Causal structure in Gauss-Bonnet Gravity

K. I. Phys. Rev. D *90*, *044037*

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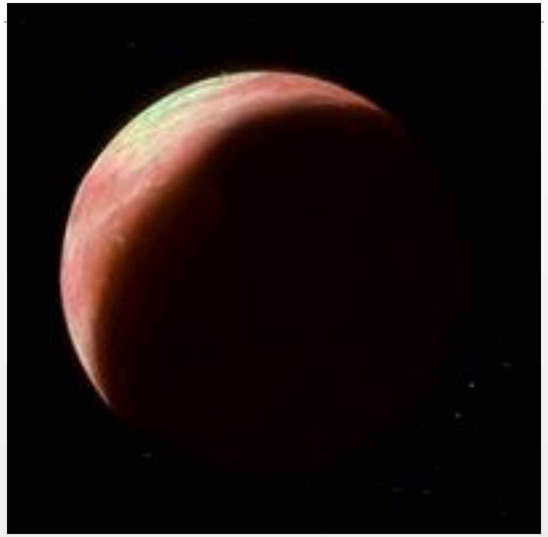
From Newton to Einstein

Newton' s gravity theory (1687)

Physics in solar system

Perihelion precession of Mercury

Dark Planet Vulcan??



Einstein' s general relativity (1916)

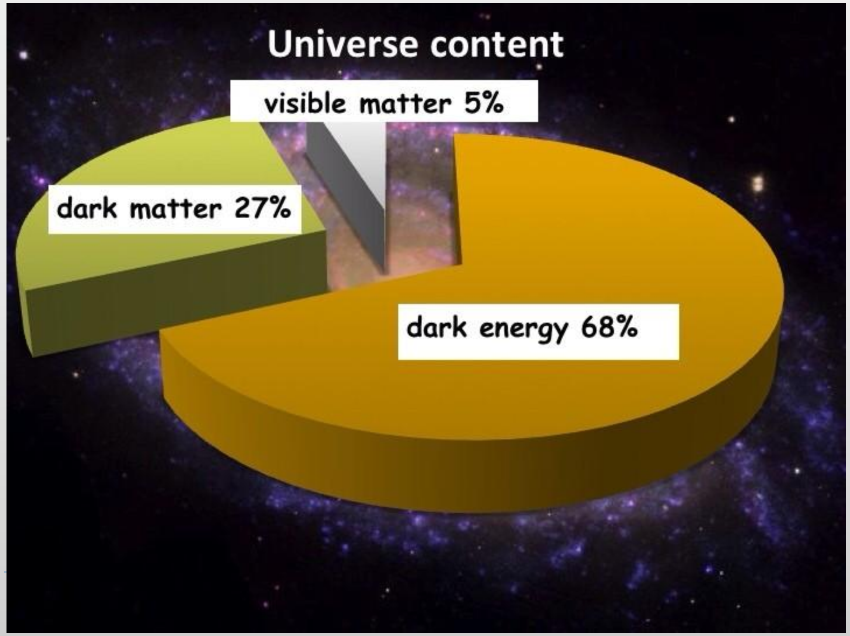
Cosmology

Cosmic acceleration
and galaxy rotation problem

Dark energy and dark matter??

?

New gravity theory?? (20??)



High Energy physics of Gravity

- General relativity

  Singularity theorem
  (Hawking, Penrose 1970)

  Singularity inside BH
  Initial singulari

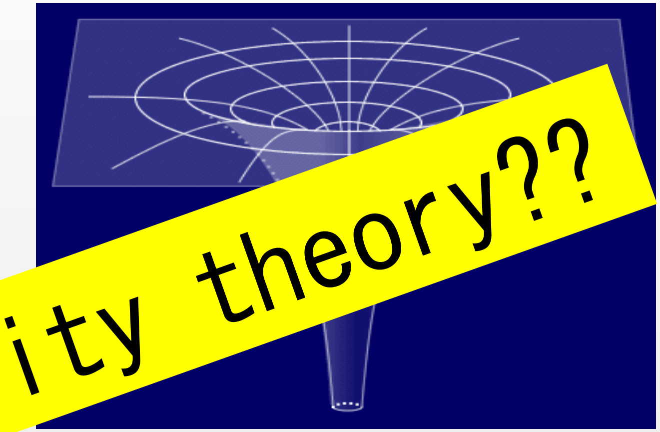
- Qu Gravity

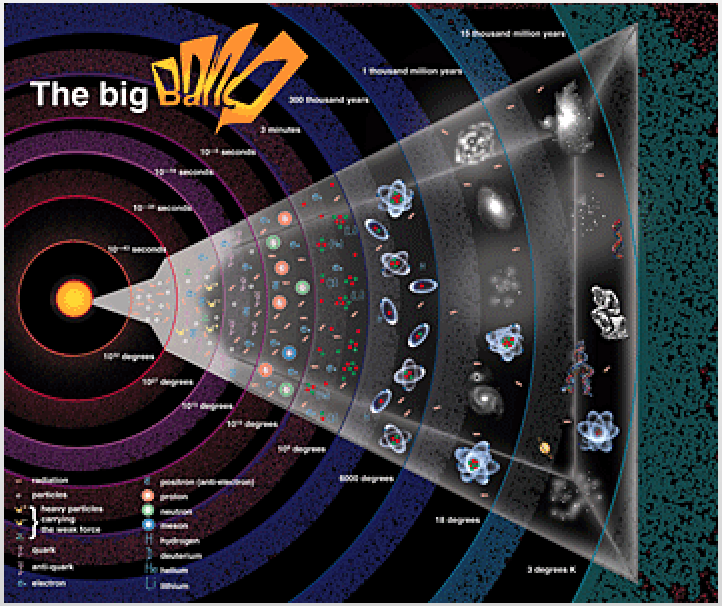
  blem of renormalization

  Algebraic structure of generators

  ⋮

Modification of gravity theory??





Modification of Gravity

- General relativity
  - Singularity problem
- Quantum gravity
  - Problem of renormalization
- Cosmology
  - Dark energy, Dark matter

Modified Gravity

- Modification of Lagrangian ： f(R), Gauss–Bonnet
- Modification of vacuum state ： ghost condensation
- Modification of concept of geometry
  - higher dimension ： Braneworld
  - other manifold ： Teleparallel gravity
- Introducing mass of graviton ： massive gravity

Consistency Check of modified gravity

0-th order （of cosmology） ： FLRW universe without perturbation

Consistency with standard cosmology
DM and DE??

1-th order ： perturbation on FLRW background

Consistency with standard cosmology
Consistency with solar system physics
Stability

Nonlinear property

Causal structure ← Main topic in this talk
Nonlinear stability

Quantization

⋮

Superluminal mode in GB

In GB gravity, superluminal gravitational waves potentially appear.

C. Aragone (1988)
Choquet-Bruhat (1988)

Superluminal modes

⇒ Causal relation based on null curve is meaningless.
We need to use the fastest propagation for analysis of causal structure.

Question

When do superluminal modes appear?
Are black holes well-defined?
Where is the horizon in the sense of causality?

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Example: Scalar field

- With **canonical** kinetic term

S. Mukohyama, J.-P. Uzan (2013)

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi + V'(\phi) = 0$$

High energy limit

$$g^{\mu\nu}k_\mu k_\nu\phi_k = 0$$

k_μ is null for metric $g_{\mu\nu}$

- With **non-canonical** kinetic term

$$(g^{\mu\nu} + \alpha\nabla^\mu\psi\nabla^\nu\psi)\nabla_\mu\nabla_\nu\phi + V'(\phi) = 0$$

High energy limit

$$\tilde{g}^{\mu\nu}k_\mu k_\nu\phi_k = 0$$

$$\tilde{g}^{\mu\nu} := g^{\mu\nu} + \alpha\nabla^\mu\psi\nabla^\nu\psi$$

k_μ is null for effective metric $\tilde{g}_{\mu\nu}$

Gravity theory

Many complications

h : graviton

- nonlinear kinetic terms $h^n \partial h \partial h$
  Nonlinear kinetic terms make the discussion non-trivial.

- constraints
  The structures of kinetic term depend on constraints..

- gauges
  Due to gauge, some variables are not physical.

General Relativity

The causal structure has been well studied,
and gravitational waves propagate with speed of light.

Extension of GR

Massive gravity, f(T) gravity

Additional constraint
No gauge

→ Superluminal modes

Y.C. Ong, K.I. J. M. Nester, P. Chen (2013)
K.I., J.-A. Gu, Y.C. Ong (2013)
K.I., Y.C. Ong (2013)
S. Deser, K.I., Y.C. Ong, A. Waldron (2013)

Gauss-Bonnet gravity

EOM :
$$G_{\mu\nu} + \{R * R\}_{\mu\nu} = 0$$

↓ linerlization

$$g^{\mu\nu}\nabla_\mu\nabla_\nu h_{\alpha\beta} + \{R\}^{\mu\nu}\nabla_\mu\nabla_\nu h_{\alpha\beta} = 0$$

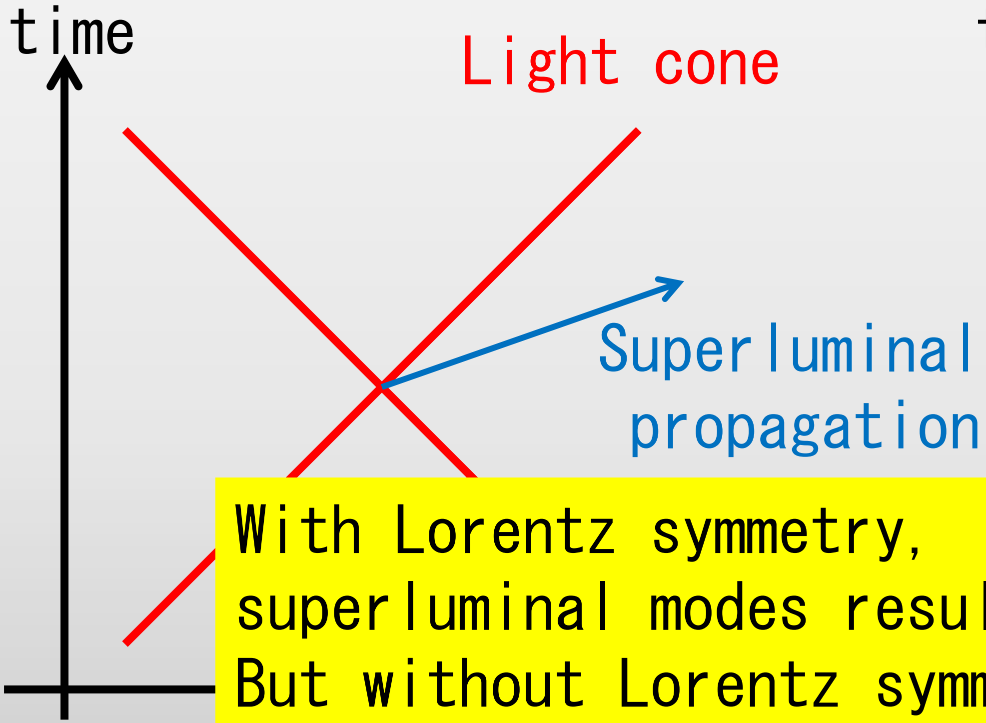
Kinetic terms are modified. → Superluminal modes

C. Aragone (1988)
Choquet-Bruhat (1988)

Superluminal mode and Acausality

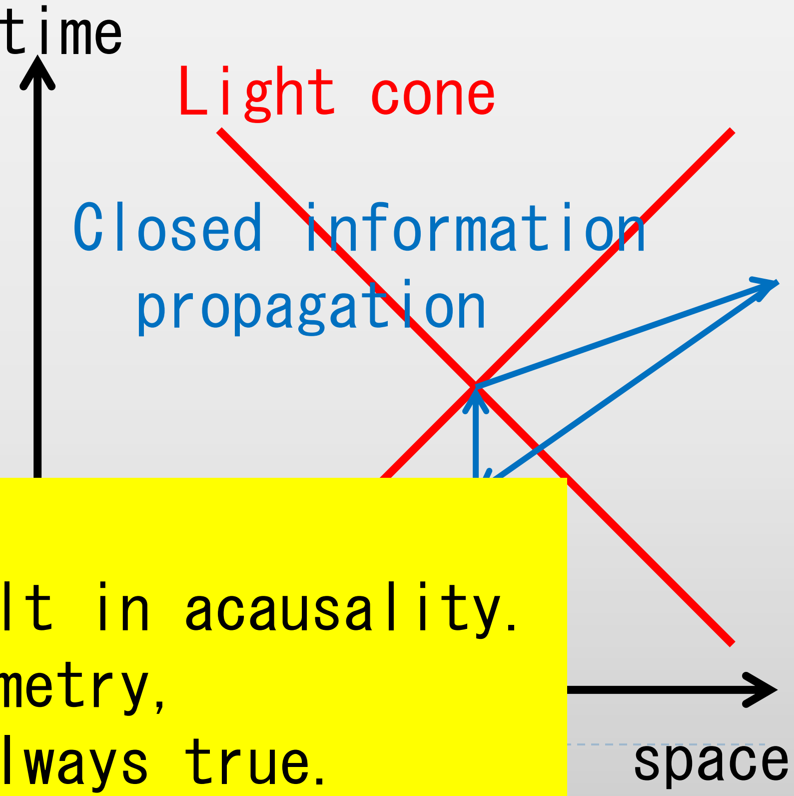
Superluminal mode

Propagation the speed of which is higher than that of light



Acausality

Pathological causal structure



With Lorentz symmetry,
superluminal modes result in acausality.
But without Lorentz symmetry,
it is not always true.

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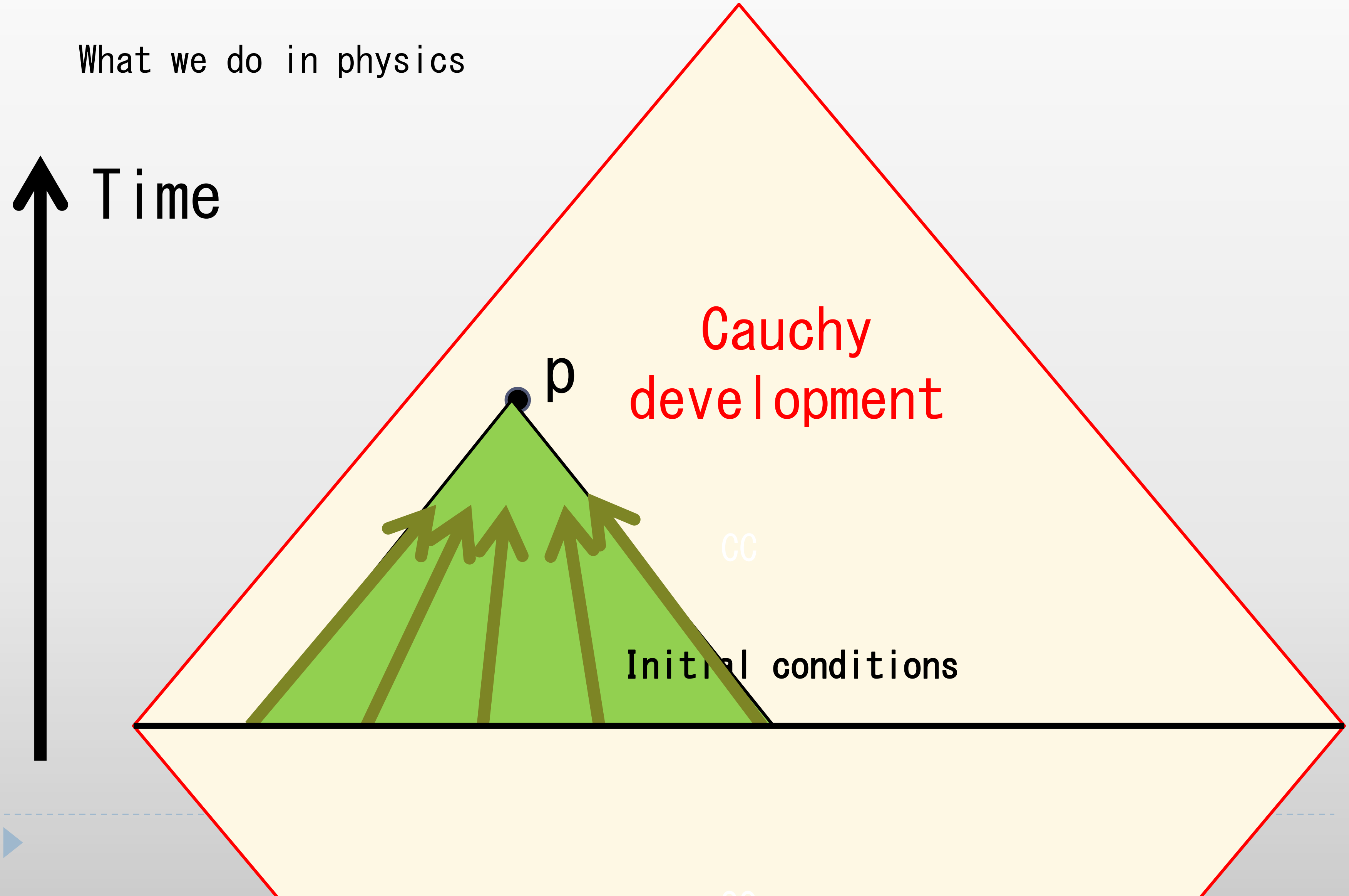
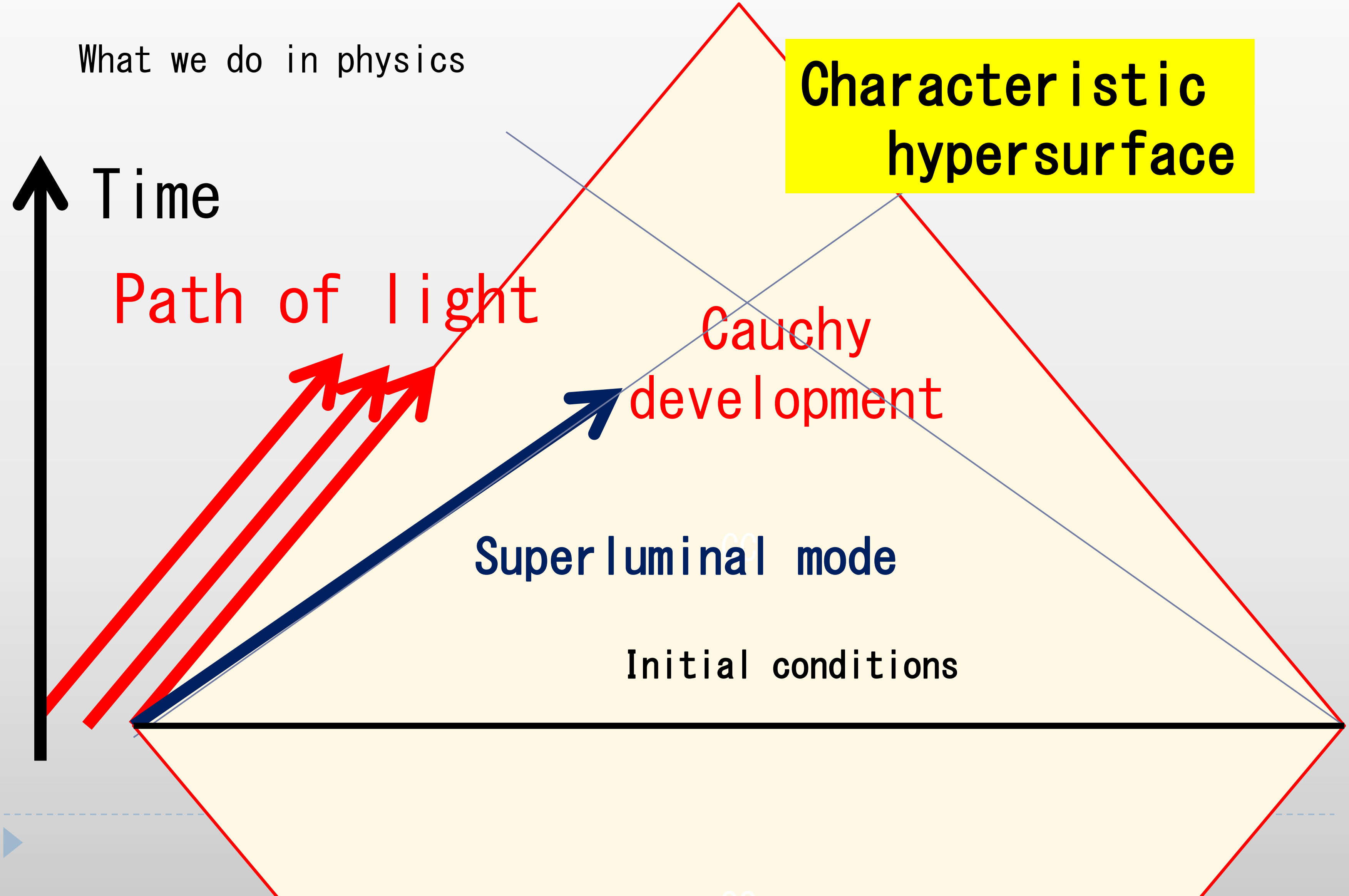
Causal structure and Propagation

What we do in physics



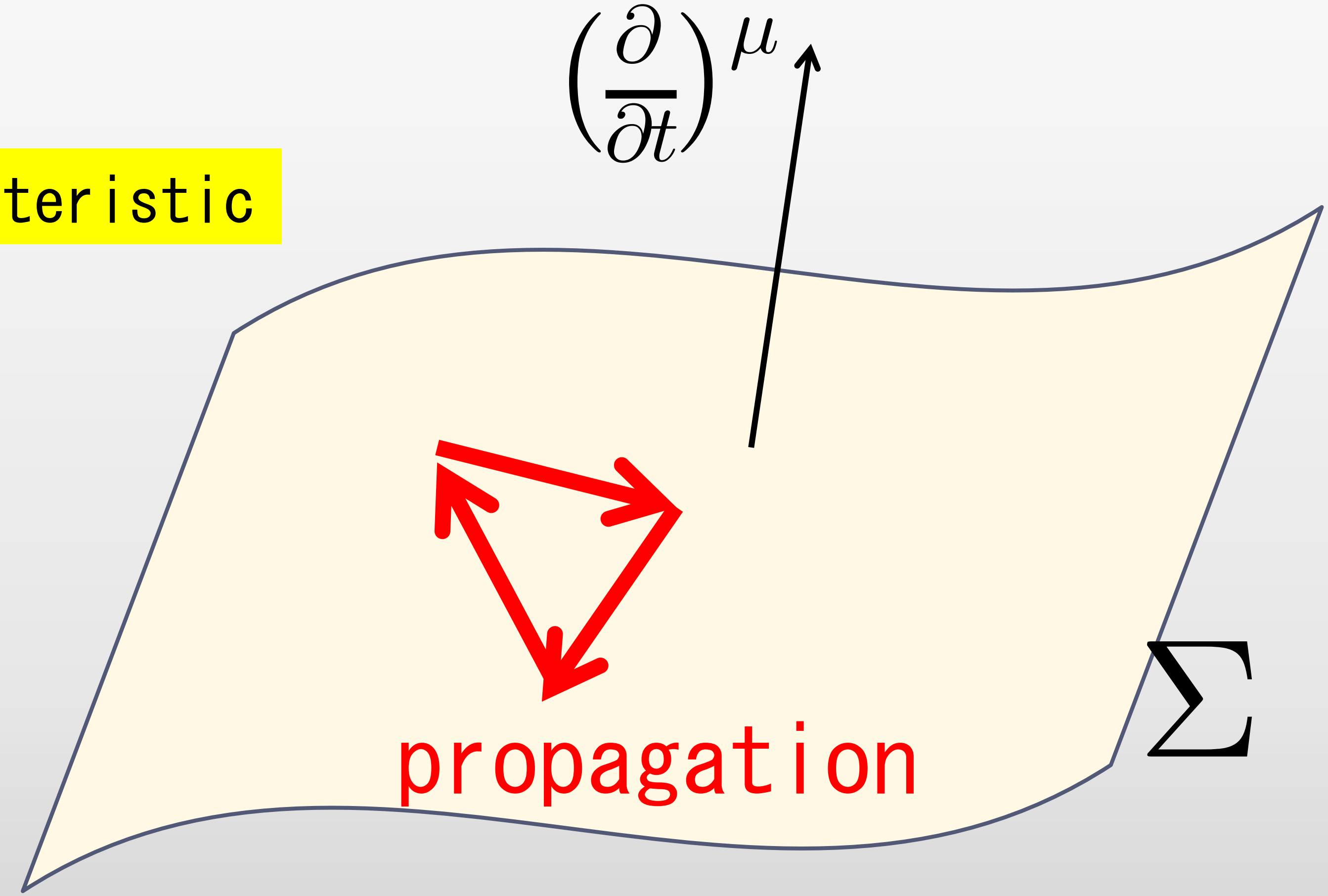
Causal structure and Propagation

What we do in physics



Characteristics

Σ is characteristic



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Characteristics (1Dim-ODE)

Quasi-linear n-th order differential equation

$$f[\phi, \partial_t\phi, \cdots, (\partial_t)^{n-1}\phi](\partial_t)^n\phi + g[\phi, \partial_t\phi, \cdots, (\partial_t)^{n-1}\phi] = 0$$

Time evolution

1, Initial condition; $\phi, \partial_t\phi, \cdots, (\partial_t)^{n-1}\phi$ at $t = t_0$

If $f[\phi, \partial_t\phi, \cdots, (\partial_t)^{n-1}\phi] \neq 0$

2, EoM ⇨ $(\partial_t)^n\phi|_{t=t_0}$

($(\partial_t)^n\phi|_{t=t_0}$ is uniquely fixed)

3, $(\partial_t)^{k-1}\phi[t = t_0 + \Delta t] = (\partial_t)^k\phi[t = t_0]\Delta t + (\partial_t)^{k-1}\phi[t = t_0]$

$(1 \leq k \leq n)$

Characteristics (1Dim-ODE)

Quasi-linear …h order differential equation

$f[\phi, \partial_t\phi,$ … $[\phi, \partial_t\phi, \cdots, (\partial_t)^{n-1}\phi] = 0$

Characteristics

Time evolution

1, Initial condition; $\phi, \partial_t\phi, \cdots, ($ … $= t_0$

If $f[\phi, \partial_t\phi, \cdots, (\partial_t)^{n-1}\phi] = 0$

2, EoM $\cancel{\Rightarrow}$ $(\partial_t)^n\phi|_{t=t_0}$

($(\partial_t)^n\phi|_{t=t_0}$ can be arbitrary)

3, $(\partial_t)^{k-1}\phi[t = t_0 + \Delta t] = (\partial_t)^k\phi[t = t_0]\Delta t + (\partial_t)^{k-1}\phi[t = t_0]$

$(1 \le k \le n-1)$

$(\partial_t)^{n-1}\phi[t = t_0 + \Delta t] = ??$

Characteristics (Single field PDE)

Quasi-linear n-th order differential equation

$$f^{\nu_1\cdots\nu_n}[\phi,\partial_\mu\phi,\cdots,\partial_{\mu1}\cdots\partial_{\mu_{n-1}}\phi]\partial_{\nu_1}\cdots\partial_{\nu_n}\phi+g[\phi,\partial_\mu\phi,\cdots,\partial_{\mu1}\cdots\partial_{\mu_{n-1}}\phi]=0$$

Time evolution

1, Initial condition on initial hypersurface Σ

$$\phi,\partial_t\phi,\cdots,(\partial_t)^{n-1}\phi \qquad \Sigma : t=t_0$$

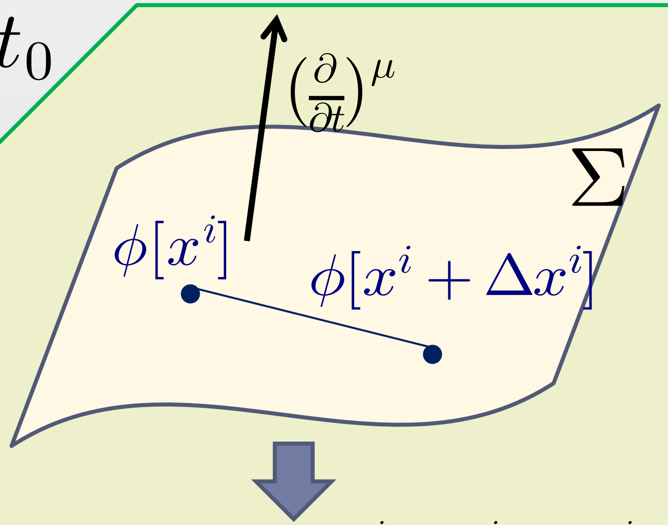
2, $\partial_i\phi,\partial_t\partial_i\phi,\partial_i\partial_j\phi,\cdots$

If $f^{t\cdots t}[\phi,\partial_\mu\phi,\cdots,\partial_{\mu_1}\cdots\partial_{\mu_{n-1}}\phi]\neq 0$

3, EoM ⇒ $(\partial_t)^n\phi|_{t=t_0}$

($(\partial_t)^n\phi|_{t=t_0}$ is uniquely fixed)

Time evolution is unique



$$\partial_i\phi=\lim_{\Delta x^i\to 0}\frac{\phi[x^i+\Delta x^i]-\phi[x^i]}{\Delta x^i}$$

Characteristics (Single field PDE)

Quasi-linear n-th order differential equation

$$f^{\nu_1\cdots\nu_n}[\phi,\partial_\mu\phi,\cdots,\partial_{\mu_1}\cdots\partial_{\mu_{n-1}}\phi]\partial_{\nu_1}\cdots\partial_{\nu_n}\phi+g[\phi,\partial_\mu\phi,\cdots,\partial_{\mu_1}\cdots\partial_{\mu_{n-1}}\phi]=0$$

Time ev

Characteristics

1, Initial

hypersurface Σ

$\phi,\partial_t\phi,\cdots,(\partial_t)$

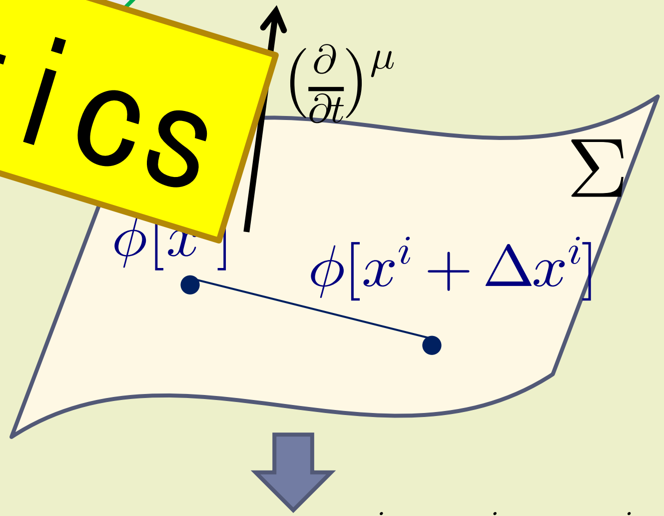
2, $\partial_i\phi,\partial_t\partial_i\phi,\partial_i\partial_j\phi,\cdots$

If $\boxed{f^{t\cdots t}[\phi,\partial_\mu\phi,\cdots,\partial_{\mu_1}\cdots\partial_{\mu_{n-1}}\phi]=0}$

3, EoM $\cancel{\Rightarrow}$ $(\partial_t)^n\phi|_{t=t_0}$

($(\partial_t)^n\phi|_{t=t_0}$ can be arbitrary)

Time evolution is not unique



$$\partial_i\phi=\lim_{\Delta x^i\to 0}\frac{\phi[x^i+\Delta x^i]-\phi[x^i]}{\Delta x^i}$$

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Gauss-Bonnet gravity

action

$$S = \int d^n x \Big[\frac{1}{2\kappa^{D-2}}\{R - 2\Lambda + \alpha(\underbrace{R^2 - 4R_{AB}R^{AB} + R_{ABCD}R^{ABCD}}_{\text{GB terms}})\} + \underbrace{L_m}_{\text{Matter contribution}}\Big]$$

EoM $$G_{AB} + \Lambda g_{AB} - \frac{\alpha}{2}H_{AB} = 2\kappa^{D-2}T_{AB}$$

$$H_{AB} := (R^2 - 4R_{CD}R^{CD} + R_{CDEF}R^{CDEF})g_{AB}$$
$$- 4(RR_{AB} - 2R_{AC}{R_B}^{C} - 2R_{ACBD}R^{CD} + R_{ACDE}{R_B}^{CDE})$$

Check the characteristic for EoM

Assumption : T_{AB} does not involve the highest order derivatives of metric

First-order formalism

EoM: Second order derivatives of metric

Introducing new variables: Levi-Civita connection

$$\Gamma_{ABC} = g_{AD}\Gamma^{D}_{BC} := \frac{1}{2}(\partial_C g_{AB} + \partial_B g_{AC} - \partial_A g_{BC}) + F[\Gamma_{EFG}, g_{EF}]$$

First order differential equation

$$R_{ABCD} = \partial_C \Gamma_{ABD} - \partial_D \Gamma_{ABC} + \cdots$$

# of variables

$$g_{AB} : \frac{(D+1)D}{2}$$

$$\Gamma_{CAB} : \frac{(D+1)D^2}{2}$$

Symmetric w.r.t A , B

Decomposition

A hypersurface Σ

$\frac{\partial}{\partial x^\mu}$ are vectors on Σ

$\frac{\partial}{\partial t}$ is independent vector from $\frac{\partial}{\partial x^\mu}$

$$\xi^A\left(\frac{\partial}{\partial x^A}\right) := \frac{\partial}{\partial t}$$

$$\zeta_A dx^A = dt$$

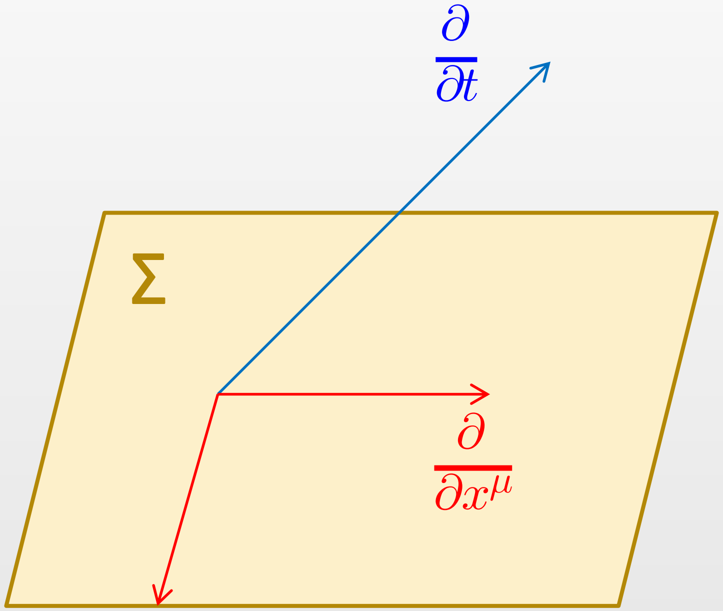
Projection operator

$$P^A_B = \delta^A_B - \xi^A \zeta_B$$

Decomposition of vector

$$V_A : V_0 := \xi^A V_A \quad , \quad V_\mu := P^A_\mu V_A$$

$$V^A : V^0 := \zeta_A V^A \quad , \quad V^\mu := P^\mu_A V^A$$



Trivial characteristics

From definition of Levi-Civita connection

$$2\Gamma_{000} = \partial_0 g_{00} \qquad 2\Gamma_{\alpha 00} = 2\partial_0 g_{0\alpha} - \partial_\alpha g_{00}$$

$$2\Gamma_{0\alpha\beta} = \partial_\alpha g_{0\beta} + \partial_\beta g_{0\alpha} - \partial_0 g_{\alpha\beta}$$

⇨ Characteristics for g_{AB}

For Levi-Civita connection

- $0 = R_{00\alpha 0} = \partial_0 \Gamma_{00\alpha} - \partial_\alpha \Gamma_{000} + \cdots$
- $0 = R_{\beta\gamma 0\alpha} - R_{0\alpha\beta\gamma}$

$$= \partial_0 \Gamma_{\beta\gamma\alpha} - \partial_\alpha \Gamma_{\beta\gamma 0} - \partial_\beta \Gamma_{0\alpha\gamma} + \partial_\gamma \Gamma_{0\alpha\beta} + \cdots$$

⇨ Characteristics for $\Gamma_{00\alpha}$, $\Gamma_{\alpha\beta\gamma}$

Using definition of Levi-Civita connection

$$\Gamma_{\alpha\beta 0} = -\Gamma_{0\alpha\beta} - \partial_\beta g_{\alpha 0}$$

⇨ Replace $\Gamma_{\alpha\beta 0}$ by $-\Gamma_{0\alpha\beta}$

Gauge modes

We have fixed time evolution of g_{AB} , $\Gamma_{00\alpha}$, $\Gamma_{\alpha\beta\gamma}$, $\Gamma_{\alpha\beta 0}$

Remaining variables are Γ_{000} , $\Gamma_{\alpha 00}$, $\Gamma_{0\alpha\beta}$

$$\begin{array}{ccccc} \Gamma_{000} & , & \Gamma_{\alpha 00} & , & \Gamma_{0\alpha\beta} \\ \uparrow & & \uparrow & & \uparrow \\ 1 & , & (D-1) & , & (D-1)D/2 \end{array}$$

DoF: 1 , $(D-1)$, $(D-1)D/2$

Diff. inv. : # of gauge DoF is D

$\partial_0\Gamma_{000}$, $\partial_0\Gamma_{\alpha 00}$ never appear in EoM. $\Rightarrow$ Gauge modes

$\Gamma_{0\alpha\beta}$ are physical.

Check the characteristics for $\Gamma_{0\alpha\beta}$!

General Relativity

$(\partial_0\Gamma_{\alpha\beta 0} = -\partial_0\Gamma_{0\alpha\beta} + \cdots)$

↓

$\partial_0\Gamma_{0\alpha\beta}$ appears only in $R_{0\alpha 0\beta}$ $(= -R_{\alpha 00\beta} = \cdots)$

$G^{AB} = R_{0\alpha 0\beta}A^{AB,\alpha\beta} + \cdots$ ⇒ $A^{AB,\alpha\beta}\partial_0\Gamma_{0\alpha\beta} + \cdots = 0$

$A^{00,\alpha\beta} = A^{0\mu,\alpha\beta} = A^{\mu 0,\alpha\beta} = 0$ $\quad A^{\mu\nu,\alpha\beta} = g^{00}(h^{\alpha\mu}h^{\beta\nu} - h^{\alpha\beta}h^{\mu\nu})$

$h^{\mu\nu} := g^{\mu\nu} - g^{0\mu}g^{0\nu}/g^{00}$ $\quad$ ($h_{\mu\nu}$ is induced metric of Σ)

Null hypersurface $\quad g^{00} = 0 \quad h_{\mu\nu} = \text{diag}\ (0, 1, 1, \cdots) \quad h^{11} = O[(g^{00})^{-1}]$

$-g^{00}h^{11}\sum_i \partial_0\Gamma_{0ii} + \cdots = 0$ $\quad (1,1)$ -component

$g^{00}h^{11}\partial_0\Gamma_{01i} + \cdots = 0$ $\quad (1,i)$ -component

$\boxed{-g^{00}h^{11}\partial_0\Gamma_{011}\delta_{ij} +} \cdots = 0$ $\quad (i,j)$ -component

1 :null direction

i :normal to 1

All component have the same kinetic term

$D(D-3)/2$ degeneracies ⇐ Gravitational wave

Gauss-Bonnet correction

$$G_{AB}+\Lambda g_{AB}-\frac{\alpha}{2}H_{AB}=2\kappa^{D-2}T_{AB}$$

$$H^{AB}=R_{0\alpha 0\beta}B^{AB,\alpha\beta}+\cdots$$

$$B^{00,\alpha\beta}=B^{0\mu,\alpha\beta}=B^{\mu 0,\alpha\beta}=0$$

$$\begin{aligned}B^{\mu\nu,\alpha\beta}=4g^{00}R_{\lambda\omega\gamma\delta}(&h^{\lambda\gamma}h^{\omega\delta}h^{\mu\nu}h^{\alpha\beta}-h^{\lambda\gamma}h^{\omega\delta}h^{\mu\alpha}h^{\nu\beta}+2h^{\lambda\mu}h^{\gamma\alpha}h^{\omega\delta}h^{\nu\beta}+2h^{\lambda\nu}h^{\gamma\alpha}h^{\omega\delta}h^{\mu\beta}\\&-2h^{\lambda\alpha}h^{\gamma\beta}h^{\omega\delta}h^{\mu\nu}-2h^{\lambda\mu}h^{\gamma\nu}h^{\omega\delta}h^{\alpha\beta}+2h^{\lambda\mu}h^{\omega\alpha}h^{\gamma\nu}h^{\delta\beta}).\end{aligned}$$

GR: $\underline{A^{AB,\alpha\beta}\partial_0\Gamma_{0\alpha\beta}+\cdots=0}$

Null hypersurface

$-g^{00}h^{11}\sum_i\partial_0\Gamma_{0ii}+\cdots=0$	$(1,1)$ -component
$g^{00}h^{11}\sum_i\partial_0\Gamma_{01i}+\cdots=0$	$(1,i)$ -component
$-g^{00}h^{11}\sum_i\partial_0\Gamma_{011}\delta_{ij}+\cdots=0$	(i,j) -component

$$\left(A^{AB,\alpha\beta}-\frac{\alpha}{2}B^{AB,\alpha\beta}\right)\partial_0\Gamma_{0\alpha\beta}+\cdots=0$$

Gauss-Bonnet correction

$$\bar{\Gamma}_{0\alpha\beta} \sim \partial_0 \Gamma_{0\alpha\beta}$$

$$-g^{00}h^{11}\left[\sum_i \bar{\Gamma}_{0ii} + 2\alpha\left(\sum_{i,k,l} R_{klkl}\bar{\Gamma}_{0ii} - 2\sum_{i,j,k} R_{ikjk}\bar{\Gamma}_{0ij}\right)\right] + \cdots = 0$$

$(1, 1)$ -component

$$g^{00}h^{11}\left[\bar{\Gamma}_{01i} + 2\alpha\left(\sum_{k,l} R_{klkl}\bar{\Gamma}_{01i} - 2\sum_{j,k} R_{ikjk}\bar{\Gamma}_{01j}\right)\right.$$

$$\left. + 8\alpha\sum_{j,k}\left(R_{1kik}\bar{\Gamma}_{0jj} - R_{1kjk}\bar{\Gamma}_{0ij} - R_{1jik}\bar{\Gamma}_{0jk}\right)\right] + \cdots = 0$$

$(1, i)$ -component

$$-g^{00}h^{11}\left[\delta_{ij}\bar{\Gamma}_{011} + 2\alpha\left(\sum_{k,l} R_{klkl}\delta_{ij} - 2R_{ikjk}\right)\bar{\Gamma}_{011} + \alpha\sum_k\left(R_{1ijk} + R_{1jik}\right)\bar{\Gamma}_{01k}\right.$$

$$+4\alpha\left\{\delta_{ij}\sum_{k,l}\left(R_{1k1k}\bar{\Gamma}_{0ll} - R_{1k1l}\bar{\Gamma}_{0kl}\right)\right.$$

$$\left.\left. + \sum_k\left(R_{1i1k}\bar{\Gamma}_{0kj} + R_{1j1k}\bar{\Gamma}_{0ki} - \boxed{R_{1k1k}\bar{\Gamma}_{0ij}} - R_{1i1j}\bar{\Gamma}_{0kk}\right)\right\}\right] + \cdots = 0$$

(i, j) -component

Example1: All degeneracies are resolved

$$R_{ijkl} = R_{1ijk} = 0$$

$$R_{1i1j} = C\delta_{ij}$$

$$-g^{00}h^{11}\sum_i \bar{\Gamma}_{0ii} + \cdots = 0$$ $(1,1)$ -component

$$g^{00}h^{11}\sum_i \bar{\Gamma}_{01i} + \cdots = 0$$ $(1,i)$ -component

$$-g^{00}h^{11}\left[\delta_{ij}\bar{\Gamma}_{011} + 4\alpha(D-4)C\left(\delta_{ij}\sum_k \bar{\Gamma}_{0kk} \boxed{-\bar{\Gamma}_{0ij}}\right)\right] + \cdots = 0$$

(i,j) -component

Characteristic hypersurface is not null.

The speed of graviton is not that of light.

Expample2: still degenerated

$$R_{1i1j} = R_{1ijk} = 0$$

$$-g^{00}h^{11}\left[\delta_{ij}\bar{\Gamma}_{011} + 2\alpha\left(\sum_{k,l} R_{klkl}\delta_{ij} - 2R_{ikjk}\right)\bar{\Gamma}_{011}\right] + \cdots = 0$$ (i, j) -component

Only $\bar{\Gamma}_{011}$ appears.
Characteristic hypersurface is still null.

Killing horizon

⇒ $\partial_1 g_{ij} = 0, \qquad \partial_1^2 g_{ij} = 0 \qquad \text{and} \qquad \partial_1\partial_k g_{ij} = 0,$

⇒ $R_{1i1j} = R_{1ijk} = 0$

Killing horizon is exactly causal edge.

Dynamical case

(spherically symmetric case)

$R_{AB}U^AU^B = 0$

U^A : Radial null vector

Null hypersurface tangent to U^A is characteristics.

$R_{AB}U^AU^B \neq 0$

Modification to characteristic equation

$$G_{AB} + \Lambda g_{AB} - \frac{\alpha}{2} H_{AB} = 2\kappa^{D-2} T_{AB}$$

balanced

Deviate from null

If curvature is enough small,
we can neglect the contribution from GB term

Dynamical case (spherically symmetric case)

$R_{AB}U^AU^B > 0$ Subluminal

$R_{AB}U^AU^B < 0$ Superluminal

H. Maeda, M. Nozawa (2008)

Einstein Branch

$$T_{AB}U^AU^B \geq 0 \Rightarrow R_{AB}U^AU^B \geq 0$$

GB Branch

$$T_{AB}U^AU^B \geq 0 \Rightarrow R_{AB}U^AU^B \leq 0$$

On Einstein Branch with Null Energy condition
(This might be physical condition)

⇒ No superluminal mode, but subluminal modes appear

⇒ No acausality, but gravitational Cherenkov happens

Summary

In GB gravity, gravitational propagation
potentially becomes superluminal.

Causal structure based on null curve is meaningless.
We need to analyze it by using the fastest propagation.

Causal structure is analyzed with method of characteristics,
which is old-fashioned but powerful technique.

Killing horizon is the event horizon in the sense of causality.

On spherically symmetric background,

$R_{AB}U^AU^B = 0$ luminal.

$R_{AB}U^AU^B > 0$ subluminal. ⇨ gravitational Cherenkov

$R_{AB}U^AU^B < 0$ superluminal. ⇨ Acausality? Need to check it.