Quantum Dimension as Entanglement Entropy in 2D RCFTs

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Outline

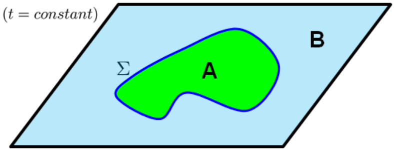
- General background of Entanglement Entropy (EE).
- Holographic Entanglement Entropy (HEE).
- Setup of Entanglement Entropy (EE).
- Quantum dimension and EE.
- Summary.

Basics of Entanglement Entropy

- Recently, (Renyi) entanglement entropy ((R)EE) has wide interest in theoretical physics.
- It is useful to study the distinctive features of various quantum state in condensed matter physics. For example: phase transition in Condensed matter system, probe of Fermi liquid (logarithmic behavior of EE) vs. non-Fermi-liquid behavior,etc..
- (Renyi) entanglement entropy is expected to be an important quantity which may shed light on the mechanism behind the AdS/CFT corresponds.
- Probe of quenches and thermalization processes, propagation of entanglement.
- Probe of phase transition between confinement and deconfinement in QCD.
- ...

Basics of Entanglement Entropy

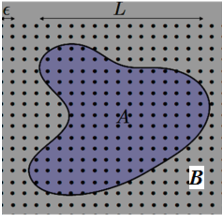
- General diagnostic: divide quantum system into two parts (A and B) and use entropy as measure of correlations between subsystems



- In QFT, typically introduce a (smooth) boundary or entangling surface Σ which divides the space into two separate regions (A and B).
- Integrate out degrees of freedom in outside region (B). Remaining dof are described by a density matrix ρ_A.
- For Entangled pure state: $|\psi\rangle = \sum_i \lambda_i |i\rangle_A |i\rangle_B$
- Entanglement leads to mixture of the reduced density matrix: $\rho_A = \mathrm{Tr}_B|0\rangle\langle 0| = \sum_i |\lambda_i|^2 |i\rangle\langle i|$
- Best way to quantify amount of entanglement is by von Neumann entropy $S_{EE} = -\mathrm{Tr}(\rho_A \log \rho_A)$.
- For this case $|\psi\rangle$, $S_{EE} = -\mathrm{Tr}(\rho_A \log \rho_A) = \sum_i |\lambda_i|^2 \log |\lambda_i|^2$. And we can defined EE for mixed state in similar way.

Basics of Entanglement Entropy

- EE is a central concept in quantum statistical mechanics and quantum information theory.
- Entanglement is a ubiquitous phenomenon in quantum systems.
- Especially, we here consider a lattice system in $D-1$ spatial dimensions. Let the characteristic length L of subsystem A much lager than size of lattice ϵ.



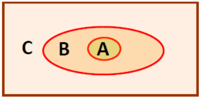
In a generic state, $S_{EE}(A) \sim \log \dim H_A \sim (\frac{L}{\epsilon})^{D-1}$
In the ground state (or other low-lying pure state), $S_{EE}(A) \sim \log \dim H_{\partial A} \sim (\frac{L}{\epsilon})^{D-2}$. Typically in 2D, $S_{EE}(A) \sim (\frac{L}{\epsilon})$.

- One can also take $\epsilon \to 0$ to obtain the QFT description.

Properties of EE in QFT

- There are some nice Properties (which are related to von Neumann algebras):

  For pure state $S_A = S_B$, otherwise $S_A \neq S_B$.
  Strong subadditivity: $S_{A+B+C} + S_B \leq S_{A+B} + S_{B+C}$.



  Subadditivity: $S_{A+B} \leq S_A + S_B$. This property corresponds to triangle inequality in Von Neumann algebra.

- $\mathrm{Tr}_{subsystem}$ corresponds seminorm which a function from a vector space (Hilbert space especially) to real number.

Properties of EE in QFT

- The standard thermal entropy is obtained as a particular case of EE: i.e. A=total space.
- The density matrix of total system can be expressed by

$$\rho = \frac{e^{-\beta H}}{Z}$$

where $Z = \mathrm{Tr} e^{-\beta H}$. One can obtain

$$\begin{aligned} S &= \frac{\partial}{\partial n} \log \left[\mathrm{Tr}[\rho^n] \right] |_{n=1} = -\frac{\partial}{\partial n} \left(\log \left[\mathrm{Tr}[e^{-\beta n H}] - nZ \right] \right) \\ &= \beta \langle H \rangle + \log Z = \beta (E - F) = S_{thermal} \end{aligned} \quad (1)$$

Replica to calculate EE in QFT

- How to calculate EE in quantum system.
- Firstly, one should introduce the Renyi entropy as following

$$S_A^n = -\frac{\log tr_A \rho_A^n}{n-1}.$$

Where the $\rho_A^n = Pe^{-\int_0^{2\pi n} d\tau H_{b,n}(\tau)}$.

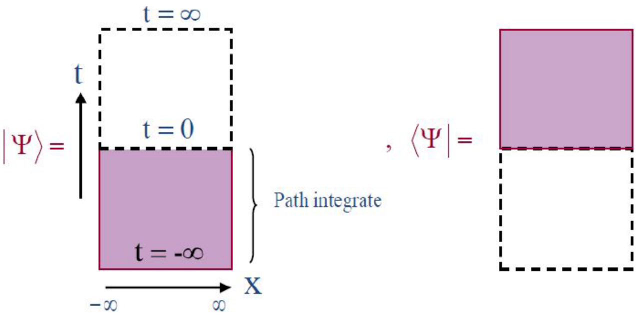
- A basic method of calculating EE in QFTs is so called the replica method.

$$S_A = -\frac{\partial \mathrm{Tr}(\rho_A)^n}{\partial n}|_{n=1} = \lim_{n\to 1} S_A^n$$

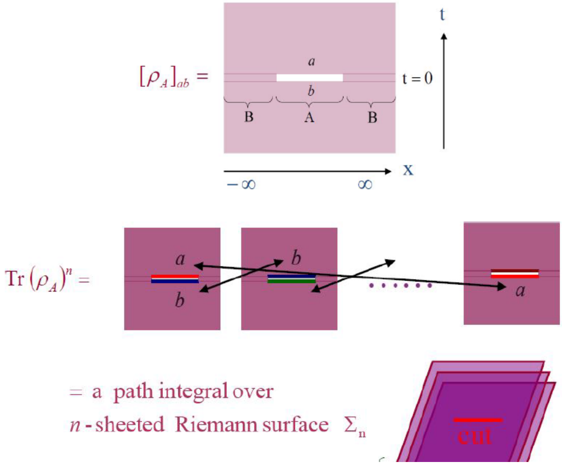
- The relation provides a practical way to compute EE in field theory, although it is difficult.

Replica trick

- The standard way is to use replica trick [J. Callan et.al. 9401072].
- Here, we only focus on the 2D CFT, which provides more analytic results.
- In Euclidean path-integral, the ground state wave-functional is represented by [T. Takayanagi's lecture in 7th Asian winter school]

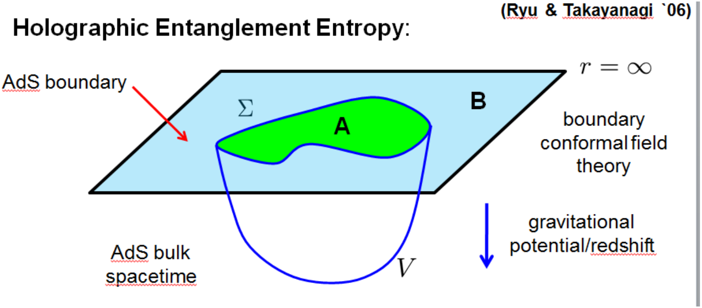


After gluing boundaries successively, one can obtain the $\mathrm{Tr}(\rho_A)^n = [\rho_A]_{ab}[\rho_A]_{bc}\ldots[\rho_A]_{ka}$. This procedure can be shown explicitly as following



In this way, one can obtain the following $\mathrm{Tr}(\rho_A)^n = \frac{Z_n}{Z_1^n}$ where Z_n is partition function on the n-sheeted Riemann surface Σ_n. We will show how to use this trick to study EE explicitly in this talk.

Holographic Entanglement Entropy



The holographic entanglement entropy of a subsystem A on the boundary is given by the area of the ($t = const$) bulk minimal surface γ_A

$$S_A = \frac{Area(\gamma_A)}{4G}, \quad \partial\gamma_A = \partial A$$

For higher derivative gravity, this functional of EE should be modified.

Extensive ways to check HEE

- leading contribution yields area law $S_{EE} \sim \frac{\text{Area}}{\text{cut off}^{d-2}}$
- recover known results for d=2 CFT [Holzhey, Larsen and Wilczek; Calabrese and Cardy] : $S_{EE} = \frac{c}{3}\log(\frac{C}{\pi\delta}\sin(\frac{\pi l}{C}))$.
- $S_A = S_{\bar{A}}$ in a pure state, where the A and $\bar{A}$ share the same entangled surface.
- strong sub-additivity [Headrick and Takayanagi]: $S_{A+B} \leq S_A + S_B$
- for even d, connection of universal/logarithmic contribution in S_{EE} to central charges of boundary CFT, eg, in $d = 4$
- New proof given by [Lewkowycz and Maldacena]
- Generalization of Euclidean path integral calc's for S_{BH}, extended to "periodic" bulk solutions without Killing vector. Where breaking the $U(1)$ Isometry time direction.
- For AdS/CFT, just translates replica trick for boundary CFT to bulk and then

$$\Delta\tau = 2\pi \to 2\pi n \longrightarrow \log Z(n) = \log \text{Tr}\left[\rho^n\right] = -I_{grav}(n)$$

$$\longrightarrow S = -n\partial_n \left[\log Z(n) - n\log Z(1)\right]\Big|_{n=1}$$

- at n=1, linearized gravity eom demand: induced curvature vanishing. The Euclidean time circle shrinks to zero on an extremal surface in bulk.

Motivation: ‘First Law’

- First law of thermodynamics: $TdS = dE$. In a generic quantum system which are far from the equilibrium, can we find the analogous relation between the EE (information) and energy of A:

$$T_{ent} dS_A = dE_A \ \ ?$$

- The first study in field theory in [F. C. Alcaraz, M. I. Berganza, G. Sierra, PRL 106, 201601]
- First holographic studied in [Jyotirmoy Bhattacharya, Masahiro Nozaki, Tadashi Takayanagi, Tomonori Ugajin, PRL 110, 091602]
- More general studies given by [Nozaki,Numasawa,Prudenziati,Tadashi Takayanagi 13],[Wu-zhong Guo, S.H, Jun Tao, 13](Higher derivative gravity)[S.H, Danning Li, Jun-Bao Wu, 13](Non-conformal cases with full backreaction) [Bhattacharya,Tadashi Takayanagi, 13]...

Motivation: ‘First Law’

- The entanglement temperature is

$$T_{ent} = \frac{c}{l}, \tag{2}$$

where c is constant and l is characteristic length of small subsystem.

The constant c is universal in that it only depends on the shape of the subsystem A in pure AdS background.[Jyotirmoy Bhattacharya, Masahiro Nozaki, Tadashi Takayanagi, Tomonori Ugajin, PRL 110, 091602]

The constant c is no longer universal which is highly depending on the vacuum of boundary field theory.[Wu-zhong Guo, S.H, Jun Tao, 13][S.H, Danning Li, Jun-Bao Wu, 13]

Motivation: 'First Law'

- More Recent Progresses:

  The first law can be simply expressed as follows [Blanco-Casini-Hung-Myers 13,Wong-Klich-Pando Zayas-Vaman 13]: $\Delta S_A = \Delta H_A$

  The perturbative Einstein eq. is equivalent to a constraint of HEE: $(\partial_l^2 - \partial_l - \partial_x^2 - \frac{3}{l^2})\Delta S_A = \langle O \rangle \langle O \rangle$.[Nozaki-Numasawa-Prudenziati-TT 13, Bhattacharaya-TT 13]

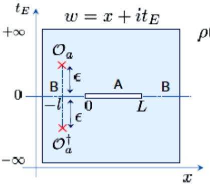
  Moreover, the first law was shown to be equivalent to the perturbative Einstein eq (Pure AdS).[Lashkari-McDermott-Raamsdonk 13, Faulkner-Guica-Hartman-Myers- Raamsdonk 13]

- All these studies show that Entanglement temperature can only be well defined in low excited quantum states with small size of subsystem region.
- In this paper, we try to consider EE of the excited states with large size limit and We try to figure out properties of EE from different point of view.

Our main result

- In next two slices, I want to show our setup and main results roughly in this paper. Firstly, let us to go to the setup

We study EE for states excited by local operators and its time evolution:



$$\rho(t) = e^{-itH}e^{-\epsilon H}\mathcal{O}_a(0,-l)|0><0|\mathcal{O}_a^\dagger(0,-l)e^{-\epsilon H}e^{itH}$$
$$= \mathcal{O}_a(w_1,\bar{w}_1)|0><0|\mathcal{O}_a^\dagger(w_2,\bar{w}_2)$$

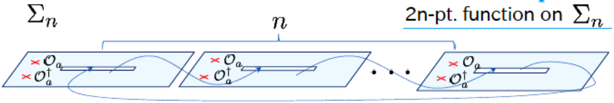
ϵ : the UV regulator for the operator ($\epsilon \longrightarrow 0$ finally)

$$\Delta S_A^{(n)} = S_A^{(n)} - S_{A\,0}^{(n)}$$

$\mathrm{Tr}{\rho_A}^n$

$$= \frac{1}{1-n}\log\left[\frac{\langle\mathcal{O}_a^\dagger(w_1,\bar{w}_1)\mathcal{O}_a(w_2,\bar{w}_2)\cdots\mathcal{O}_a^\dagger(w_{2n-1},\bar{w}_{2n-1})\mathcal{O}_a(w_{2n},\bar{w}_{2n})\rangle_{\Sigma_n}}{\left\langle\mathcal{O}_a^\dagger(w_1,\bar{w}_1)\mathcal{O}_a(w_2,\bar{w}_2)\right\rangle_{\Sigma_1}^n}\right]$$

2n-pt. function on Σ_n



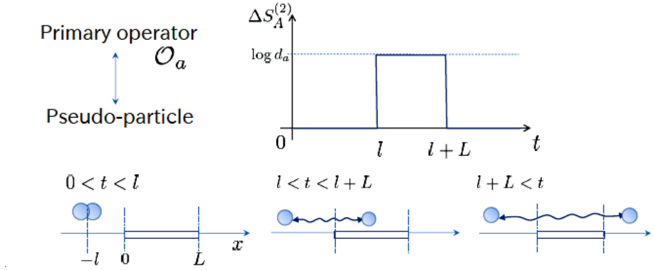
Our main result

- In this slice, we can go to final result.

2D rational CFT

Primary Operator $\mathcal{O}_a$ d_a : Quantum Dimension

$$\Delta S_A^{(n)} = \log d_a \qquad (l < t < l + L)$$

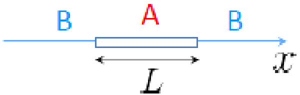


Setup

Our main motivation

- In this talk, we just only consider 1+1 dimension space time. Our system can be shown as follow carton

$$|\Psi> \qquad \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$



For small subsystem $(L \to 0)$ ⟷ For large subsystem $(L \to \infty)$

- → The 1st law for EE
  $T_{ent} \cdot \Delta S_A = \Delta E_A$
- → Universal property at IR

- → We cannot expect any universal properties like the 1st law
- → Study solvable explicit models by direct field theory calculations

Our main motivation

- The EE has been studied in D=2,4,6 for free scalar fields.

  - EE for excited states by local operators $\longrightarrow$ New class !

  $|\Psi>=\mathcal{O}(x)|0>$ [Nozaki-Numasawa-Takayanagi 14]

- In this paper, we would like to study EE in 2 dimensional Rational CFT. Our main result is n^{th} Reni entropy of state excited by Local operator $O(x)$ corresponds to quantum dimension of $O(x)$.
- The definition of $\Delta S_A^{(n)}$ is defined by the excess of REE

$$\Delta S_A^{(n)} = S_A^{(n)}[|\Psi\rangle] - S_A^{(n)}[|0\rangle], \tag{3}$$

where $S_A^{(n)}[|\Psi\rangle]$ denote the n-th Renyi entanglement entropy for the state $|\Psi\rangle$ with the subsystem A. Thus $\Delta S_A^{(n)}$ measures the increased amount of the entropy compared with the ground state $|0\rangle$.

EE for Excited State

- Where REE for $|\Psi(t)\rangle = e^{-itH-\epsilon H}O(-l)|0\rangle$,

$$S^{(n)}[|\Psi(t)\rangle] = \frac{1}{1-n}\log\left[\frac{\int \phi O^{+}(x_1)O(x_2)...O^{+}(x_{2n-1})O(x_{2n})e^{-S}}{(\int \phi O^{+}(x_1)O(x_2)e^{-S})^n}\right] \tag{4}$$

- Where REE for $|0\rangle$,

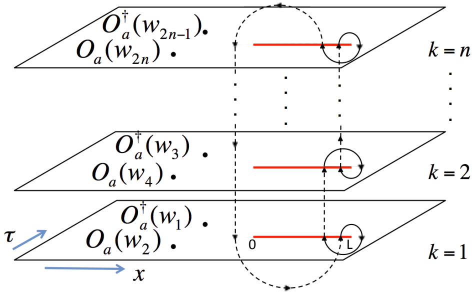
$$S^{(n)}[|0\rangle] = \frac{1}{1-n}\log\frac{Z_n}{Z_1^n} \tag{5}$$

- In the end, we find that $\Delta S_A^{(n)}$ can be computed as

$$\begin{aligned}\Delta S_A^{(n)} &= \frac{1}{1-n}\Bigg[\log\left\langle \mathcal{O}_a^{\dagger}(w_l,\bar{w}_1)\mathcal{O}_a(w_2,\bar{w}_2)..\mathcal{O}_a(w_{2n},\bar{w}_{2n})\right\rangle_{\Sigma_n} \\ &\quad -n\log\left\langle \mathcal{O}_a^{\dagger}(w_l,\bar{w}_1)\mathcal{O}_a(w_2,\bar{w}_2)\right\rangle_{\Sigma_1}\Bigg],\end{aligned} \tag{6}$$

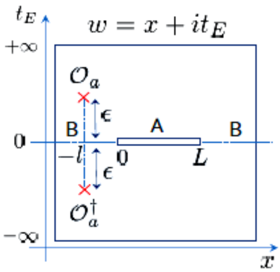
EE for Excited State

- Where (w_{2k+1}, w_{2k+2}) for $k = 1, 2, \ldots, n-1$ are $n-1$ replicas of (w_1, w_2) in the k-th sheet of Σ_n. We just glue all sheets with proper boundary conditions to construct Σ_n.



EE for Excited State

- To be more precise, each sheet Σ_1 is shown as following



$w_i, \bar{w}_i$ can be expressed by

$$w_1 = i(\epsilon - it) - l, \quad w_2 = -i(\epsilon + it) - l, \qquad (7)$$

$$\bar{w}_1 = -i(\epsilon - it) - l, \quad \bar{w}_2 = i(\epsilon + it) - l. \qquad (8)$$

These coordinates correspond to positions where we have inserted operators locally.

- For single sheet, we can define the reduced density matrix

$$\rho_A = \mathrm{Tr}_B[|\Psi(t)\rangle\langle\Psi(t)|]. \qquad (9)$$

EE for Excited State

- In term of introduction of replica trick, one can follow the logic

- Entanglement Entropy (EE)

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A = \lim_{n\to 1} \frac{1}{1-n} \log \mathrm{Tr}\rho_A^n$$

$$\rho_A = \mathrm{Tr}_B\, \rho$$

Renyi EE : $S_A^{(n)}$

- "Replica trick"

By using path integral, calculate $\mathrm{Tr}\rho_A{}^n$ not $\mathrm{Tr}\rho_A \log \rho_A$ then $n \to 1$

- The n-th Renyi entanglement entropy $S_A^{(n)}[|\Psi(t)\rangle]$ is defined by

$$S_A^{(n)}[|\Psi(t)\rangle] = \frac{1}{1-n} \log \mathrm{Tr}[\rho_A^n]. \tag{10}$$

- The causality argument tells us that $\Delta S_A^{(n)} = 0$ for $t < l$ and $t > L$. Especially we will be interested in the late time behavior $L > t \gg l$ and we will call the final value of $\Delta S_A^{(n)}$ as $\Delta S_A^{(n)f}$.

EE for Excited State

- In order to study Reny entropy, one should make use of following conformal map to do replica trick:

$$\frac{w}{w-L} = z^n, \tag{11}$$

which maps Σ_n to Σ_1. Setting $n = 2$ and using (7), the coordinates z_i are given by (similarly $\bar{z}_i$ using (8))

$$\begin{aligned} z_1 &= -z_3 = \sqrt{\frac{l-t-i\epsilon}{l+L-t-i\epsilon}}, \\ z_2 &= -z_4 = \sqrt{\frac{l-t+i\epsilon}{l+L-t+i\epsilon}}. \end{aligned} \tag{12}$$

It is useful to define the cross ratios $(z, \bar{z})$

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad \bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}, \tag{13}$$

where $z_{ij} = z_i - z_j$.

EE for Excited State

- We would like to study the behavior of $(z, \bar{z})$ in the limit $\epsilon \to 0$. When $0 < t < l$ or $t > L + l$, we find $(z, \bar{z}) \to (0, 0)$:

$$z \simeq \frac{L^2 \epsilon^2}{4(l-t)^2(L+l-t)^2}, \quad \bar{z} \simeq \frac{L^2 \epsilon^2}{4(l+t)^2(L+l+t)^2}.$$

- In the other case $l < t < L + l$, we find $(z, \bar{z}) \to (1, 0)$:

$$z \simeq 1 - \frac{L^2 \epsilon^2}{4(l-t)^2(L+l-t)^2}, \quad \bar{z} \simeq \frac{L^2 \epsilon^2}{4(l+t)^2(L+l+t)^2}.$$

Though this limit $(z, \bar{z}) \to (1, 0)$ does not seem to respect the complex conjugate, it inevitably arises via our analytical continuation of t from imaginary to real values.

EE for Excited State

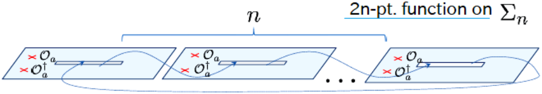
- For more general n copies of original system, we use replica trick to obtain the difference of EE between excited state and ground state.

$$\Delta S_A^{(n)} = S_A^{(n)} - S_{A\,0}^{(n)}$$

$\mathrm{Tr}\rho_A{}^n$

$$= \frac{1}{1-n}\log\left[\frac{\langle \mathcal{O}_a^\dagger(w_1,\bar{w}_1)\mathcal{O}_a(w_2,\bar{w}_2)\cdots\mathcal{O}_a^\dagger(w_{2n-1},\bar{w}_{2n-1})\mathcal{O}_a(w_{2n},\bar{w}_{2n})\rangle_{\Sigma_n}}{\left\langle \mathcal{O}_a^\dagger(w_1,\bar{w}_1)\mathcal{O}_a(w_2,\bar{w}_2)\right\rangle_{\Sigma_1}^n}\right]$$

2n-pt. function on Σ_n



Our problem has been changed to calculate multiple points Green function in 2D CFT.

EE in 2D CFT

- As a warmup, we consider some states excited by primary operators in the 2D minimal model. Firstly, we just only consider the simplest case $\Delta S^{(2)}$ with $n = 2$. To obtain $\Delta S^{(2)}$, normally, we should know the two-point Green function on Σ_1 and $2n$ Green function on Σ_n.
- The two point function on the Σ_1 looks like

$$\langle O(w_1, \bar{w}_1) O(w_2, \bar{w}_2) \rangle_{\Sigma_1} = \frac{\mathcal{N}}{|w_{12}|^{4\Delta}} = \frac{\mathcal{N}}{(2\epsilon)^{4\Delta}}, \tag{14}$$

- For $n = 2$ case, the four point green function on Σ_2 is follow

$$\begin{aligned} &\langle O(w_1, \bar{w}_1) O(w_2, \bar{w}_2) O(w_3, \bar{w}_3) O(w_4, \bar{w}_4) \rangle_{\Sigma_2} \\ &= \prod_{i=1}^{4} \left| \frac{dw_i}{dz_i} \right|^{-2\Delta} \langle O(z_1, \bar{z}_1) O(z_2, \bar{z}_2) O(z_3, \bar{z}_3) O(z_4, \bar{z}_4) \rangle_{\Sigma_1} \end{aligned} \tag{15}$$

  Due to conformal map, the difference of correlation function for primary operator on Σ_2 and Σ_1 is so called Jacobi factor. Roughly speaking, one can combine the two formula to obtain the ratio $\mathrm{Tr}\rho^2$ for $\Delta S_A^{(2)}$.

EE in 2D CFT

- Especially for our previous setup, we should use conformal map $w/(w-L) = z^n$ and obtain

$$\begin{aligned} \mathrm{Tr}\rho_A^2 &= \frac{\langle O(w_1,\bar{w}_1)O(w_2,\bar{w}_2)O(w_3,\bar{w}_3)O(w_4,\bar{w}_4)\rangle_{\Sigma_2}}{\left(\langle O(w_1,\bar{w}_1)O(w_2,\bar{w}_2)\rangle_{\Sigma_1}\right)^2} \\ &\sim |z|^{4\Delta}|1-z|^{4\Delta}\cdot G(z,\bar{z}). \end{aligned} \quad (16)$$

  Where $G(z,\bar{z})$ is related to conformal block. This formula is very important. We will make use of this formula to discuss general rational CFT.

- As we know precisely $(z,\bar{z}) \to (1,0)$ in late time limit:

$$z \simeq 1 - \frac{L^2\epsilon^2}{4(l-t)^2(L+l-t)^2}, \quad \bar{z} \simeq \frac{L^2\epsilon^2}{4(l+t)^2(L+l+t)^2}.$$

  In this limit, we can roughly expect that there maybe some nice properties of Renyi entropy.

Considering $c = 1$ CFT

- Firstly, let us consider a $c = 1$ CFT defined by a (non-compact) massless free scalar ϕ and choose two operators

$$\mathcal{O}_1 = e^{\frac{i}{2}\phi}, \quad \mathcal{O}_2 = \frac{1}{\sqrt{2}}(e^{\frac{i}{2}\phi} + e^{-\frac{i}{2}\phi}), \tag{17}$$

which have the same conformal dimension $\Delta_1 = \Delta_2 = \frac{1}{8}$.

- Then, the function $G_a(z, \bar{z})$ is found to be

$$G_1(z, \bar{z}) = \frac{1}{\sqrt{|z|\,|1 - z|}} \tag{18}$$

It is obvious that the Renyi entropy always becomes trivial $\Delta S_A^{(2)} = 0$ for the operator $\mathcal{O}_1$.

- For $\mathcal{O}_1$ is because the excited state $e^{\frac{i}{2}\phi}|0\rangle$ can be regarded as a direct product state $e^{\frac{i}{2}\phi_L}|0\rangle_L \otimes e^{\frac{i}{2}\phi_R}|0\rangle_R$ in the left-moving (L: chiral) and right-moving (R: anti-chiral) sector. Therefore it is not an entangled state.

EE in $c = 1$ CFT

- Further, let us consider the second operator in $c = 1$ 2D CFT massless free field.

$$\mathcal{O}_1 = e^{\frac{i}{2}\phi}, \quad \mathcal{O}_2 = \frac{1}{\sqrt{2}}(e^{\frac{i}{2}\phi} + e^{-\frac{i}{2}\phi}), \tag{19}$$

which have the same conformal dimension $\Delta_1 = \Delta_2 = \frac{1}{8}$.

- Then, the function $G_a(z, \bar{z})$ is found to be

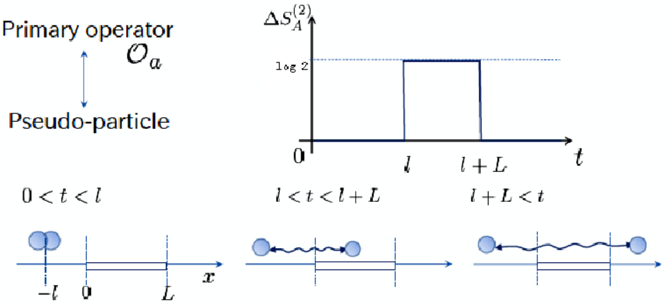
$$G_2(z, \bar{z}) = \frac{1}{2\sqrt{|z||1-z|}} \left(|z| + 1 + |1-z|\right). \tag{20}$$

- For $\mathcal{O}_2$, we find

$$\Delta S_A^{(2)} = \begin{cases} 0 & (0 < t < l, \text{ or } t > l + L), \\ \log 2 & (l < t < l + L). \end{cases} \tag{21}$$

- On the other hand, $\mathcal{O}_2$ creates a maximally entangled state (or equally Einstein-Podolsky-Rosen state): $\frac{1}{\sqrt{2}}\left(e^{\frac{i}{2}\phi_L}|0\rangle_L \otimes e^{\frac{i}{2}\phi_R}|0\rangle_R + e^{-\frac{i}{2}\phi_L}|0\rangle_L \otimes e^{-\frac{i}{2}\phi_R}|0\rangle_R\right)$, which carries the Renyi entropy $\log 2$

EE in $c = 1$ CFT


Primary operator
$\mathcal{O}_a$
Pseudo-particle
$\Delta S_A^{(2)}$
log 2
0
l
$l+L$
t
$0 < t < l$
$l < t < l + L$
$l + L < t$
$-l$
0
L
x

2D Ising Model

- A very interesting example involves the minimal model family of exactly solvable 2D CFT. The unitary minimal models are numbered by an integer m=3,4..., and describe the universality class of the multicritical Ginzburg-Landau model:

$$\mathcal{L} \sim (\partial\phi)^2 + \lambda\phi^{2m-2} \tag{22}$$

For $m = 3$, the Ising model is in the same universality class.

- The central charge of the model is

$$c = 1 - \frac{6}{m(m-1)}. \tag{23}$$

- All Virasoro primaries are scalar $O_{r,s}$ $1 \leq s \leq r \leq m-1$ whose dimension is

$$\Delta_{r,s} = \frac{(r + m(r-s))^2 - 1}{4m(m+1)} \tag{24}$$

EE in Ising model

- We consider primary operator $O_{2,2}$ in Ising model whose conformal dimension is

$$\Delta_{2,2} = \frac{3}{4m(m+1)}\Big|_{m=3} = \frac{1}{16} \tag{25}$$

This operator is also called spin operator.

- For Ising model, the conformal block of spin operator can be expressed by

$$G(z,\bar{z}) = \frac{1}{\sqrt{2}}\sqrt{\sqrt{\frac{|z|}{|1-z|}} + \frac{1}{\sqrt{|z||1-z|}} + \sqrt{\frac{|1-z|}{|z|}}}. \tag{26}$$

Using this explicit expression, one can take late time limit to obtain

$$\Delta S_A^{(2)} = \log\sqrt{2}. \tag{27}$$

EE in Ising model

- Through very very highly nontrivial calculation, we can show that $\Delta S_A^{(2)} = \Delta S_A^{(3)} = \Delta S_A^{(4)} = \ldots = \log\sqrt{2}$. Here we arrange a systematic program to check the result up to $n = 6$ by using computer.
- In terms of above exercises, it is nature to ask What is the meaning of $\sqrt{2}$.
- We have answered to this question in this paper. The $\sqrt{2}$ is exact quantum dimension of spin operator σ in Ising model.

EE in Minimal model

- Before answering the question, we would like to know what happen for general rational CFTs. In general CFTs, the function $G(z,\bar{z})$ can be expressed using the conformal blocks:

$$G_a(z,\bar{z}) = \sum_b (C^b_{aa})^2 F_a(b|z)\bar{F}_a(b|\bar{z}), \tag{28}$$

where b runs over all primary fields. In our normalization, the conformal block $F_a(b|z)$ behaves in the $z \to 0$ limit:

$$F_a(b|z) = z^{\Delta_b - 2\Delta_a}(1 + O(z)), \tag{29}$$

Δ_b is the conformal dimension of $\mathcal{O}_b$.

- Since we found $(z,\bar{z}) \to (0,0)$ when $0 < t < l$ or $t > l + L$, we get the behavior $G_a(z,\bar{z}) \simeq |z|^{-4\Delta_a}$, as the dominant contribution arises when $b = 0$ i.e. when $\mathcal{O}_b$ coincides with the identity $\mathcal{O}_0 (\equiv I)$ operator. We get $\Delta S^{(2)}_A = 0$, as expected from the causality argument.

EE in Minimal model

- To analyze the entropy when the causality condition $l < t < l + L$ is satisfied, we need to apply the fusion transformation, which exchanges z_2 with z_4 (or equally z with $1 - z$):

$$F_a(b|1-z) = \sum_c F_{bc}[a] \cdot F_a(c|z), \tag{30}$$

where $F_{bc}[a]$ is a constant, called Fusion matrix [G.Moore and N. Seiberg, '88][E.Verlinde, '88]. In the limit $(z, \bar{z}) \to (1, 0)$, we obtain

$$G_a(z, \bar{z}) \simeq F_{00}[a] \cdot (1-z)^{-2\Delta_a} \bar{z}^{-2\Delta_a}. \tag{31}$$

Therefore we find the following expression from (16):

$$\Delta S_A^{(2)} = -\log F_{00}[a]. \tag{32}$$

- Moreover, in rational CFTs, based on the arguments of bootstrap relations of correlations functions [G.Moore and N. Seiberg, '88], it was shown in [E.Verlinde, '88] that $F_{00}[a]$ coincides with the inverse of the quantity called quantum dimension d_a:

$$F_{00}[a] = \frac{1}{d_a} = \frac{S_{00}}{S_{0a}}, \tag{33}$$

where S_{ab} is the modular S matrix of the rational CFT we consider.

What is quantum dimension

- Here we just list the standard alternative definition of quantum dimension in Minimal model.

Quantum Dimension d_a ? ?

2D CFT

d_a → Maximal eigenvalue of N_{ab}^c

$$\mathcal{O}_a \cdot \mathcal{O}_b = \sum_c N_{ab}^c \, \mathcal{O}_c$$: Fusion rule

of the primary fields in $\overbrace{\mathcal{O}_a \cdots \mathcal{O}_a}^{N} = \sum_c (N_a \cdots N_a)_a^c \, \mathcal{O}_c \sim (d_a)^N$ $(N \to \infty)$

$\sum_c (N_a \cdots N_a)_a^c$ → M_N

$$\longrightarrow \log d_a = \lim_{N\to\infty} \frac{\log M_N}{N}$$

⇨ Quantum Dimension $d_a =$ " The effective d.o.f. of $\mathcal{O}_a$ "

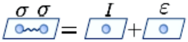
What is quantum dimension

- Especially in Ising model, one can easily work out quantum dimension of spin operator σ.

Quantum Dimension $d_a =$ " The effective d.o.f. of $\mathcal{O}_a$ "

Ising model

$$\mathcal{O}_a = \{I, \sigma, \varepsilon\} \qquad \begin{cases} \varepsilon \cdot \varepsilon = I \\ \sigma \cdot \sigma = I + \varepsilon \\ \varepsilon \cdot \sigma = \sigma \end{cases}$$



$$\sigma^{2N} = (I+\varepsilon)^N = 2^{N-1} I + 2^{N-1} \varepsilon \longrightarrow \log d_\sigma = \lim_{N\to\infty} \frac{\log 2^N}{2N} = \log\sqrt{2}$$

$$\longrightarrow d_\sigma = \sqrt{2} \qquad \text{Similarly} \quad d_I = d_\varepsilon = 1$$

- Comment: In the Ising model (i.e.(4,3) minimal model), there are three primary operators: the identity I, the spin σ and the energy operator ψ. Since the quantum dimension is 1 for I and ψ, $\Delta S_A^{(n)}$ is always vanishing for these. However, for the spin operator σ, we find $\Delta S_A^{(n)} = \log\sqrt{2}$ for any n as $d_\sigma = \sqrt{2}$.

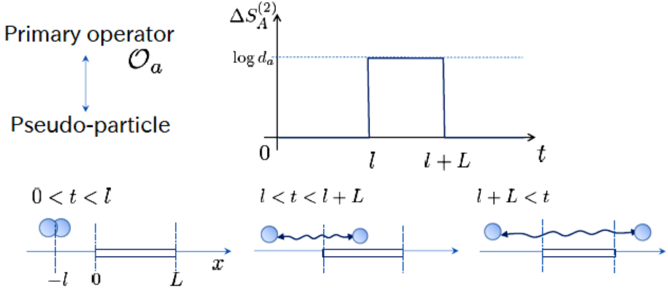
Our claim

- We propose following result to our experiences.

2D rational CFT

Primary Operator $\mathcal{O}_a$ d_a : Quantum Dimension

$$\Delta S_A^{(n)} = \log d_a \qquad (l < t < l + L)$$



Proof of Conjecture

Essence of the Derivation

- In the left part, we would like to prove the claim. Firstly, we would like to start from $n = 2$ to show $\Delta S_{EE}^{(2)}(A) = \log d_Q$

$$\frac{\left\langle \mathcal{O}_a^\dagger(w_1,\bar{w}_1)\mathcal{O}_a(w_2,\bar{w}_2)\mathcal{O}_a^\dagger(w_3,\bar{w}_3)\mathcal{O}_a(w_4,\bar{w}_4)\right\rangle_{\Sigma_2}}{\left\langle \mathcal{O}_a^\dagger(w_1,\bar{w}_1)\mathcal{O}_a(w_2,\bar{w}_2)\right\rangle_{\Sigma_1}^2} = |z|^{4\Delta}|1-z|^{4\Delta}\cdot G_a(z,\bar{z})$$

conformal map $z_i^2 = \frac{w_i}{w_i - L}$ cross ratio z

For general CFTs

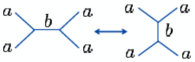
$$G_a(z,\bar{z}) = \sum_b \left(C_{aa}^b\right)^2 F_a(b|z)\bar{F}_a(b|\bar{z})$$

$F_a(b|z)$, $\bar{F}_a(b|\bar{z})$ → conformal blocks

$$F_a(b|z) \simeq z^{\Delta_b - 2\Delta} + \cdots \quad (z \to 0)$$

$$F_a(b|z) = \sum_c F_{bc}[a]\, F_a(c|1-z)$$

: crossing symmetry



- For $n = 2$ case, one can see $z \to 1 - z$ corresponding to interchange z_2 and z_4. This is crucial rule to generalize this proof to arbitrary n.

Essence of the Derivation

$(z, \bar{z}) \to (1, 0)$: late time limit $(l < t < l + L)$

$$G_a(z, \bar{z}) \to (1 - z)^{-2\Delta} (\bar{z})^{-2\Delta} F_{II}[a]$$

$$\longrightarrow \Delta S_A^{(2)} = -\log F_{II}[a]$$

For rational CFTs

$$d_a = \frac{1}{F_{II}[a]} = \frac{S_{Ia}}{S_{II}}$$

S_{ab} : modular S-matrix

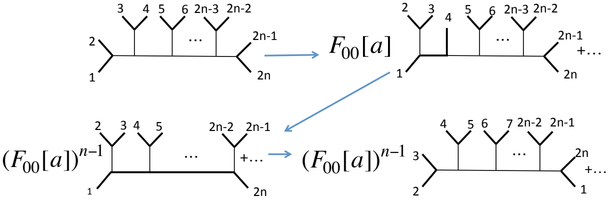
Similarly, for general n

$$\Delta S_A^{(n)} = \log d_a$$

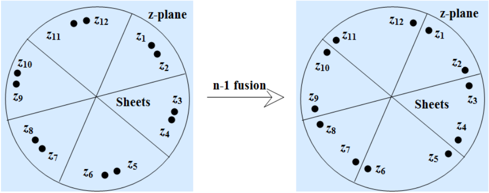
Note :
- The contribution reflects a global structure of conformal blocks
  reflects "topology" of CFTs
- Due to the analytic continuation of t ,
  z and $\bar{z}$ can have different values in the late time limit

Graph proof

- One can generalize this procedure to arbitrary n to show $\Delta S_{EE}^{(n)}(A) = \log d_Q$. The following procedure can be described by the above cartoon.



- One alternative way to understand this procedure.



Graph proof

- If we just repeat $n-1$ times the fusion transformation. Thus we obtain

$$\langle \mathcal{O}_a(z_1,\bar{z}_1)\mathcal{O}_a(z_2,\bar{z}_2)\dots\mathcal{O}_a(z_{2n},\bar{z}_{2n})\rangle_{\Sigma_1}$$
$$\simeq (F_{00}[a])^{n-1}\cdot\left[\prod_{k=0}^{n-1}(z_{2k+1}-z_{2k})(\bar{z}_{2k+1}-\bar{z}_{2k+2})\right]^{-2\Delta_a}.$$

Finally, the ratio at late time limit is computed to be $(F_{00}[a])^{n-1}=(d_a)^{1-n}$ And $\Delta S^{(n)}(A)=\frac{1}{1-n}\log(d_a)^{1-n}$.

Summary

We derived the simple formula which is applicable to both Renyi ($n \geq 2$) and von-Neumann ($n = 1$) entanglement entropy for primary operator excitations at late time.

Intuitively, this result fits nicely with the fact that the quantum dimension is a measure of the number of elementary fields included in a given primary field.

The essence of this calculation was that the time evolution performs the fusion transformation only in left-moving sector.

If we consider a product of primary operators $\prod_a (\mathcal{O}_a)^{n_a}$, we obtain $\Delta S_A^{(n)} = \sum_a n_a \log d_a$, using the sum rule in [Nozaki,14]. The quantum dimension d_a satisfies $d_a d_b = \sum_c N_{ab}^c d_c$.

Note that the topological entanglement entropy defined in the 3d topological theories also has the same contribution $\log d_a$ from anyons, in terms of its equivalent 2d (chiral) rational CFT which lives on their boundary. In this sense, our results formally look like a holographic dual of topological entanglement entropy. However, in our results, this contribution arises in dynamical systems defined by two dimensional rational CFTs, where their real time evolutions played an important role.

Comments

One may ask how about the irrational CFT, for example: Liouvile Theory. For irrational CFT, there are infinite number of conformal blocks. One can also study EE in this frame work. One difficult is that there are multiple contour integrals and it seems no way to extract leading contribution to EE in late time limit. Also the definition of quantum dimension in Liouvile theory is also subtle.

One may also consider much more complicated operators which are composed by primary operators with different way, linear combination, product, etc. In principle, it can be realized. To resolve such kind of problem highly depend on explicit formulas for conformal block in some rational CFT. One may make use of conformal boostrap to solve this problem. At this stage, we just move on some of them. If you have any comments and suggestions, please let me know.

One can also consider local operator and EE in WZW, D1-D5 system...

How about Higher dimensional generalization? Are there any characteristic quantities like quantum dimension.

From holographic point of view, how to realize this in gravity side?

Thanks for your attention!