Ω-deformation and quantization

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July 8, 2014 at Kavli IPMU

Based on arXiv:1405.6714

Overview

Motivation

Intriguing phenomena in 4d $\mathcal{N} = 2$ supserymmetric gauge theories:

1. Ω-deformation on $\mathbb{R}^3 \times S^1$
   $\leftrightarrow$ quantization of integrable systems [Nekrasov–Shatashvili]
2. loop operator VEVs on $\mathbb{R}^3 \times_\varepsilon S^1$
   $\leftrightarrow$ deformation quantization on a hyperkähler manifold

   [Gaiotto–Moore–Neitzke, Ito–Okuda–Taki]

Derivations of (1) have been given using

- brane quantization [Nekrasov–Witten]
- topological strings/matrix models [Aganagic et al., Bonelli–Maruyoshi–Tanzini]

Is there a unified framework for understanding both phenomena?

Answer

Yes.

I will discuss an approach to quantization

- that explains the 4d phenomena
- based on a deformation of a 3d TQFT
- viewed as a 2d TQFT

The 3d TQFT is Rozansky–Witten theory.

The 2d TQFT is a B-twisted Landau–Ginzburg model.

The idea is similar to [Luo–Tan–JY], where we considered $\mathcal{N} = 2$ gauge theories on $\mathbb{R} \times S^2 \times S^1$ and found that they quantize real integrable systems.

From 4d to 3d

Consider a 4d $\mathcal{N} = 2$ gauge theory.

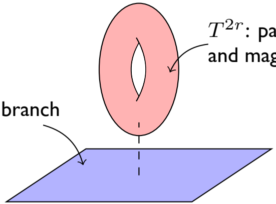
We go down to 3d:

1. Compactify on S^1.
2. In IR, we get a 3d abelian gauge theory on the Coulomb branch.
3. In 3d, abelian gauge fields are dual to periodic scalars.
4. Dualization gives an $\mathcal{N} = 4$ sigma model.

This setup was studied by Gaiotto–Moore–Neitzke in connection with wall-crossing phenomena.

Geometry of $\mathcal{M}$

The target $\mathcal{M}$ is a torus fibration:



$\mathcal{M}$ is hyperkähler:

- $\mathbb{CP}^1$ of complex structures $aI + bJ + cK$, $a^2 + b^2 + c^2 = 1$, with $I^2 = J^2 = K^2 = IJK = -1$
- metric g that is Kähler with respect to J_ζ for all $\zeta \in \mathbb{CP}^1$

For theories in "class $\mathcal{S}$," the target is the Hitchin moduli space.

Geometry of $\mathcal{M}$

In I, $\mathcal{M}$ is a complex integrable system:

- complex version of phase space in classical mechanics
- holomorphic symplectic form $\Omega_I = \omega_J + i\omega_K$
- $\mathcal{B}$ holomorphic, T^{2r} holomorphic Lagrangian

There are complex coordinates $(a^i) \in \mathcal{B}$, $(z_i) \in T^{2r}$ in which the Poisson bracket is given by

$$\{z_i, z_j\} = \{a^i, a^j\} = 0, \quad \{z_i, a^j\} = \delta^i_j.$$

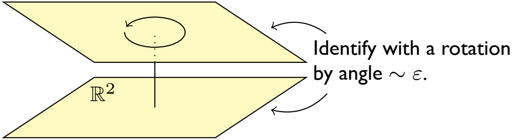
Integrability: $\frac{1}{2}\dim_{\mathbb{C}} \mathcal{M}$ commuting conserved momenta $\{a^i\}$

Quantization by Ω-deformation [Nekrasov–Shatashvili, Nekrasov–Witten]

Start with an $\mathcal{N} = 2$ gauge theory on $\mathbb{R}^3 \times S^1$.

Turn on an Ω-deformation:

1. Lift to a 5d theory on $\mathbb{R}^3 \times S^1 \times S^1_R$.
2. Replace $\mathbb{R}^2 \times S^1_R$ by $\mathbb{R}^2 \times_\varepsilon S^1_R$:



3. Take $R \to 0$ to go back to 4d.

Quantization by Ω-deformation [Nekrasov–Shatashvili, Nekrasov–Witten]

The original theory has a TQFT sector. (Donaldon–Witten theory)

After the Ω-deformation, this sector becomes quasi-TQFT.

The quasi-TQFT is equivalent to quantum mechanics on $L \subset \mathcal{M}$.

L is a symplectic submanifold, determined by the boundary condition.

The Planck constant $\hbar \propto \varepsilon$:

$$[z_i, z_j] = [a^i, a^j] = 0, \quad [z_i, a^j] \propto i\varepsilon\delta^i_j.$$

Quantization by twisting of the spacetime [Gaiotto–Moore–Neitzke, Ito–Okuda–Taki]

Start again with an $\mathcal{N} = 2$ gauge theory on $\mathbb{R}^3 \times S^1$.

Define supercharges $Q^\zeta \propto Q + \zeta G_4, \ \zeta \in \mathbb{CP}^1$.

For $\zeta \neq 0$, ∞, there are Q^ζ-invariant line operators $\mathcal{L}_a^\zeta$.

$\mathcal{L}_a^\zeta$ realize the algebra of holomorphic functions on $(\mathcal{M}, J_\zeta)$:

1. Wrap them on $\{p_a\} \times S^1 \subset \mathbb{R}^3 \times S^1$.
2. We can actually move p_a around freely.
3. Taking them far apart, we find

$$\langle \mathcal{L}_1^\zeta \cdots \mathcal{L}_n^\zeta \rangle = \langle \mathcal{L}_1^\zeta \rangle \cdots \langle \mathcal{L}_n^\zeta \rangle.$$

4. $\langle \mathcal{L}_a^\zeta \rangle$ are holomorphic functions on $\mathcal{M}$. (framed BPS indices)

Quantization by twisting of the spacetime [Gaiotto–Moore–Neitzke, Ito–Okuda–Taki]

Twist the spacetime: $\mathbb{R}^3 \times S^1 \to \mathbb{R} \times \mathbb{R}^2 \times_\varepsilon S^1$:

Now $\mathcal{L}_a^\zeta$ must be inserted at $(t_a, 0) \in \mathbb{R} \times \mathbb{R}^2$ in order to be Q^ζ-invariant.

Ordering is well-defined: $\mathcal{L}_1^\zeta(t_1)\mathcal{L}_2^\zeta(t_2) \neq \mathcal{L}_2^\zeta(t_1)\mathcal{L}_1^\zeta(t_2)$.

One finds

The algebra of holomorphic functions on $\mathcal{M}$ gets quantized:

$$\langle \mathcal{L}_1^\zeta(t_1) \cdots \mathcal{L}_n^\zeta(t_n) \rangle = \langle \mathcal{L}_1^\zeta \rangle \star \cdots \star \langle \mathcal{L}_n^\zeta \rangle.$$

$\star$: noncommutative multiplication with $\hbar \sim \varepsilon$.

Main results

Consider RW theory, a TQFT based on 3d $\mathcal{N} = 4$ sigma model.

Let the spacetime $\mathbb{R} \times \Sigma$.

Let the target be a hyperkähler manifold X.

Pick a complex structure I on X.

We can construct an Ω-deformation of RW theory.

The construction involves an Ω-deformation of a B-twisted LG model with infinite-dimensional target.

Main results

Take $\Sigma = D$, a disk.

Choose $L \subset X$ that is of type (A, B, A):

- Lagrangian with respect to ω_I
- holomorphic in J
- Lagrangian with respect to ω_K

Use L as the support of a brane placed on ∂D.

The Ω-deformed RW theory is equivalent to QM on L.

It seems closely related to brane quantization of Gukov–Witten.

Main results

Furthermore,

the 4d phenomena are two special cases with the same target.

The two cases just differ by the choice of complex structure.

The case (1) (the NS correspondence) should be equivalent to the Nekrasov–Witten approach.

I'll explain things by going 2d $\to$ 3d $\to$ 4d.

2d: Ω-deformation of B-twisted LG models

Ω-deformation in 4d

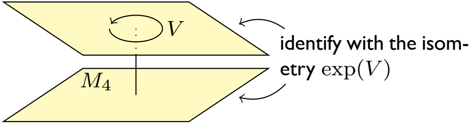
Topologically twisted 4d $\mathcal{N} = 2$ gauge theory:

- scalar supercharge Q, with $Q^2 = 0$, used as a BRST operator
- TQFT – invariant under deformations of the metric

Pick a Killing vector field V on the spacetime 4-manifold M_4.

Use V to introduce the Ω-deformation:

1. Lift to a 5d theory on $M_4 \times S^1$.
2. Replace $M_4 \times S^1$ by $M_4 \times_V S^1$:



3. Shrink the S^1 to go back to 4d.

Ω-deformation in 4d

Topologically twisted 4d $\mathcal{N} = 2$ gauge theory:

- scalar supercharge Q, with $Q^2 = 0$, used as a BRST operator
- TQFT – invariant under deformations of the metric

Pick a Killing vector field V on the spacetime 4-manifold M_4.

Ω-deformed twisted theory:

- $Q^2 = L_V$, with L_V acting as the Lie derivative $\mathcal{L}_V$ on fields
- quasi-TQFT – invariant under deformations of the metric as long as V remains to be a Killing vector field

Ω-deformation of B-twisted LG model

B-twisted LG model:

- scalar supercharge Q, with $Q^2 = 0$, used as a BRST operator
- TQFT – invariant under deformations of the metric

Pick a Killing vector field V on the worldsheet Σ.

Ω-deformed B-twisted LG model:

- $Q^2 = L_V$, with L_V acting as the Lie derivative $\mathcal{L}_V$ on fields
- quasi-TQFT – invariant under deformations of the metric as long as V remains to be a Killing vector field

Ω-deformation of B-twisted LG model

Input data:

- worldsheet (Σ, h)
- target (Y, g), a (curved) Kähler manifold
- superpotential W, a holomorphic function on Y
- Killing vector field V on Σ

Field content:

- bosonic field: $\Phi\colon \Sigma \to Y$
- fermions: scalar $\eta^{\bar{\imath}}$, 1-form ρ^i, 2-form $\mu^{\bar{\imath}}$
- bosonic auxiliary 2-forms: F^i, $\overline{F}^{\bar{\imath}}$

We use $\mu^{\bar{\imath}}$ instead of the scalar θ_i in ordinary B-twisted models:

$$\mu^{\bar{\imath}} \sim g^{\bar{\imath}j} \star \theta_j .$$

Ω-deformation of B-twisted LG model

For Y flat, $Q = Q_{V=0} + V^\mu G_\mu$, with G the 1-form supercharge:

$$\begin{aligned} \delta\phi^i &= \iota_V \rho^i, & \delta\bar{\phi}^{\bar{\imath}} &= \eta^{\bar{\imath}}, \\ \delta\rho^i &= \mathrm{d}\phi^i + \iota_V F^i, & \delta\eta^{\bar{\imath}} &= V(\bar{\phi}^{\bar{\imath}}), \\ \delta F^i &= \mathrm{d}\rho^i, & \delta\mu^{\bar{\imath}} &= \overline{F}^{\bar{\imath}}, \\ & & \delta\overline{F}^{\bar{\imath}} &= \mathrm{d}\iota_V \mu^{\bar{\imath}}. \end{aligned}$$

Compare with the 4d formula for abelian gauge group:

$$\begin{aligned} \delta\phi &= \iota_V \psi, & \delta\bar{\phi} &= \eta, \\ \delta\psi &= \mathrm{d}\phi + \iota_V F_A, & \delta\eta &= V(\bar{\phi}), \\ \delta A &= \psi, & \delta\chi &= iH, \\ & & \delta H &= -i\mathcal{L}_V \chi. \end{aligned}$$

It makes sense to call the auxiliary field F!

Ω-deformation of B-twisted LG model

The action $S = S_0 + S_W$.

S_0 is the sigma model action:

- Q-exact
- contains the metrics g on Y and h on Σ

S_W is the superpotential part:

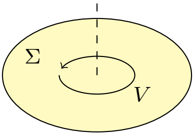
- not Q-exact, but Q-invariant assuming $\partial\Sigma = \emptyset$
- independent of g and h

The theory is invariant under

- overall rescaling of g
- deformations of h as long as V remains Killing

Branes for Ω-deformed LG model

Suppose $\partial\Sigma = S^1$ and $V|_{\partial\Sigma} = \varepsilon\partial_\varphi$:



We find $\delta S_W \neq 0$.

To recover Q-invariance, change

$$S_W \to S_W - \frac{i}{\varepsilon}\int_{\partial\Sigma} W \mathrm{d}\varphi.$$

Now $\delta S_W = 0$.

Branes for Ω-deformed LG model

But the boundary term

$$-\frac{i}{\varepsilon}\int_{\partial\Sigma} W \mathrm{d}\varphi.$$

is not bounded in general.

To remedy this, place a brane supported on $\gamma \subset Y$ and impose

- $\operatorname{Im} W$ is constant on γ

We can set $\operatorname{Im} W = 0$ by shift $W \to W + W_0$.

For a reason that will become clear shortly, we also impose

- γ is a Lagrangian submanifold

The brane is more analogous to A-branes than B-branes.

Localization of Ω-deformed LG model

Localize the path integral for $\Sigma = D$:

1. Send $g \to \infty$.
2. The path integral localizes to constant maps Φ_0.
3. The 1-loop determinant is independent of Φ_0 if γ is Lagrangian.
4. For $\Phi = \Phi_0$, only the boundary term survives in S.
5. No fermion zero modes by the boundary condition.

We obtain the localization formula

$$\langle \mathcal{O} \rangle = \int_\gamma \mathrm{d}\Phi_0 \exp\Big(\frac{2\pi i}{\varepsilon} \operatorname{Re} W(\Phi_0)\Big) \mathcal{O}(\Phi_0).$$

3d: Quantization via Ω-deformed RW theory

Ω-deformation of RW theory

Consider RW theory, a TQFT based on $\mathcal{N} = 4$ sigma model, with

- spacetime $\mathbb{R} \times \Sigma$.
- target X hyperkähler (as opposed to complex symplectic)

Pick a complex structure on X, say I.

View the theory as a B-twisted LG model on Σ:

- $Y = \mathrm{Map}(\mathbb{R}, X)$, with complex structure induced from I
- The terms with ∂_t are provided by the superpotential

$$W(\Phi) = \frac{1}{2} \int_{\mathbb{R}} \Phi^* \Lambda, \quad \Omega_I = \mathrm{d}\Lambda.$$

We can Ω-deform the theory.

Branes for Ω-deformed RW theory

Suppose $\partial\Sigma = S^1$.

Recall the conditions on the support of brane γ:

1. $\operatorname{Im} W$ is constant on γ
2. γ is Lagrangian

To satisfy these we take $\gamma = \operatorname{Map}(\mathbb{R}, L)$, with $L \subset X$ such that

1. $\operatorname{Im} \Omega_I = \omega_K = 0$ on L
2. L is Lagrangian with respect to ω_I

It follows that L is of type (A, B, A); the brane is similar to (A, B, A)-branes in $\mathcal{N} = (4, 4)$ sigma models.

Localization of the Ω-deformed RW theory

Take $\Sigma = D$. The localization formula

$$\langle \mathcal{O} \rangle = \int_\gamma \mathrm{d}\Phi_0 \exp\Big(\frac{2\pi i}{\varepsilon} \operatorname{Re} W(\Phi_0) \Big) \mathcal{O}(\Phi_0)$$

translates into

$$\langle \mathcal{O} \rangle = \int_{\mathrm{Map}(\mathbb{R}, L)} \mathcal{D}\Phi_0 \exp\Big(\frac{i\pi}{\varepsilon} \int_{\mathbb{R}} \Phi_0^* \operatorname{Re} \Lambda \Big) \mathcal{O}(\Phi_0).$$

If $\operatorname{Re} \Omega_I = \omega_J = \mathrm{d}p^a \wedge \mathrm{d}q_a$, then the Lagrangian is $p^a \mathrm{d}q_a$.

The Ω-deformed RW theory on $\mathbb{R} \times D$ is equivalent to QM on (L, ω_J) with $\hbar \propto \varepsilon$.

Note that (L, ω_J) is a Kähler submanifold of X, hence symplectic.

Localization of the Ω-deformed RW theory

What about the observables?

The SUSY transformations

$$\delta\phi^i = \iota_V \rho^i, \qquad \delta\bar{\phi}^{\bar{\imath}} = \eta^{\bar{\imath}},$$
$$\delta\rho^i = \mathrm{d}\phi^i + \iota_V F^i, \quad \delta\eta^{\bar{\imath}} = V(\bar{\phi}^{\bar{\imath}})$$

show $Q \leftrightarrow \bar{\partial}$, $\eta^{\bar{\imath}} \leftrightarrow \mathrm{d}\bar{\phi}^{\bar{\imath}}$ at zeros of V.

Thus $H^{0,q}(X;\mathbb{C}) \subset Q$-cohomology.

The localization sets fermions to zero; only the $q = 0$ part survives.

The Ω-deformation quantizes the algebra of holomorphic functions.

This is a deformation quantization.

4d: Applications to $\mathcal{N} = 2$ gauge theory

Quantization by Ω-deformation

Let's derive the Nekrasov–Shatahsvili correspondence:

1. Consider a twisted $\mathcal{N} = 2$ gauge theory on $\mathbb{R} \times D \times S^1$.
2. Ω-deform the theory.
3. By topological invariance, we can shrink the S^1.
4. We get the Ω-deformed RW theory on $\mathbb{R} \times D$ whose target is the complex integrable system $(\mathcal{M}, \Omega_I)$.
5. It is QM on $L \subset \mathcal{M}$, specified by the brane.

We conclude:

The Ω-deformation quantizes (L, ω_J).

Quantization by Ω-deformation

We can derive the Bethe/gauge correspondence:

1. Take L to be the locus $\operatorname{Im} a_{D,i} = \theta_{m,i} = 0$.
2. The QM Lagrangian is $-\operatorname{Re} a_{D,i} \mathrm{d}\theta_e^i$.
3. Integrating over θ_e^i imposes $\operatorname{Re} a_{D,i}/\hbar = \mathbb{Z}$.
4. Combined with $\operatorname{Im} a_{D,i} = 0$, we obtain

$$\exp\left(\frac{2\pi i}{\hbar} a_{D,i}\right) = \exp\left(\frac{\partial \widetilde{W}}{\partial a^i}\right) = 1,$$

where $\widetilde{W} = 2\pi i \mathcal{F}/\hbar$ and $\mathcal{F}$ is the deformed prepotential.

This is the Bethe equations of the integrable system, with $\widetilde{W}$ identified with the Yang–Yang function.

Quantization by twisting of the spacetime

Now we derive the second correspondence:

1. Consider a twisted $\mathcal{N} = 2$ gauge theory on $\mathbb{R} \times D \times_\varepsilon S^1$.
2. Wrap $\mathcal{L}_a^\zeta$ on the S^1 at $\{t_a\} \times \{0\} \in \mathbb{R} \times D$.
3. The VEV is an index, so we can shrink the S^1.
4. We get the Ω-deformed RW theory on $\mathbb{R} \times D$ with target $(\mathcal{M}, J_\zeta)$.
5. $\mathcal{L}_a^\zeta$ descend to local operators, namely holomorphic functions on $\mathcal{M}$, and their algebra is quantized.

Twisting the spacetime quantizes the algebra of holomorphic functions on $(\mathcal{M}, J_\zeta)$ generated by SUSY loop operators.

Concluding remarks

Summary

In this talk I discussed

- Ω-deformation of B-twisted LG models in 2d
  - branes are analogous to A-branes
  - localization formula on a disk
- Ω-deformation of RW theory in 3d
  - branes are similar to (A, B, A)-branes
  - the Ω-deformed RW theory on $\mathbb{R} \times D$ quantizes a symplectic submanifold of the hyperkähler target space
- applications to $\mathcal{N} = 2$ gauge theory in 4d
  - Ω-deformation on $\mathbb{R} \times D \times S^1$ quantizes the integrable system $(\mathcal{M}, \Omega_I)$ associated with the Coulomb branch
  - loop operator VEVs on $\mathbb{R} \times D \times_\varepsilon S^1$ quantize the algebra of holomorphic functions on $(\mathcal{M}, J_\zeta)$, $\zeta \neq 0, \infty$

Future directions

Possible directions for future research:

- Ω-deformation of mirror symmetry

  The A-model side compute vortex partition functions. Reproduced by B-twisted LG models?

- Ω-deformation of gauged RW theory

  Constructed by Kapustin & Saulina. A TQFT version of $\mathcal{N} = 4$ sigma model with Chern–Simons coupling, constructed by Gaiotto–Witten. Lead to "equivariant" quantization?

- quantization of Seiberg–Witten curve [Fucito et al., ...]
- wall-crossing?

Work in progress (with Y. Luo, M.-C. Tan and Q. Zhao):

- Ω-deformation of B-twisted gauge theories

  Application to the 3d/3d correspondence between 3d SCFT and complex CS [Dimofte et al., Terashima–Yamazaki]. The idea is similar to [JY, Cordova–Jafferis, Lee–Yamazaki].