + \frac{1}{2}\mathcal{H}$
- ▷ (0,2) world-sheet SUSY likes Chern connection, but ... that requires G to pick up non-trivial space-time Lorentz and gauge transformations (and α' shifts in susy transformations)
- ▷ (Narain) T-duality

- Lessons of α' expansion:

- ▷ it is not physically correct to treat the heterotic space-time equations of motion, truncated to include just the leading order α' corrections, as a closed system
- ▷ simultaneously consider the α' expansion for both the solution and the equations of motion

I. Dualities and higher derivative couplings

T-duality - coord-independent $O(n, n)$ transformation (perturbative symmetry) in a background with n isometries v^i leading to **topology change**. For type II theories:



Correspondence space $Y = X \times_M \tilde{X}$:

- ◇ a circle bundle over X with first Chern class $\pi^*(c_1(\tilde{X}))$
- ◇ a circle bundle over \tilde{X} with first Chern class $\tilde{\pi}^*(c_1(\tilde{X}))$ ($\mathcal{L}_v H = 0 \Rightarrow d(\iota_v H) = 0$)

T-duality:

$$\pi_* H = c_1(\tilde{X}) \quad \tilde{\pi}_* \tilde{H} = c_1(X) \quad \in H^2(M, \mathbb{Z})$$

- Start with $dH = 0$ and

$$\mathcal{L}_v g_{10} = 0 = \mathcal{L}_v H :$$

$$\begin{array}{ccc}
 S^1 \hookrightarrow & X_{10} & \\
 & \downarrow \pi & \\
 & X_9 &
 \end{array}
 \quad de = \pi^* T \quad (\mathcal{L}_v e = 0)$$

$$H = \pi^* h_3 + \pi^* \tilde{T} \wedge e$$

$$dH = 0 \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \bullet \quad dh_3 = \tilde{T} \wedge T = \frac{1}{4} [(T_+)^2 - (T_-)^2] \\ \bullet \quad d\tilde{T} = 0 \end{array} \right.$$

- $T_{\pm} = T \pm \tilde{T}$
- Locally $H = dB = d(b_2 + b_1 \wedge e) \quad \Rightarrow \quad \tilde{T} = db_1 \quad \text{and} \quad h_3 = db_2 - b_1 \wedge T$
- h_3 - invariant under T-duality (b_2 is not!)
- Note $b_2 \rightarrow b_2 + d\lambda_1 + \lambda_0 T$, $b_1 \rightarrow b_1 + d\lambda_0$ and h_3 is gauge invariant

- Now turn to $dH = \frac{\alpha'}{4} [\text{tr } R^2(\Omega_+) - \text{tr } F^2]$

denote $X_4(\Omega_+, A) \equiv \text{tr } R^2(\Omega_+) - \text{tr } F^2 = \pi^* \tilde{X}_4 + \pi^* \tilde{X}_3 \wedge e$

$$dX_4 = 0 \quad \Leftrightarrow \quad \begin{cases} \bullet & d\tilde{X}_3 = 0 \\ \bullet & d\tilde{X}_4 - \tilde{X}_3 \wedge T = 0 \end{cases}$$

If $\mathcal{L}_v X_3^{(0)} = 0 \quad \Rightarrow \quad \tilde{X}_3 \text{ is exact: } \tilde{X}_3 = d\tilde{X}_2$

$$dH = \frac{\alpha'}{4} X_4(\Omega_+, A) \quad \Leftrightarrow \quad \begin{cases} \bullet & d\tilde{H}_2 = \frac{\alpha'}{4} \tilde{X}_3 \quad \Rightarrow \quad \tilde{T} = H_2 - \frac{\alpha'}{4} \tilde{X}_2 \\ \bullet & dh_3 = \frac{\alpha'}{4} [\tilde{X}_4 - \tilde{X}_2 \wedge T] - T \wedge \tilde{T} \end{cases}$$

$$\begin{aligned} -8\pi^2 \left(\tilde{X}_4 - \tilde{X}_2 \wedge T \right) &= R^{\alpha\beta}(\omega_+) \wedge R^{\alpha\beta}(\omega_+) - \frac{1}{2} R^{\alpha\beta}(\omega_+) \wedge \hat{T}_+^{\alpha\gamma} \hat{T}_+^{\beta\delta} e^\gamma \wedge e^\delta \\ &+ \frac{1}{2} (\nabla_\gamma \hat{T}_+^{\alpha\delta} + \frac{1}{2} h_\gamma^{\alpha\rho} \hat{T}_+^{\rho\delta}) (\nabla_\epsilon \hat{T}_+^{\alpha\iota} + \frac{1}{2} h_\epsilon^{\alpha\sigma} \hat{T}_+^{\sigma\iota}) e^\gamma \wedge e^\delta \wedge e^\epsilon \wedge e^\iota \\ &+ \mathcal{O}(\alpha') - F \wedge F \end{aligned}$$

- missing RF^2 and $(\nabla F)^2$ terms
- \hat{T}_+ (“graviphoton”) terms - OK with SUSY, but ...O(n) vs. O(n,n+16)
- $(\alpha')^2$ terms? $\hat{T}_+ = T + \tilde{H}_2 = T_+ + \frac{\alpha'}{4} \tilde{X}_2$

problems?

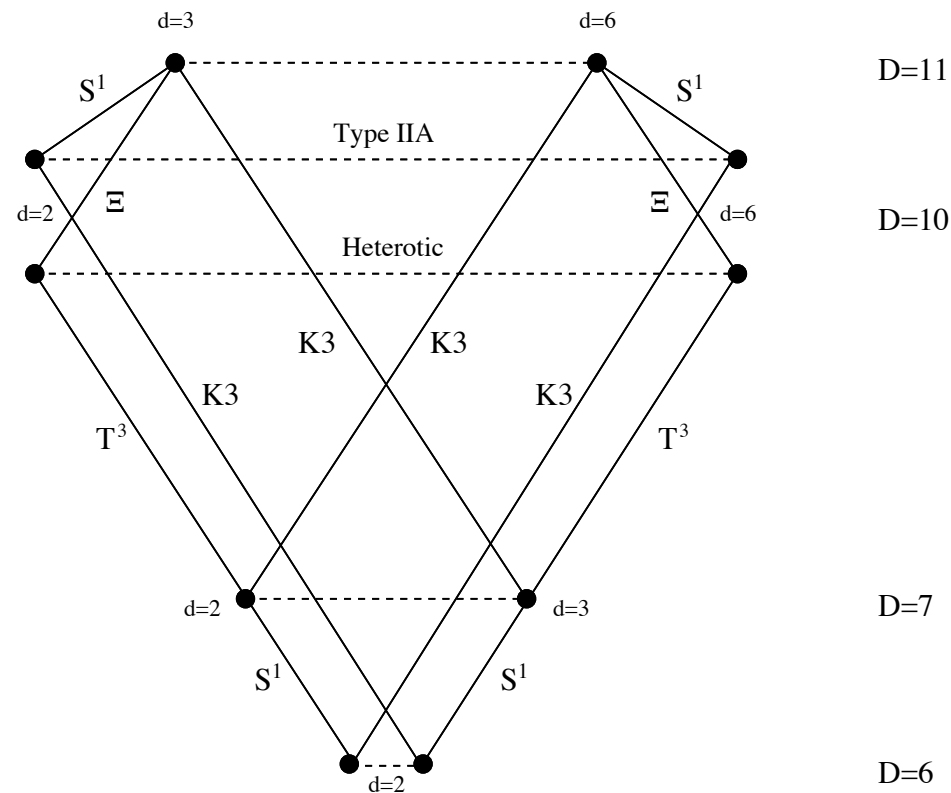
Computing on \mathbb{T}^n vs \mathbb{R}^n (generic point in moduli space)

- Appearance of couplings that vanish in decompactification limit
- $T_+^i \leftarrow V_I^i \mathcal{F}^I$ where $i = 1, \dots, n$ and $I = 1, \dots, 2n + 16$
- V_I^i is $O(n, n + 16)/O(n) \times O(n + 16)$ (part of) coset element
 - ◊ $V = \{V_I^i; V_I^\alpha\}$ with $\alpha = 1, \dots, n + 16$
 - ◊ $V_I^i = \mathbb{I}_n + \tilde{V}_I^i(\phi)$
 - ◊ T_+^i couplings start from 3-pt, the generic \mathcal{F}^I - form 4-pt
- no $(\alpha')^2$ terms
- Three-level $B\mathcal{F}^2 \mapsto (CGG)_M$; $BR\mathcal{F}^2$ and $B(\mathcal{D}\mathcal{F})^2 \mapsto ???$

Reduction of heterotic GS couplings

- (on $K3$ these give rise to GS couplings in 6d (1, 0) theory $\sim \alpha'$)
- on \mathbb{T}^n do not give rise to terms $\sim \alpha'$ but ...
- do **not** vanish! $\Rightarrow \iota_{v_1} \dots \iota_{v_n} (B \wedge X^{\text{GS}}) \neq 0$

Six-dimensional Heterotic/IIA duality



- $H^{\text{het}} = e^{2\phi} * H^{\text{IIA}}, \quad g_{\mu\nu}^{\text{het}} = e^{2\phi} g_{\mu\nu}^{\text{IIA}}, \quad \phi = -\varphi^{\text{IIA}}$
- Het. BI : $dH_3 = \frac{\alpha'}{4} (\text{tr } R^2 - \text{tr } F^2) \Leftrightarrow$ Type II EOM ($\Leftarrow B \wedge (F^I \wedge F^J d_{IJ} - \text{tr } R^2)$)
 - ◇ $B \wedge F \wedge F$ descends from 11d $C_3 \wedge G_4 \wedge G_4$ ($d_{iJ} = \int_{K3} \omega_I \wedge \omega_J$)
 - ◇ $B \wedge \text{tr } R^2$ descends from 11d $C_3 \wedge X_8(\Omega^{\text{LC}})$
- Ω_+ and the duality?

Heterotic effective action

$$e^{-1} \mathcal{L} = e^{-2\phi} \left[R + 4\partial\phi^2 - \frac{1}{12} H_{\mu\nu\rho}^2 - \frac{1}{4} \alpha' \text{tr} F_{\mu\nu}^2 + \frac{1}{8} \alpha' R_{\mu\nu\lambda\sigma}(\Omega_+) R^{\mu\nu\lambda\sigma}(\Omega_+) \right],$$

Dirac operator in the susy transformations has $\Omega_- = \Omega - \frac{1}{2} \mathcal{H}$ (similar to the sign flip in the local expressions for index theorems for Dirac operation with $\Omega \neq \Omega^{LC}$)

$$R(\Omega_+) = R(\Omega + \frac{1}{2} \mathcal{H}) = R(\Omega) + \frac{1}{2} d\mathcal{H} + \frac{1}{4} \mathcal{H} \wedge \mathcal{H}, \quad \mathcal{H}^{ab} = H_{\mu}{}^{ab} dx^{\mu}.$$

H has a non-trivial Bianchi identity

$$dH = \frac{1}{4} \alpha' \text{tr} R(\Omega_+) \wedge R(\Omega_+) - \frac{1}{4} \alpha' \text{tr} F \wedge F.$$

The equations of motion (up to $(\alpha')^2$ terms) :

$$\begin{aligned} R - 4\partial\phi^2 + 4\Box\phi - \frac{1}{12} H_{\mu\nu\rho}^2 - \frac{1}{4} \alpha' \text{tr} F_{\mu\nu}^2 + \frac{1}{8} \alpha' R_{\mu\nu\lambda\sigma}(\Omega_+) R^{\mu\nu\lambda\sigma}(\Omega_+) &= 0, \\ R_{\mu\nu} + 2\nabla_{\mu} \nabla_{\nu} \phi - \frac{1}{4} H_{\mu\rho\sigma} H_{\nu}{}^{\rho\sigma} - \frac{1}{4} \alpha' \text{tr} F_{\mu\rho} F_{\nu}{}^{\rho} + \frac{1}{4} \alpha' R_{\mu\lambda\rho\sigma}(\Omega_+) R_{\nu}{}^{\lambda\rho\sigma}(\Omega_+) &= 0, \\ d(e^{-2\phi} * H) &= 0, \\ e^{2\phi} d(e^{-2\phi} * F) + A \wedge *F - *F \wedge A + F \wedge *H &= 0 \end{aligned}$$

Two (1, 1) supergravities in 6D:

- $(1, 1)_{\text{het}}$: Non-trivial BI and single Dirac operator
- $(1, 1)_{\text{IIA}}$: $dH = 0$ and two Dirac operators $\partial + \Omega_{\pm} \rightarrow D_{\pm}$

Dualisation of $dH = \frac{1}{4}\alpha' \text{tr} R(\Omega_+) \wedge R(\Omega_+) + \dots = 2\pi^2\alpha'[\overline{X}_4 + \underline{X}_4]$:

- $\overline{X}_4(\text{het}) \rightarrow \overline{X}_4(\text{IIA})$ (up to terms vanishing on-shell)
- $\underline{X}_4(\text{het}) \rightarrow \text{CP-even terms in IIA}$

On IIA side: $d(e^{2\phi} * H + \alpha' * T) = 2\pi^2\alpha' \overline{X}_4$

- $\alpha' d * T$ comes from the variation of (one-loop) $\epsilon_4 \epsilon_4 R^2$ and $t_4 t_4 R^2$ terms in 6D IIA theory (lin. duality: $H = 0 \Rightarrow T \rightarrow 0$ & $\overline{X}_4 \rightarrow X_4$)

- CS term in $(1, 1)_{\text{IIA}}$: $\sim \alpha' B_2 \wedge \overline{X}_4$

$$2e^{-1} \delta \mathcal{L}_{\text{CP-even}} = \begin{cases} R_{\mu\nu\rho\sigma}(\Omega_+)^2 + E_4(\Omega_+) + 4R_{\mu\nu\rho\sigma}(\Omega_+) H^{\mu\rho a} H^{\nu\sigma a} - 4R_{\mu\nu}(\Omega_+) H^{2\mu\nu} \\ + \frac{2}{3} R(\Omega_+) H^2 + \frac{1}{9} (H^2)^2 - \frac{2}{3} H^4 \end{cases}$$

- Matched by 4-pt function calculation! **NO** $\mathcal{O}(\alpha')$ corrections to 6d duality dictionary

Summary of type II $(\alpha')^3$ couplings:

	No B	With B
e-o	$\frac{1}{8}(t_8\epsilon_{10} + \epsilon_{10}t_8)BR^4$	$\frac{1}{8}(t_8\epsilon_{10} + \epsilon_{10}t_8)BR^4(\Omega_+)$
+	$= B \wedge X_8(\Omega^{\text{LC}})$	$= \frac{1}{8}t_8\epsilon_{10}B(R^4(\Omega_+) + R^4(\Omega_-))$
o-e	$= \frac{1}{192(2\pi)^4}B \wedge (\text{tr } R^4 - \frac{1}{4}(\text{tr } R^2)^2)$	$= \frac{1}{2}B \wedge [X_8(\Omega_+) + X_8(\Omega_-)]$ $= \frac{1}{192 \cdot (2\pi)^4}B \wedge (\text{tr } R^4 - \frac{1}{4}(\text{tr } R^2)^2 + \text{exact terms})$
e-e	$t_8t_8R^4$	$t_8t_8R^4(\Omega_+) = t_8t_8R^4(\Omega_-)$
o-o	$\frac{1}{8}\epsilon_{10}\epsilon_{10}R^4$	$\frac{1}{8}\epsilon_{10}\epsilon_{10} (R(\Omega_+)^4 + \frac{8}{3}H^2R(\Omega_+)^3 + \dots)$ $= \frac{1}{8}\epsilon_{10}\epsilon_{10} (R(\Omega_-)^4 + \frac{8}{3}H^2R(\Omega_-)^3 + \dots)$

- ◇ Connection with torsion: $\Omega_{\pm\mu}{}^{\alpha\beta} = \Omega_{\mu}{}^{\alpha\beta} \pm \frac{1}{2}H_{\mu}{}^{\alpha\beta}$ (where $\mathcal{H}^{\alpha\beta} = H_{\mu}{}^{\alpha\beta}dx^{\mu}$)
- ◇ Curvature: $R(\Omega_{\pm}) = R \pm \frac{1}{2}d\mathcal{H} + \frac{1}{4}\mathcal{H} \wedge \mathcal{H}$
- ◇ New kinematic structures in o-o sector

Couplings can be lifted to eleven dimensions

- All expressions even in $H: H^2 \rightarrow G^2$ with an extra pair of summed indices
- lifting ambiguities: more terms in eleven dimensions than in ten

Reduction of heterotic GS couplings on \mathbb{T}^3 vs. reduction of $C_3 \wedge X_8$ on $K3$

$$B \wedge X^{\text{GS}}(R, F) \Rightarrow \left\{ \begin{array}{l} \bullet d_{IJKLM} a^I \wedge da^J \wedge F^K \wedge F^L \wedge F^L \Rightarrow 0 \\ \diamond d_{IJK} F^I \wedge R_2^{ab} \wedge (F_1)_a^J \wedge \nabla(F_1)_b^K \\ \diamond d_{IJK} F^I \wedge R_2^{ab} \wedge R_2^{cd} \wedge F_{ac}^J \wedge \nabla F_{bd}^K + \dots \end{array} \right\} \Leftarrow C \wedge \overline{X_8(\Omega, G)}$$

- 0 for four-derivative terms at generic lattice $\Gamma_{3,19}$ points. Non-zero at enhancement points
- $C_3 \wedge X_8$ cannot give rise to four derivative terms beyond $C \wedge \text{tr } R^2$
 - ▷ $R\mathcal{F}^2$ and $(D\mathcal{F})^2$ terms in heterotic BI are matched by $\overline{X_8(\Omega, G)}$
- ◇ d_{IJK} are computable on heterotic side
- ◇ on M-theory involve integrals dependent on $K3$ metric (e.g. $\int \omega_2^I \wedge \omega_{ab}^J R_2^{bc} \omega_{ca}^K$)
- ◇ $C_3 \wedge X_8(\Omega, G)$ gets fixed!

II. Torsional heterotic backgrounds

Conditions for supersymmetry (up to $\sim \mathcal{O}(\alpha')$):

- Internal space:

$$\omega^3 = \frac{3i}{4} \Omega \wedge \bar{\Omega}, \quad \omega \wedge \Omega = 0$$

space with trivial canonical bundle ($SU(3)$ structure)

- dilaton:

$$d(e^{-\phi} \omega^2) = 0, \quad d(e^{-\phi} \Omega) = 0$$

internal manifold is complex

- H -flux:

$$H = i(\bar{\partial} - \partial)\omega = 0$$

- Gauge fields:

$$\mathcal{F} \wedge \Omega = 0 = \mathcal{F} \wedge \bar{\Omega}, \quad \omega^2 \wedge \mathcal{F} = 0$$

Hermitean YM

▷ Susy + Bianchi identity \Rightarrow solution

Solutions

- ◇ Leading ($\sim \mathcal{O}(\alpha')$) corrections to the geometry in CY compactifications
- Theorem: for 4d $\mathcal{N} = 2$ supersymmetry need internal CFT with with $c = 9$ and $(0, 4)$ susy + a pair of $U(1)$'s ($c = 3$ with $(0, 2)$ susy)
 - ▷ \mathbb{T}^2 fibration over hyper-Kähler base
 - ▷ Generic internal space for het. strings in $\mathcal{N} = 2$ heterotic/type II duality (not $\mathbb{T}^2 \times K3$)
 - ▷ the target-space necessarily has string-scale cycles; flux solutions do not have a 10d large radius limit, they do have an *eight-dimensional* large radius limit
 - ▷ Duality Het/ $M \times \mathbb{T}^2$ vs. F theory/ $M \times K3_e$:
 - ◇ eight-dimensional theory (min. susy) with $O(2, 18, \mathbb{Z})$ symmetry
 - ◇ Non-trivial $G_4 \Rightarrow$ nontrivial $\mathbb{T}^2 \hookrightarrow X \xrightarrow{\pi} M$
 - ◇ $G_4 = 0$ for $X = \mathbb{T}^4$, so X only be $K3$
 - ◇ F-theory: $G_4 = \gamma \wedge \gamma'$ (γ and γ' - $(1, 1)$ primitive forms on M and $K3_e$)
 - ▷ $G_4 = \gamma \wedge \gamma' + \Omega_0 \times \bar{\Omega}'_0 + c.c.$ only $\mathcal{N} = 1$ susy is preserved
 - ◇ eight-dimensional *graviphotons* are affected!
 - ◇ deformations of C.S. in $\mathcal{N} = 2$ case either preserve all susy or break all susy

$$\mathbb{T}^2 \hookrightarrow X \xrightarrow{\pi} K3$$

- on the K3 base

$$\omega_0^2 = \frac{1}{2} \Omega_0 \bar{\Omega}_0, \quad \omega_0 \Omega_0 = \Omega_0^2 = 0$$

$(\omega_0, \Omega_0, \mathcal{A})$ - Calabi-Yau structure

- On X :

$$\omega_X = e^{2\phi} \omega_0 + \frac{i\mathbf{a}}{2} \Theta \bar{\Theta}, \quad \Omega_X = e^{2\phi} \sqrt{\mathbf{a}} \Omega_0 \Theta, \quad \mathcal{F} = \pi^* \bar{\partial} \mathcal{A}$$

- ◇ $\mathbf{F} = \mathbf{F}^1 + i\mathbf{F}^2$ ($d\Theta^{1,2} = \pi^*(\mathbf{F}^{1,2})$, $\mathbf{F}^{1,2} \in H^2(M, 2\pi\mathbb{Z})$) must satisfy:

$$\omega_0 \wedge \mathbf{F} = 0$$

$$\Omega_0 \wedge \mathbf{F} = 0$$

- ◇ Curvature of \mathbb{T}^2 bundle:

$$\mathbf{F} = F + F', \quad F \in H^{1,1}(M), \quad F' \in H^{2,0}(M)$$

- ▷ supersymmetry:

$$\left. \begin{array}{l} F' = 0 \quad (\text{i.e. } \bar{\Omega}_0 \wedge \mathbf{F} = 0) \\ F' \neq 0 \end{array} \right\} \Rightarrow \begin{cases} \mathcal{N} = 2 \text{ (only left symmetries broken)} \\ \mathcal{N} = 1 \text{ (broken right (susy!) symmetries)} \end{cases}$$

- H -flux:

$$\begin{aligned} H &= i(\bar{\partial} - \partial)\omega_X = i\omega_0(\bar{\partial} - \partial)e^{2\phi} + \frac{\mathbf{a}}{2}(\bar{F}' - \bar{F})\Theta + \frac{\mathbf{a}}{2}(F' - F)\Theta \\ &= H_{\text{hor}} + H_I\bar{\Theta}^I = H_{\text{hor}} + \mathbf{a}(F_I^{2,0} + F_I^{0,2} - F_I^{1,1})\Theta^I \end{aligned}$$

▷ $\mathcal{N} = 2$ solution: $H = H_{\text{hor}} - \frac{\mathbf{a}}{2}(\Pi_0^{1,1}F)\bar{\Theta} - \frac{\mathbf{a}}{2}(\Pi_0^{1,1}\bar{F})\Theta$

- Quantisation of $\mathbf{a} \Leftrightarrow$ disconnection between $\mathcal{N} = 2$ and $\mathcal{N} = 1$ flux vacua:

▷ C.S. deformation of $F = F \in H^{1,1}(M)$ can generate $F' \in H^{2,0}(M)$

▷ can relate $\mathcal{N} = 2$ and $\mathcal{N} = 1$ backgrounds via deformation?!?!?

▷ cf
$$\begin{cases} H_{\text{def}} = (H_{\text{hor}})_{\text{def}} - \frac{\mathbf{a}}{2}(\Pi_s^{1,1}F + \Pi_s^{2,0}F)\bar{\Theta} - \frac{\mathbf{a}}{2}(\Pi_s^{1,1}\bar{F} + \Pi_s^{0,2}\bar{F})\Theta \\ H_{\mathcal{N}=1} = H_{\text{hor}} - \frac{\mathbf{a}}{2}(\Pi_s^{1,1}F - \Pi_s^{2,0}F)\bar{\Theta} - \frac{\mathbf{a}}{2}(\Pi_s^{1,1}\bar{F} - \Pi_s^{0,2}\bar{F})\Theta \end{cases}$$

▷ variation of complex structure of $\mathcal{N} = 2$ solution either breaks or preserves **ALL** supersymmetry

- The area of \mathbb{T}^2 \mathbf{a} is quantised in units of α'

▷ resolves 2 problems... need to turn to Bianchi Identity

Heterotic BI and connections with torsion

Fu and Yau showed existence of solutions to susy equations with H flux and BI:

$$dH_3 = 2i\partial\bar{\partial}\omega_X = \frac{\alpha'}{4}(\text{tr } R^2(\Sigma_{\text{Chern}}) - \text{tr } F^2)$$

▷ Chern (canonical Hermitean) connection on complex manifolds:

$$\Sigma = \begin{pmatrix} \Sigma_\nu^\mu & 0 \\ 0 & \bar{\Sigma}_{\bar{\nu}}^{\bar{\mu}} \end{pmatrix} = \begin{pmatrix} dz^\lambda g_{\nu\bar{\lambda},\lambda} g^{\bar{\lambda}\mu} & 0 \\ 0 & d\bar{z}^{\bar{\lambda}} g_{\lambda\bar{\nu},\bar{\lambda}} g^{\bar{\mu}\lambda} \end{pmatrix}$$

⊕! When connection is Chern, both sides of the BI are $(2, 2)$ forms

▷ Curvature two-form for the connection with torsion ($\Omega_+ = \Sigma + T$):

$$R_+ = \begin{pmatrix} \bar{\partial}\Sigma - \bar{T}T & \bar{\partial}T - \bar{\Sigma}T \\ \partial\bar{T} - \Sigma\bar{T} & \partial\bar{\Sigma} - T\bar{T} \end{pmatrix} + \begin{pmatrix} 0 & \partial T - T\Sigma \\ \bar{\partial}\bar{T} - \bar{\Sigma}\bar{T} & 0 \end{pmatrix} = R_{(1,1)} + R_{(2,0)} + R_{(0,2)}$$

⊖! $R_{(2,0)} \neq 0$ for $F' \neq 0$ ($F' = F_{(2,2)}$) problems for $\mathcal{N} = 1$ solution

⊖!! To satisfy EOM $\mathcal{O}(\alpha')$ two-form R needs to satisfy HYM. $R(\Sigma)$ does not!

\mathbb{T}^2 area \mathbf{a} is quantised in units of α'

- Bianchi Identity

$$2i\partial\bar{\partial}e^{2\phi}\omega_0 + \mathbf{a}\partial\bar{F}'\Theta + \mathbf{a}\bar{\partial}F'\bar{\Theta} = \frac{\alpha'}{4} \left[\text{tr } R_+^2 - \text{tr } \mathcal{F}^2 + \frac{4\mathbf{a}}{\alpha'}(F\bar{F} - F'\bar{F}') \right] + \mathcal{O}(\alpha'^2)$$

▷ For $\mathbf{a} \sim \alpha'$ all solved by

$$\partial\bar{F}' = 0 \quad \Rightarrow \quad \begin{cases} \bullet F' = \lambda\Omega_0 \text{ for } \lambda = \text{const} \\ \bullet \partial F = 0 \end{cases}$$

▷ $\phi = \alpha' f/4 \Rightarrow i\partial\bar{\partial}f = \frac{1}{4} [\text{tr } R_+^2 - \text{tr } \mathcal{F}^2 + \frac{4\mathbf{a}}{\alpha'}(F\bar{F} - F'\bar{F}')] + \mathcal{O}(\alpha')$

▷ In $\mathcal{N} = 2$ case, Ω_+ is *horizontal*. No α' expansion of dilaton is needed. Direct map to Fu-Yau solution (higher α' vanish for $\mathcal{N} = 2$?!)

- HYM for R_+

▷ $R_{(2,0)} \sim \mathcal{O}(\alpha')$ provided $\partial F = 0$

▷ Can show $\omega_X^2 \wedge R_+^{1,1} = \frac{i\mathbf{a}}{2}\Theta \wedge \bar{\Theta} \wedge \omega_0 \wedge R_+^{1,1} = \mathcal{O}(\alpha'^2)$

III. Generalised geometry for heterotic strings

Generalised complex structure (GCG)

- GCG $\mathcal{J} : T \oplus T^* \longrightarrow T \oplus T^*$ ($\mathcal{J}^2 = -1$; $\mathcal{J}^\dagger \mathcal{I} \mathcal{J} = \mathcal{I}$)
 - ◊ Structure group: $\Rightarrow \text{U}(3,3)$
- **GCS integrable:** $\pi_+[\pi_-(v), \pi_-(w)]_{\text{Lie}} = 0 \mapsto \Pi_+[\Pi_-(X), \Pi_-(Y)]_C = 0$ with
Courant bracket:

$$[v + \xi, w + \eta] = [v, w]_{\text{Lie}} + \left\{ \mathcal{L}_v \eta - \mathcal{L}_w \xi - \frac{1}{2} d(\iota_v \eta - \iota_w \xi) \right\}$$

(Courant closes on $L_{\mathcal{J}}$ – the i-eigenbundles of \mathcal{J} .)

- Closed B-transform $(v_1, \rho_1) \mapsto e^B(v_1, \rho_1) = (v_1, \rho_1 + \iota_{v_1} B)$ is an auto-morphism of Courant : $[e^B(v_1, \rho_1), e^B(v_2, \rho_2)] \mapsto e^B[(v_1, \rho_1), (v_2, \rho_2)]$
- Twisting: $d \mapsto d - H \wedge$, $[\cdot, \cdot]_C \mapsto [\cdot, \cdot]_C + \underline{\iota_v \iota_w H}$
- Replacing Lie bracket by Courant allows to extend Riemannian objects to generalised objects (e.g. **generalised connection**)

Generalised tangent bundle :

$$0 \longrightarrow T^*M \longrightarrow E \xrightarrow{\pi} TM \longrightarrow 0,$$

Sections of E:

$$X = \begin{pmatrix} v \\ \xi \end{pmatrix} \longmapsto X' = e^{-B} X = \begin{pmatrix} \mathbb{I} & 0 \\ -B & \mathbb{I} \end{pmatrix} \begin{pmatrix} v \\ \xi \end{pmatrix} = \begin{pmatrix} v \\ \xi - \iota_v B \end{pmatrix}.$$

- Note $(v, \xi) \rightarrow (v, \xi - \iota_v d\Lambda)$
- Courant on $E \rightarrow$ *twisted* Courant on $T \oplus T^*$

On $L_\phi \otimes E$ (ϕ , g and B define $S^\pm(E) \cong L_\phi \otimes \Lambda^{\text{even/odd}} T^*M$):

- Courant does not define an unambiguous gen. Riemman but ...unique

$$\hat{R}_{ab} = R_{ab} - \frac{1}{4} H_{acd} H_b{}^{cd} + 2\nabla_a \nabla_b \phi + \frac{1}{2} e^{2\phi} \nabla^c (e^{-2\phi} H_{cab}) \text{ and}$$

$$\hat{R} = R + 4\nabla^2 \phi - 4(\partial\phi)^2 - \frac{1}{12} H^2 \quad \leftrightarrow \quad \text{LBT: } S^{\text{NS}} \epsilon = 4 (D^a D_a - D^2) \epsilon$$

- The dynamics and supersymmetry transformations of type II supergravity theories are captured by a (torsion-free) generalised connection

- α' corrections to geometry
 - HS ($dH \neq 0$): $\Rightarrow B \rightarrow B + d\Lambda + \frac{\alpha'}{4} d^{-1} \delta d^{-1} X_4(\Omega_+, A)$
- } ← Extensions of GTB

Two simple ideas:

- Find an extended space on which one has a closed 3-form!
- ◇ Generalise $U(1)$ fibration $S^1 \hookrightarrow X \xrightarrow{\pi} M$ case:
 - ▷ $dH = 0$ on $X \Rightarrow dh_3 = \tilde{T} \wedge T = \frac{1}{4} [(T_+)^2 - (T_-)^2]$ on M
 - ▷ h_3 is gauge invariant!

Generalised heterotic tangent space is built as a double fibration:

$$0 \longrightarrow \mathfrak{g} \longrightarrow C \longrightarrow TM \longrightarrow 0 ,$$

$$0 \longrightarrow T^*M \longrightarrow E \longrightarrow C \longrightarrow 0$$

- ◇ \mathfrak{g} is the adjoint bundle given a principle G -bundle
- ◇ Locally $E \simeq TM \oplus T^*M \oplus \mathfrak{g}$
- ◇ **Obstruction:** $p_1(\mathfrak{g}) = 0$

Heterotic effective action

- (generalised) Lichnerowicz theorem: $(D^A D_A - D^2) \epsilon = \frac{1}{4} S \epsilon$ (S tensorial!)
- ◇ $S = R + 4 \nabla^2 \phi - 4 (\partial \phi)^2 - \frac{1}{12} H^2 - \frac{\alpha'}{4} \text{tr} \hat{\mathcal{F}}^2$ the (bosonic part of) sugra
- ◇ iff $\frac{1}{6} \nabla_{[a} H_{bcd]} - \frac{\alpha'}{8} \text{tr} \hat{\mathcal{F}}_{[ab} \hat{\mathcal{F}}_{cd]} = 0$
- ◇
$$\left. \begin{aligned} \delta \psi_a &= D_a \epsilon = \nabla_a \epsilon \mp \frac{1}{8} H_{abc} \gamma^{bc} \epsilon \\ \delta \zeta_\alpha &= D_\alpha \epsilon = -\frac{1}{8} \sqrt{2\alpha'} \hat{\mathcal{F}}_{ab\alpha} \gamma^{ab} \epsilon \end{aligned} \right\} \leftarrow \text{covariant derivative } (A = \{a, \alpha\})$$
- ◇ $\delta \lambda = D \epsilon = (\gamma^a \nabla_a \mp \frac{1}{24} H_{abc} \gamma^{abc} - \gamma^a \partial_a \phi) \epsilon \leftarrow \text{Dirac operator}$

Gravitational terms (obstruction to E) ?

- ◇ take $G \rightarrow G_{\text{gauge}} \times O(d)$. This splits $E = \tilde{C}_+ \oplus \tilde{C}_{\mathfrak{g}} \oplus \tilde{C}_-$
- ◇ reduce the structure group of E to $O(d) \times G \times O(d) \subset O(d + \dim(\mathfrak{g})) \times O(d)$
- ◇ Identify $O(d) \in G$ with $O(d)$ in \tilde{C}_+
 - ▷ Works only for $\hat{\mathcal{A}} = \Omega_{\pm} = \omega^{\text{LC}} - \frac{1}{2} \mathcal{H}$!!!
 - ▷ Flip of the sign in $\mathcal{O}(\alpha')$ effective action wrt $D_a : \Omega_- \longrightarrow \Omega_+$!!!
 - ▷ Bismut (GCG) vs SUSY

Some open questions:

- Tests of 10 and 11-d couplings:
 - ▷ fixing the ambiguities
 - ▷ higher orders in α'
 - ▷ susy transformations
- Lower dimensions and less supersymmetry:
 - ▷ recent progress in construction of four-dimensional $\mathcal{N} = 2$ higher-derivative terms
 - ▷ Implications for consistency (swampland)
 - ▷ subleading terms in AdS/CFT
- Construction of generic $\mathcal{N} = 1$ heterotic flux backgrounds
- ★ Can generalised geometry capture the systematics of the string (perturbation) theory?
- ★ Can there be more news from old dualities?