Gauge Theories with Rigid Supersymmetry in d=4,5

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1401.3266, 1411.xxxx (with Diego Rodriguez-Gomez), 1411.xxxx (with Yiwen Pan)

1. Introduction

Rigid supersymmetry is used to construct gauge theories, localisation to calculate observables.

Rigid supersymmetry

The general procedure

1) Pick a supergravity theory (off-shell); pick a manifold.
2) Solve Killing spinor equation (KSE).

$$\delta(\text{gaugino}) = 0 = \delta(\text{dilatino})$$

3) This fixes background fields in the Weyl multiplet.

See also: Bawane, Bonelli, Ronzani, Tanzini (last week).

Festuccia, Seiberg; Dumitrescu, Closset, Komargodski; Klare, Tomasiello, Zaffaroni; …

Localisation I

$$\delta = \delta_{\mathrm{SUSY}} + \delta_{\mathrm{BRST}}, \qquad \delta^2 = \pounds_V + G_a .$$

The localisation argument

$$Z(t) = \int [d\phi] \mathcal{O} e^{-S - t\delta V} \qquad \Rightarrow \qquad \frac{d}{dt} Z(t) = 0.$$

$$Z = \sum_{\delta V|_B = 0} (\text{1-loop fluctuations})$$

$$\delta V|_B = 0 \quad \Leftrightarrow \quad \delta(\text{fermions}) = 0.$$

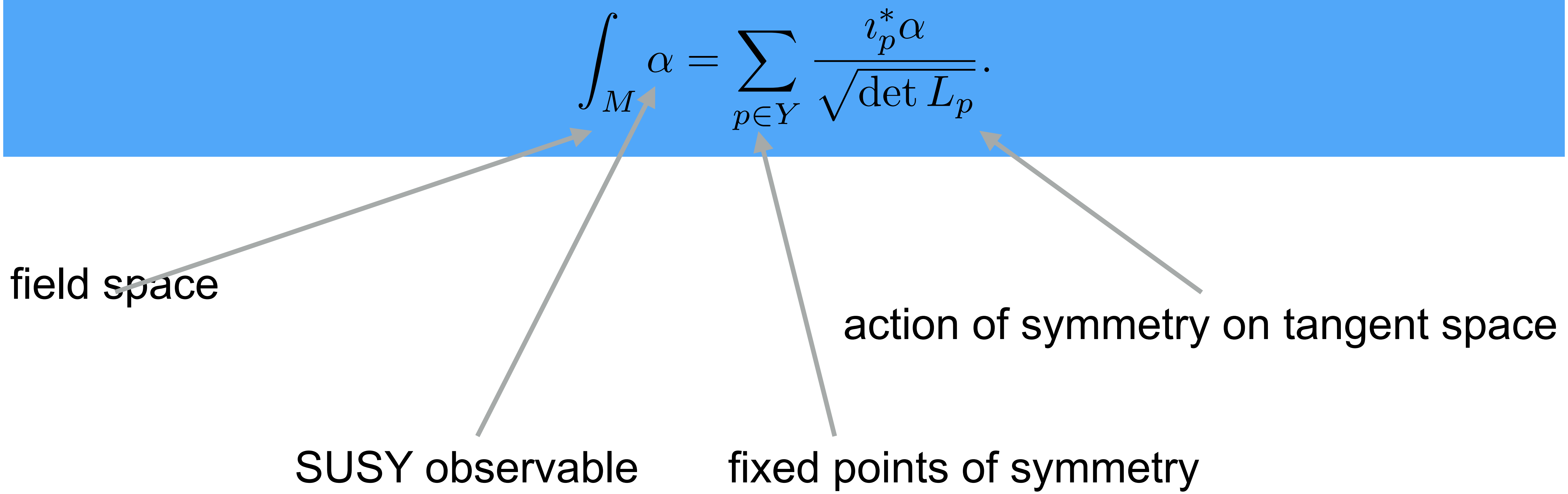
Sometimes

$$Z = Z_{\mathrm{cl}} Z_{\text{1-loop}} Z_{\mathrm{inst}}$$

Pestun; Kapustin, Willett, Yaakov; Gomis, Okuda; Cassani, Martelli; Alday, Richmond, Sparks; Kim, Kim, Lee, Park; Hosomichi, Seong, Terashima; Källen; Qiu, Zabzine; …

Localisation II

Atiyah-Bott-Berline-Vergne localisation theorem (generalised to infinite dim.)



$$\int_M \alpha = \sum_{p\in Y} \frac{\imath_p^* \alpha}{\sqrt{\det L_p}}.$$

field space

SUSY observable

fixed points of symmetry

action of symmetry on tangent space

For us, this means:

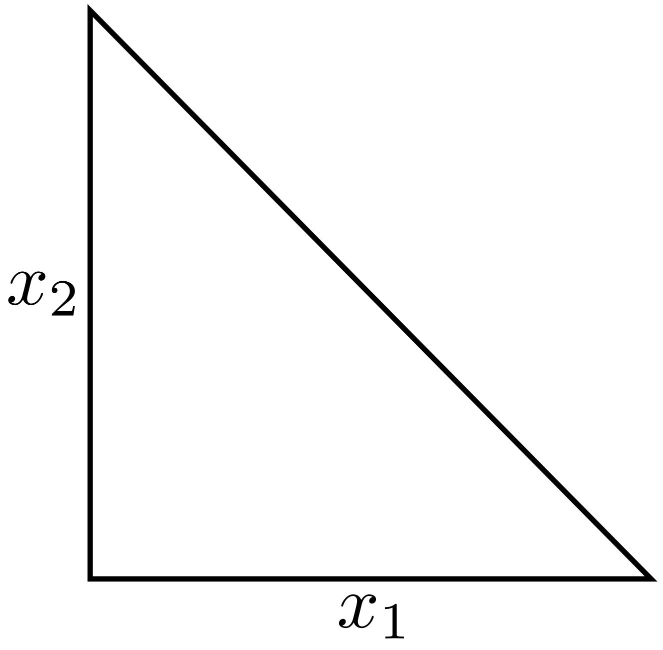
Evaluate super-determinants at localisation loci.

Toric Kähler manifolds and the Delzant construction

Toric Kähler manifold: A closed, connected, 2n dim. Kähler manifold with an effective Hamiltonian action of the real n-torus.

Today:

Delzant polytope and “potential” $g(x) = g_P(x) + h(x)$



$$\begin{aligned}\omega &= \sum_i dx_i \wedge dy_i\,,\\ ds^2 &= (\partial_i\partial_j g) dx_i \otimes dx_j + (\partial_i\partial_j g)^{-1} dy_i \otimes dy_j\,,\\ V &= p\partial_{y_1} + q\partial_{y_2}\,.\end{aligned}$$

Guillemin; Abreu

On holomorphy

Complex manifolds

$$T_{\mathbb{C}}M = T^{1,0} \oplus T^{0,1}, \qquad [T^{1,0}, T^{1,0}] \subseteq T^{1,0}, \qquad \exists \bar{\partial}.$$

Almost Cauchy-Riemann structure

$$T^{1,0} \cap T^{0,1} = \{0\}$$

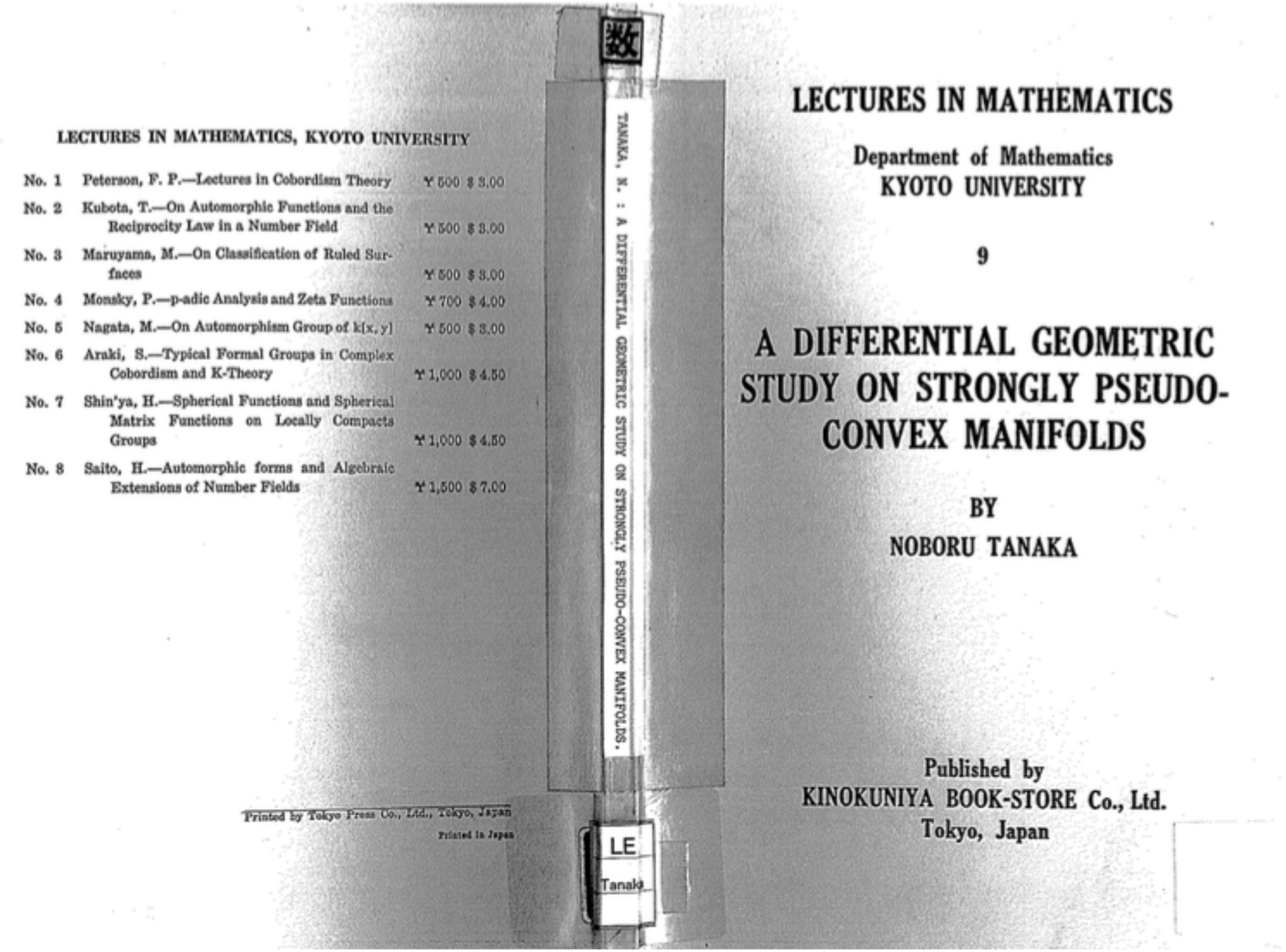
… of hypersurface type

$$T_{\mathbb{C}}M = T^{1,0} \oplus T^{0,1} \oplus R$$

Integrability

$$[T^{1,0}, T^{1,0}] \subseteq T^{1,0}, \qquad [T^{1,0} \oplus R, T^{1,0} \oplus R] \subseteq T^{1,0} \oplus R.$$

Last time I was here …



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A DIFFERENTIAL GEOMETRIC STUDY ON STRONGLY PSEUDO-CONVEX MANIFOLDS

BY

NOBORU TANAKA

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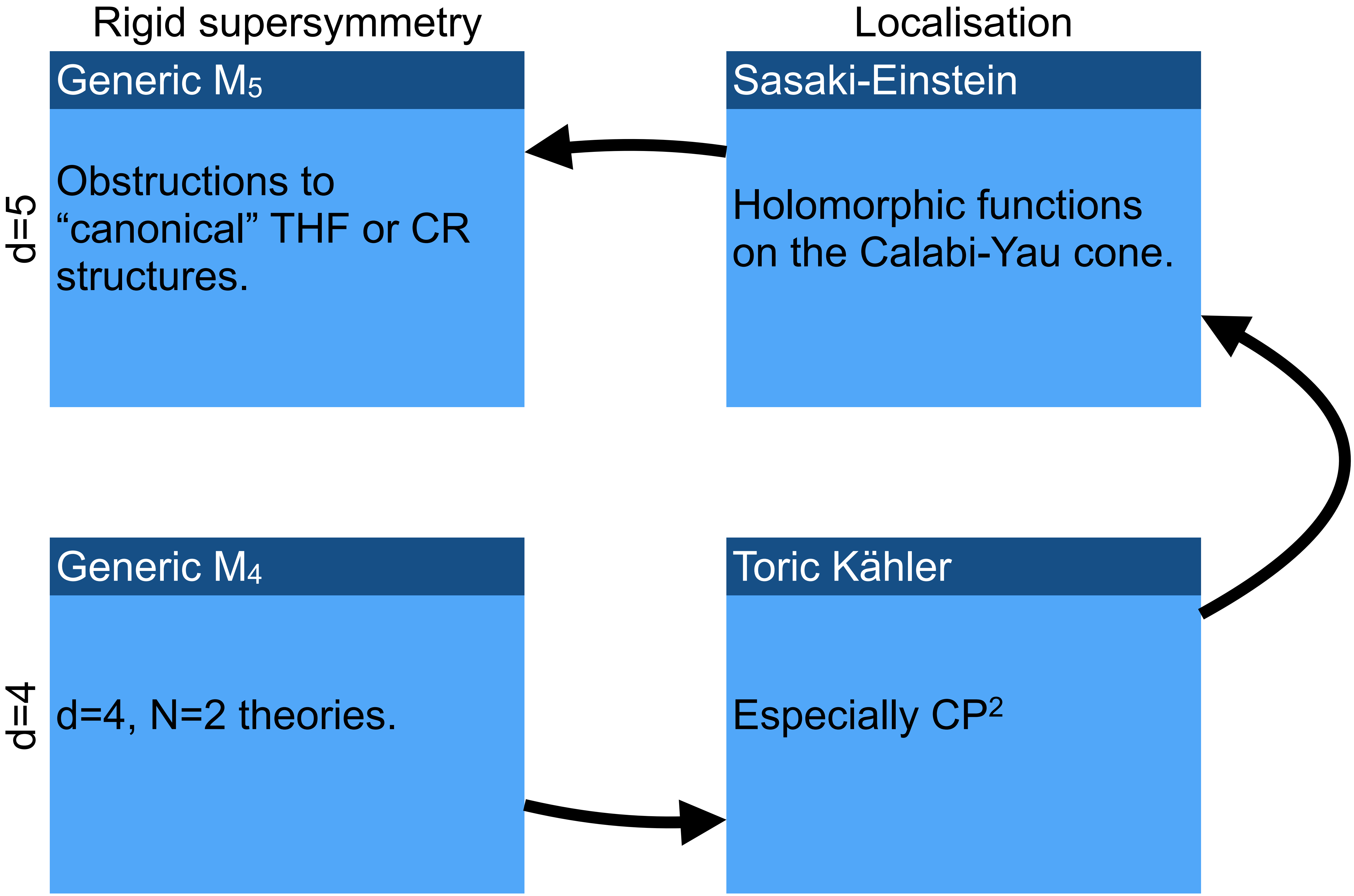
Outline



Rigid supersymmetry

Localisation

d=5

Generic M_5

Obstructions to “canonical” THF or CR structures.

Sasaki-Einstein

Holomorphic functions on the Calabi-Yau cone.

d=4

Generic M_4

d=4, N=2 theories.

Toric Kähler

Especially CP^2

What to expect:

Examples of localisation loci:

$$F^- = 0, \qquad F^-|_{V=0} = 0, \qquad F = 0.$$

One loop determinants simplify due to holomorphy.

There is some evidence of factorisation.

Some examples are known to be “quasi topological”.

4d & 5d results should be compatible.

2. Super Yang-Mills theory on toric Kähler manifolds

With focus on CP^2.

with Diego Rodriguez-Gomez

N=2, d=4 supergravity

$$\underbrace{g_{mn}, \quad T^{\pm}_{mn}, \quad d, \quad \mathcal{A}_m, \quad \mathcal{A}^i_{mj}, \quad \psi^i_m, \quad \chi^i}_{\text{Weyl multiplet}} \qquad \underbrace{\phi, \quad \bar{\phi}, \quad \mathcal{A}_m, \quad \Omega^i_{\pm}, \quad Y_{ij}}_{\text{Vector multiplet}}$$

Killing spinor equations:

$$\delta\psi^i_{+m} = \nabla_m \epsilon^i_+ + \frac{\imath}{2} A_{mx} \sigma^{xi}{}_j \epsilon^j_+ + \frac{1}{2} A_{m4} \epsilon^i_+ + \frac{\imath}{4} T^+_{mn} \gamma^n \epsilon^i_- - \gamma_m \eta^i_- ,$$

$$\delta\psi^i_{-m} = \dots , \qquad \delta\chi^i_+ = \dots , \qquad \delta\chi^i_- = \dots .$$

Solutions

Only one chirality preserved: Any four manifold.
Both chiralities: Existence of a conformal Killing vector.

$$\nabla_{(m} V_{n)} = \lambda g_{mn}$$

Klare and Zaffaroni; see also: Bawane, Bonelli, Ronzani, Tanzini

The Killing spinors

$$\delta\psi^i_{+m} = \nabla_m\epsilon^i_+ + \frac{\imath}{2}A_{mx}\sigma^{xi}{}_j\epsilon^j_+ + \frac{1}{2}A_{m4}\epsilon^i_+ + \frac{\imath}{4}T^+_{mn}\gamma^n\epsilon^i_- - \gamma_m\eta^i_- .$$

On any Kähler manifold

$$\nabla_m\epsilon^1_- = -\frac{\imath}{2}(A_{\mathrm{Ric}})_m\epsilon^1_-$$

Negative chirality solution

$$\epsilon^2_- = \imath C(\epsilon^1_-)^*, \qquad \mathcal{A}_3 = A_{\mathrm{Ric}}$$

Both chiralities

$$\epsilon^i_+ = \imath V^m\gamma_m\epsilon^i_-, \quad T^+ = -2dV.$$

Symplectic form is anti self-dual!

$$\omega \in \Omega^-$$

The cohomological theory

The “usual” complex

$$\begin{aligned} &Z \in \{\bar{\phi}, A, \chi\}, && Z' \in \{\eta, \Psi, H\}, \\ &\delta Z = Z', && \delta Z' = \pounds_V Z + G_{\phi - V^2\bar{\phi} - \imath_V A}[Z] \end{aligned}$$

Gauginos are mapped to p-forms:

$$\Omega^i_{\pm} \quad \leftrightarrow \quad \eta \in \Omega^0, \Psi \in \Omega^1, \chi \in \Omega^-$$

The auxiliary triplet

$$Y_{ij} \quad \leftrightarrow \quad H \in \Omega^-$$

The localisation locus

$$\begin{aligned}\delta\eta &= D_V\bar{\phi} - [\phi,\bar{\phi}],\\ \delta\Psi &= \imath_V(F+\bar{\phi}dV) + D\phi - V^2 D\bar{\phi},\\ \delta\chi &= -2\imath(F+\bar{\phi}dV)^- + \frac{\imath}{2}\mathcal{M}_-^{ij}Y_{ij} - 4\imath(D\bar{\phi}\wedge V)^-,\\ (\mathcal{M}_-^{ij} &= -\imath\epsilon_-^i C\gamma_{(2)}\epsilon_-^j).\end{aligned}$$

Contour rotation

$$(Y_{ij})^* = -Y^{ij}$$

Localisation locus

$$\phi_1 = \alpha_1, \quad \phi_2 = \frac{\alpha_2}{1+V^2}, \quad F = -\alpha_1 dV, \quad (A = -\alpha_1 V), \quad Y_{ij} = Y_{ij}[\alpha_2, V].$$

Complex: Gauge transformation along *complex* gauge parameter.

Without contour rotation (and without positive chirality) - e.g.:

$$F = -\frac{\imath}{2}Y_{12}\omega.$$

Compare: Hama and Hosomichi

Gauge fixing

BRST + SUSY complex

$$\delta = \delta_{\mathrm{SUSY}} + \delta_{\mathrm{BRST}},$$

$$\delta Y = Y', \qquad \delta Y' = (\pounds_v + G_{a_0})Y, \qquad \delta a_0 = 0.$$

Gauge fixing term

$$\begin{aligned}\delta V_{\mathrm{g.f.}} &= \imath(b, d^\dagger \mathcal{A}) - \imath(\tilde{c}, d^\dagger \Psi) - \imath(\tilde{c}, d^\dagger d_{\mathcal{A}} c) \\ &\quad + \imath(b, b_0) - \imath(\tilde{c}, c_0) - (c, \tilde{c}_0) + \left(\rho, \tilde{a}_0 - \frac{\xi_2}{2} a_0\right).\end{aligned}$$

Corrected localisation locus

$$\delta c = a_0 - (\alpha_1 + \imath\alpha_2) - \frac{g}{2}[c, c],$$

$$a_0 = \imath a_0^E \quad \Rightarrow \quad \alpha_1 = 0.$$

Note: Alternative choice seems possible. $\alpha_2 = 0.$

Then, some later computations become problematic.

Partition function — general structure

Recall

$$Z = \sum_{\delta V|_B = 0} (\text{1-loop fluctuations})$$

Additional singular instanton solutions at V = 0

$$\int_{\mathfrak{g}} [da_0^E] Z_{\text{cl}}(a_0^E) Z_{\text{1-loop}}(a_0^E) Z_{\text{instantons}}(a_0^E).$$

Weyl integration formula:

$$\frac{1}{|W|} \frac{\text{vol}(G)}{\text{vol}(T)} \int_{\mathfrak{t}} [da_0^E] \prod_{\beta > 0} \langle a_0^E, \beta \rangle^2 Z_{\text{cl}}(a_0^E) Z_{\text{1-loop}}(a_0^E) Z_{\text{instantons}}(a_0^E).$$

Partition function — classical action

$$\mathcal{L} = d\phi\bar{\phi} + \nabla_m^A \phi \nabla^{Am}\bar{\phi} + \frac{1}{8} Y^i_{\ j} Y^j_{\ i} - g[\phi, \bar{\phi}]^2 + \frac{1}{8} F_{mn} F^{mn}$$
$$- \frac{1}{4}(\phi F_{mn} T^{+mn} + \bar{\phi} F_{mn} T^{-mn}) - \frac{1}{16}(\phi^2 T^+_{mn} T^{+mn} + \bar{\phi}^2 T^-_{mn} T^{-mn}).$$

The classical contribution

$$S_{\mathrm{cl}} = \frac{(a_0^E)^2}{4 g_{\mathrm{YM}}^2} \int_{M_4} \mathrm{vol} \left(\frac{dV}{1+V^2} \right)^2 , \qquad Z_{\mathrm{tree}}(a_0^E) = e^{-S_{\mathrm{cl}}}$$

Partition function — one-loop determinant

$$Z_{\text{1-loop}}(a_0^E) = \frac{\det_{\text{fermions}} \delta^2}{\det_{\text{bosons}} \delta^2} = \sqrt{\frac{\det_L \Omega^{2,0} \det_L \Omega^{0,0}}{\det_L \Omega^{1,0}}} \sqrt{\frac{\det_L \Omega^{0,2} \det_L \Omega^{0,0}}{\det_L \Omega^{0,1}}} \frac{1}{\det_L H^0}.$$

Modes are related by Dolbeault operator.

$$\ldots \xrightarrow{\bar\partial} \Omega^{0,q-1} \xrightarrow{\bar\partial} \Omega^{0,q} \xrightarrow{\bar\partial} \Omega^{0,q+1} \xrightarrow{\bar\partial} \ldots .$$

1-loop contribution evaluates Dolbeault cohomologies.

$$Z_{\text{1-loop}}(a_0^E) = \sqrt{\frac{\det_L H^{0,2} \det_L H^{0,0}}{\det_L H^{0,1}}} \sqrt{\frac{\det_L H^{2,0} \det_L H^{0,0}}{\det_L H^{1,0}}} = 1.$$

This is essentially the equivariant index of the torus action.

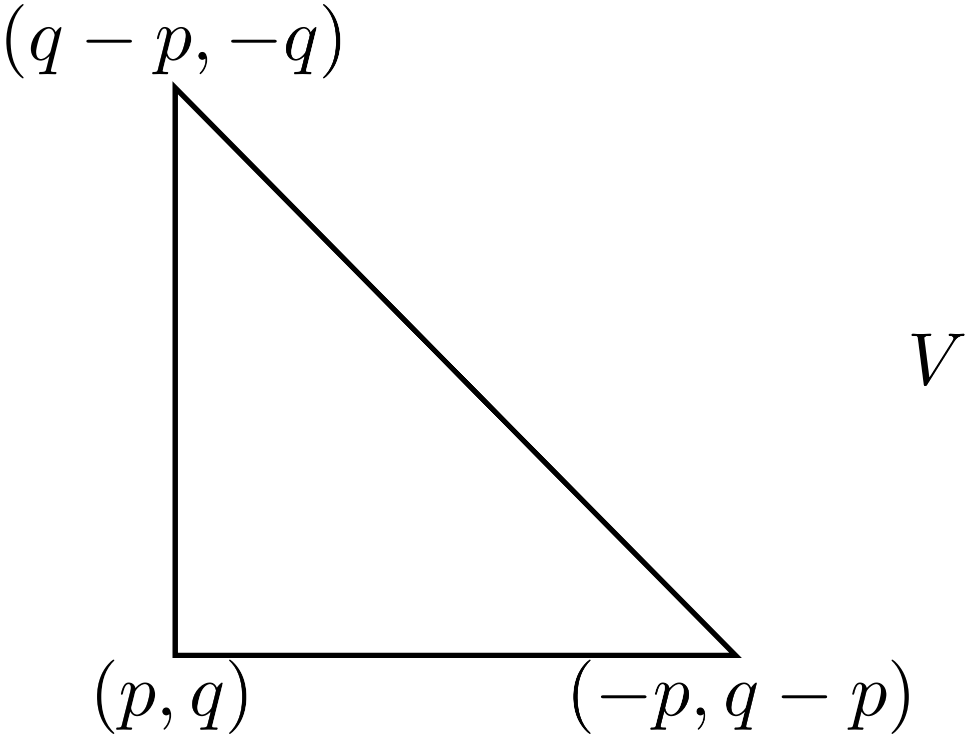
Note: It’s essential that the isometry commutes with the complex structure.

Partition function — instanton contribution

Locally near e.g. (0,0) vertex

$$\epsilon^i_+ = -\imath p(Y_1\gamma_1 - X_1\gamma_2)\epsilon^i_- - \imath q(Y_2\gamma_3 - X_2\gamma_4)\epsilon^i_-.$$

Omega background near vertices of Delzant polytope.
Contribute Nekrasov partition function with suitable equivariant parameters.



$$V = p\partial_{y_1} + q\partial_{y_2}$$

Partition function - perturbative action - CP^2

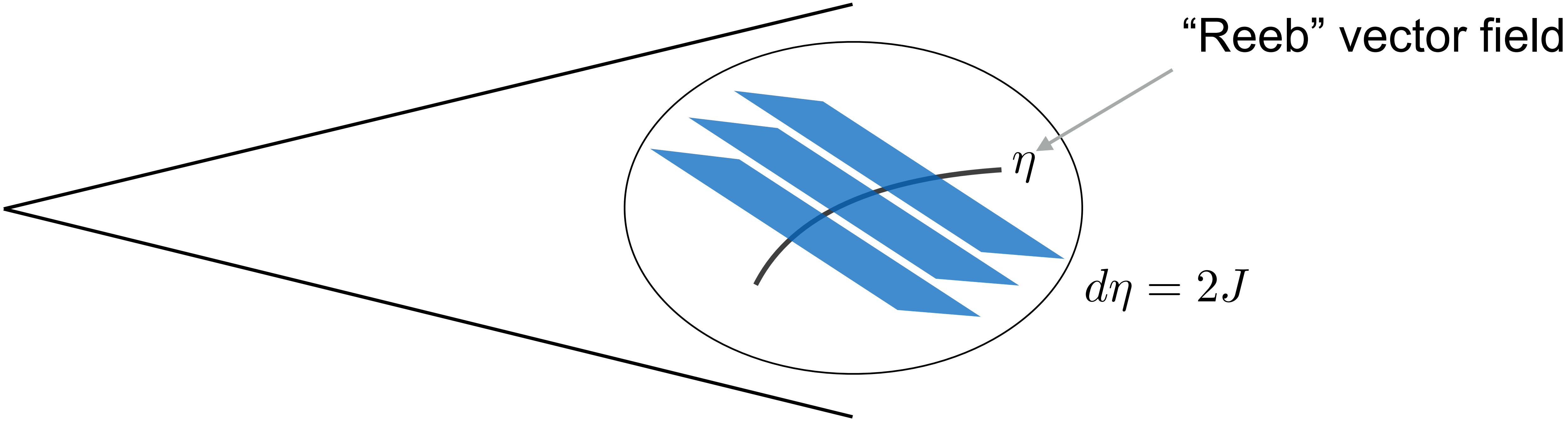
$$\left(\frac{dV}{1+V^2}\right)^2 = 8\frac{p^2(5x_1^2-4x_1+1)+q^2(5x_2^2-4x_2+1)+2pq(5x_1x_2-x_1-x_2)}{[2p^2(x_1^2-x_1)+2q^2(x_2^2-x_2)+4pqx_1x_2-1]^2}.$$

$$\begin{aligned} S_{\mathrm{cl}}^{\mathbb{CP}^2} =& \frac{4(a_0^E)^2\pi^2}{g_{\mathrm{YM}}^2}\frac{1}{pq(p-q)\sqrt{(p^2+2)(q^2+2)[(p-q)^2+2]}} \\ & \Bigg\{ -\sqrt{(q^2+2)[(p-q)^2+2]}(5p^2-2pq+2q^2+9)\mathrm{arctanh}\ \frac{p}{\sqrt{p^2+2}} \\ & +\sqrt{(p^2+2)[(p-q)^2+2]}(5q^2-2pq+2p^2+9)\mathrm{arctanh}\ \frac{q}{\sqrt{q^2+2}} \\ & +\sqrt{(p^2+2)(q^2+2)}(5p^2-8pq+5q^2+9)\mathrm{arctanh}\ \frac{p-q}{\sqrt{(p-q)^2+2}}\Bigg\}. \end{aligned}$$

3. Super Yang-Mills theories on Sasakian manifolds

The perturbative partition function counts holomorphic functions on the Calabi-Yau cone.

Sasakian geometry



Decomposition of the tangent bundle

$$J^2 = -1 + \eta \otimes \eta,$$
$$T_{\mathbb{C}}Y = T^{1,0}Y \oplus T^{0,1}Y \oplus \mathbb{C}\eta$$

Gauntlett, Martelli, Sparks, Waldram; Boyer, Galicki

Kohn-Rossi cohomology

Integrability

$$[T^{1,0}Y, T^{1,0}Y] \subseteq T^{1,0}Y \qquad \Rightarrow \qquad d = \partial_b + \bar{\partial}_b + \eta \wedge \pounds_\eta$$

Kohn-Rossi cohomology

$$\dots \xrightarrow{\bar{\partial}_b} \Omega^{p,q-1} \xrightarrow{\bar{\partial}_b} \Omega^{p,q} \xrightarrow{\bar{\partial}_b} \Omega^{p,q+1} \xrightarrow{\bar{\partial}_b} \dots \qquad H^{p,q}_{\bar{\partial}_b}(Y)$$

- Cohomology groups can often be calculated on the CY cone.

$$H^{0,0}_{\bar{\partial}_b}(Y) \cong H^0(\mathcal{O}_{C(Y)})$$

N=1 super Yang-Mills on Sasaki-Einstein manifolds

Very similar to d=4, N=2 theory.

- One scalar less, one dimension more!
- Complex reduces to the “topological” (=Witten) complex.

Localisation locus

Contact instantons $$(1 + \imath_\eta \star)F = 0, \qquad \imath_\eta F = 0.$$

Källen, Qiu, Zabzine

The partition function

$$Z = \int_{\imath \mathfrak{t}} dx Z_{\text{class}}(x)\, \text{sdet}'(-\imath \pounds_\eta + G_x) Z_{\text{inst.}}(x)$$

Perturbative partition function

Evaluated in terms of holomorphic functions on Calabi-Yau cone.

$$\text{sdet}'\, L = \left(\det{}'_{H^{0,0}_{\bar\partial_b}}(L) \det{}_{H^{0,0}_{\partial_b}}(L+3) \det{}'_{H^{0,0}_{\partial_b}}(L) \det{}_{H^{0,0}_{\bar\partial_b}}(L-3) \right)^{\frac{1}{2}}$$

J.S.

Toric Sasaki-Einstein (conjecture):
Contact instantons at closed orbits of torus action.

Qiu, Zabzine

4. The geometry of five dimensional theories

CR structures and THFs in d=5 can be obstructed.

with Yiwen Pan

d=5, *N*=1 Supergravity

Weyl multiplet (- a scalar)

SU(2)$_R$ triplet   SU(2)$_R$ R-symmetry   U(1)   two-form

$$D_m \xi_I = t_I{}^J \gamma_m \xi_J + A_{mI}{}^J \xi_J + \mathcal{F}_{mn} \gamma^n \xi_I + \frac{1}{2} \gamma_{mpq} \mathcal{V}^{pq} \xi_I$$

Kugo, Ohashi; Zucker

KSEs have been solved locally.

Imamura, Matsuno

Algebraic properties

Spinor bilinears

$$\eta_m = \xi_I \gamma_m \xi^I, \qquad \Theta_{IJmn} = \xi_I \gamma_{mn} \xi_J.$$

su(2) isomorphism

$$SO(5) \to SO(4) \cong SU(2)_+ \times SU(2)_- \xleftrightarrow{\Theta_{mnI}{}^{J}} SU(2)_{\mathcal{R}}$$

Family of almost contact metric structures

$$\imath_\eta \Theta_{IJ} = 0, \quad \left[\frac{m^{IJ}\Theta_{IJ}}{\sqrt{\det m}}\right]^2 = -1 + \eta \otimes \eta, \quad T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M \oplus \mathbb{C}\eta.$$

Candidate geometries

Transversally holomorphic foliation (THF)

$$[T^{1,0}M \oplus \mathbb{R}\eta, T^{1,0}M \oplus \mathbb{R}\eta] \subseteq T^{1,0}M \oplus \mathbb{R}\eta$$

d=3: Closset, Dumistrescu, Festuccia, Komargodski

Compare

$$S^1 \times S^2 \text{ vs. } S^1 \times S^4$$

Integrable Cauchy-Riemann structure (CR)

$$[T^{1,0}M, T^{1,0}M] \subseteq T^{1,0}M$$

Hypo structure

Conti, Salamon

Contact geometry

Qiu, Zabzine

Integrability conditions

$$X \in T^{1,0}M \Leftrightarrow X^m({m_I}^J - \imath {\delta_I}^J \sqrt{\det m})\gamma_m \xi_J = 0.$$

(This requires reality conditions)

Necessary conditions

$$m_{IJ} = t_{IJ}, \qquad {\Pi_m}^n D_n {t_I}^J = 0.$$

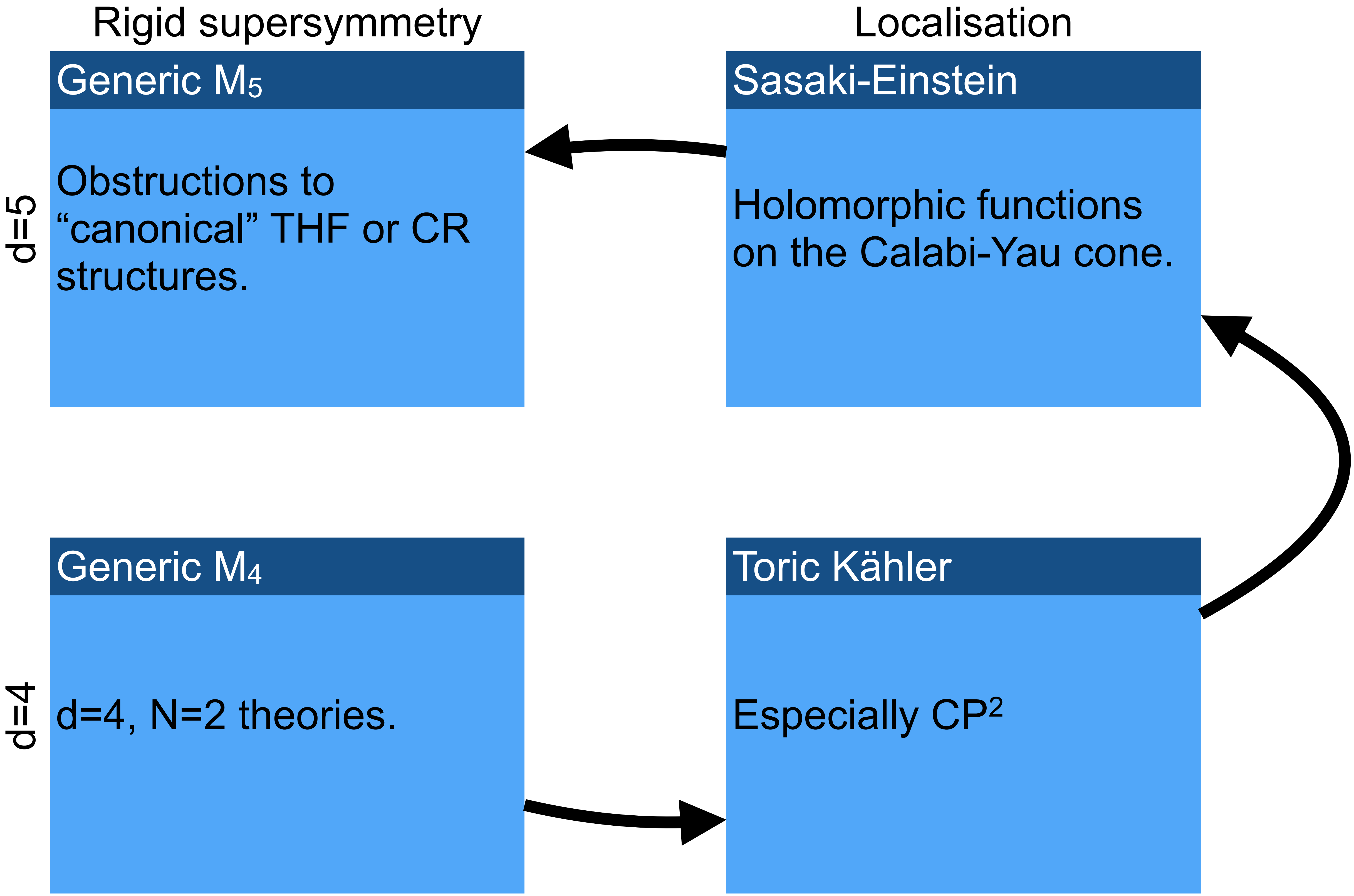
Additional conditions for sufficiency

$$(\mathcal{F} + \mathcal{V})^{2,0} = 0 \qquad \text{CR}$$

$$(\mathcal{F} + 2\mathcal{V})^{2,0} = 0 \quad (\Leftrightarrow D_R t_{IJ} = 0) \qquad \text{THF}$$

Yiwen Pan, J.S.

Summary



Rigid supersymmetry

Localisation

d=5

Generic M_5

Obstructions to "canonical" THF or CR structures.

Sasaki-Einstein

Holomorphic functions on the Calabi-Yau cone.

d=4

Generic M_4

d=4, N=2 theories.

Toric Kähler

Especially CP^2

Directions for future work

- d=4:
  - Factorisation.
  - Edges of polytope.
  - N=1.
- d=5:
  - Analysis of hypo structures; general classification.
  - Examples.
  - Contact instantons.
- Supersymmetry, odd-dim. holomorphy, and holography.

ありがとうございます。

Additional Material

Killing spinor equations

$$\begin{aligned}
\delta\psi^i_{+m} &= \nabla_m\epsilon^i_+ + \frac{\imath}{2}A_{mx}\sigma^{xi}{}_j\epsilon^j_+ + \frac{1}{2}A_{m4}\epsilon^i_+ + \frac{\imath}{4}T^+_{mn}\gamma^n\epsilon^i_- - \gamma_m\eta^i_-,\\
\delta\psi^i_{-m} &= \nabla_m\epsilon^i_- + \frac{\imath}{2}A_{mx}\sigma^{xi}{}_j\epsilon^j_- - \frac{1}{2}A_{m4}\epsilon^i_- + \frac{\imath}{4}T^-_{mn}\gamma^n\epsilon^i_+ - \gamma_m\eta^i_+,\\
\delta\chi^i_+ &= \frac{\imath}{6}(\nabla^m + A^{m4})T^+_{mn}\gamma^n\epsilon^i_- - \frac{1}{6}dA_4\cdot\gamma\epsilon^i_+ + \frac{D}{3}\epsilon^i_+ + \frac{\imath}{12}\gamma\cdot T^+\eta^i_+\\
&\quad + \frac{\imath}{6}\left(\partial_{[m}A^x_{n]} + \frac{1}{2}A^y_m A^z_n\epsilon^{yzx}\right)\gamma^{mn}\sigma^{xi}{}_j\epsilon^j_+,\\
\delta\chi^i_- &= \frac{\imath}{6}(\nabla^m - A^{m4})T^-_{mn}\gamma^n\epsilon^i_+ + \frac{1}{6}dA_4\cdot\gamma\epsilon^i_- + \frac{D}{3}\epsilon^i_- + \frac{\imath}{12}\gamma\cdot T^-\eta^i_-\\
&\quad + \frac{\imath}{6}\left(\partial_{[m}A^x_{n]} + \frac{1}{2}A^y_m A^z_n\epsilon^{yzx}\right)\gamma^{mn}\sigma^{xi}{}_j\epsilon^j_-.
\end{aligned}$$

SUSY variations

$$
\begin{aligned}
\delta\phi^I &= -\frac{\imath}{2}\epsilon^i_+ B\Omega^I_{i+}, \\
\delta\bar{\phi}^I &= \frac{\imath}{2}\epsilon_{-i} B\Omega^{Ii}_-, \\
\delta\mathcal{A}^I_m &= \frac{1}{2}\epsilon^{ij}\epsilon_{i-}B\gamma_m\Omega^I_{j+} + \frac{1}{2}\epsilon_{ij}\epsilon^i_+ B\gamma_m\Omega^{Ij}_-, \\
\delta\Omega^{Ii}_+ &= \imath\gamma^a D_a\phi^I\epsilon^i_- - \frac{1}{4}\gamma^{ab}\left(F^{I+}_{ab} - \frac{1}{2}\bar{\phi}^I T^+_{ab}\right)\epsilon^i_+ + \frac{1}{2}Y^{Ii}{}_j\epsilon^j_+ - g\phi^J\bar{\phi}^K f_{JK}{}^I\epsilon^i_+ \\
&\quad + 2\imath\phi^I\eta^i_+ - g\alpha^J\Omega^{Ki}_+ f_{JK}{}^I, \\
\delta\Omega^{Ii}_- &= -\imath\gamma^a D_a\bar{\phi}^I\epsilon^i_+ + \frac{1}{4}\gamma^{ab}\left(F^{I-}_{ab} - \frac{1}{2}\phi^I T^-_{ab}\right)\epsilon^i_- - \frac{1}{2}Y^{Ii}{}_j\epsilon^j_- - g\phi^J\bar{\phi}^K f_{JK}{}^I\epsilon^i_- \\
&\quad - 2\imath\bar{\phi}^I\eta^i_- - g\alpha^J\Omega^{Ki}_- f_{JK}{}^I, \\
\delta Y^I_{ij} &= \epsilon_{(i-}B\gamma^a D_a\Omega^I_{j)+} + \epsilon_{ik}\epsilon_{jl}\epsilon^{(k}_+ B\gamma^a D_a\Omega^{l)I}_- + 2\imath g\epsilon_{k(i}\left(\epsilon_{j)-}B\phi^J\Omega^{kK}_- + \epsilon^k_+ B\bar{\phi}^J\Omega^K_{j)+}\right) f_{JK}{}^I.
\end{aligned}
$$

Localisation locus

$$S_{\rm loc} = \int \mathrm{Tr}(\, \delta\bar{\Omega}^i_- \, \delta\Omega^i_- + \delta\bar{\Omega}^i_+ \, \delta\Omega^i_+ \,),$$

$$\delta\bar{\Omega}^i_- \, \delta\Omega^i_- + \delta\bar{\Omega}^i_+ \, \delta\Omega^i_+ = \aleph\,(1+V^{-2})\,|\imath_V\, D\phi|^2 + \aleph\,(1+V^2)\,[\phi,\,\bar{\phi}]^2 + \frac{\aleph}{4}\,|\mathcal{M}_-\,|^2 + \frac{\aleph\,V^2}{4}\,|\mathcal{M}_+\,|^2$$

$$\mathcal{M}_+^{mn} = (F^{mn})^+ - \frac{\bar{\phi}}{2}\,(T^+)^{mn} + \frac{i}{\aleph\,V^2}\,D_a\bar{\phi}\,\epsilon_+^{\bar{i}}\,\gamma^{mn}\,\gamma^a\,\epsilon_-^i + \frac{1}{2\,\aleph\,V^2}\,Y^i{}_j\,\epsilon_+^{\bar{i}}\,\gamma^{mn}\,\epsilon_+^j,$$

$$\mathcal{M}_-^{mn} = (F^{mn})^- - \frac{i}{\aleph}\,D_a\bar{\phi}\,\epsilon_-^{\bar{i}}\,\gamma^{mn}\,\gamma^a\,\epsilon_+^i - \frac{1}{2\,\aleph}\,Y^i{}_j\,\epsilon_-^{\bar{i}}\,\gamma^{mn}\,\epsilon_-^j + \frac{i}{2\,\aleph}\,\bar{\phi}\,\epsilon_-^{\bar{i}}\,\gamma^{mn}\,\gamma^k\nabla_k\epsilon_+^i.$$

Signs of Ghosts

$$
\begin{aligned}
V_{\text{g.f.}} &= \left(\tilde{c}, \imath d^{\dagger}\mathcal{A} + \imath b_0\right) \pm \left(c, \tilde{a}_0 - \frac{\xi_2}{2}a_0\right), \\
\delta V_{\text{g.f.}} &= \imath(b, d^{\dagger}\mathcal{A}) - \imath(\tilde{c}, d^{\dagger}\Psi) - \imath(\tilde{c}, d^{\dagger}d_{\mathcal{A}}c) \\
&\quad + \imath(b, b_0) - \imath(\tilde{c}, c_0) \mp (c, \tilde{c}_0) \pm \left(\rho, \tilde{a}_0 - \frac{\xi_2}{2}a_0\right),
\end{aligned}
$$

$$
\pm\left(\rho, \tilde{a}_0 - \frac{\xi_2}{2}a_0\right) = \mp\frac{\xi_2}{2}\left(a_0 - \sigma - \frac{g}{2}[c,c], a_0 - \frac{2}{\xi_2}\tilde{a}_0\right),
$$

$$
\frac{\xi_2}{2}\left(a_0^E + \imath\sigma + \frac{\imath g}{2}[c,c], a_0^E + \frac{2\imath}{\xi_2}\tilde{a}_0\right) \rightarrow \frac{1}{2\xi_2}\left[\tilde{a}_0 - \frac{\xi_2}{2}\left(\sigma + \frac{g}{2}[c,c]\right)\right]^2,
$$

$$
\delta c = a_0 - (\alpha_1 + \imath\alpha_2) - \frac{g}{2}[c,c].
$$

Previously we chose the upper signs. Were we to change this, we would have to wick rotate

$$
\tilde{a}_0 \rightarrow \imath\tilde{a}_0^E \Rightarrow \alpha_2 = 0.
$$

Holomorphy conditions

Niejenhuis tensor

- tedious, yet useful

Spinorial conditions

X∈T^{1,0} iff

- CR $$X^a {H_I}^J \gamma_a \xi_J = 0$$
- THF $$X^a {\Pi_a}^b {H_I}^J \gamma_b \xi_J = 0$$
- where $${H_I}^J = {t_I}^J + \imath \sqrt{\det t} {\delta_I}^J$$

- Note: $H_I{}^J$ has zero determinant - i.e. picks a single spinor.