Towards a holographic Bose–Hubbard model

Mitsutoshi Fujita (YITP, Kyoto U.)

Collaborators: S. Harrison, A. Karch, R. Meyer, and N. Paquette

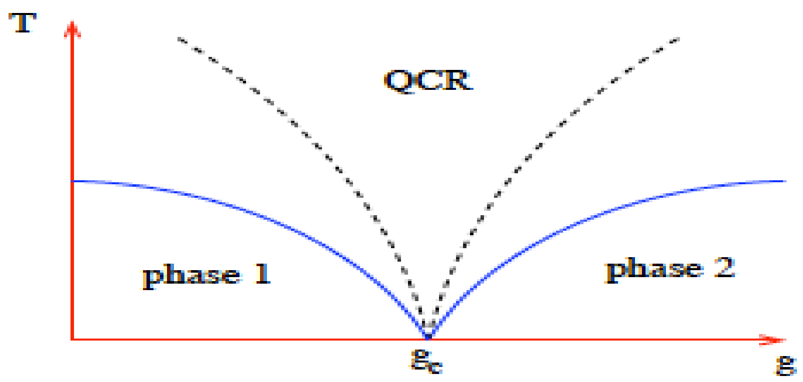
arXiv: 1411.7899 [hep-th]

Gauge/gravity correspondence applied for condensed matter physics

Realizing strong coupled many-body systems such as QGP

- String theory embedding (symmetry, stable, cft, etc.)

Realizing an analog of the particle-vortex duality, superfluid/insulator transitions, superconductivity, and the field theory with Lifshitz point



c.f. *Herzog-Kovtun-Sachdev-Son ``07*
Hartnoll-Herzog-Horowitz ''08

Typical phase structure of the quantum critical region (QCR) from *Herzog ``09*

CP^1 model: phase1=Higgs
phase 2= Coulomb

A motivation:
To introduce the holographic lattice

Many applications of holographic model have been assumed the translation symmetry

◆ Infinite DC conductivity unlike real materials

The translation invariance breaking system appeared recently by introducing the periodic function of the chemical potential.

$$\mu(x) = \bar{\mu}[1 + A_0 \cos(k_0 x)]$$

Horowitz, Santos, D. Tong '12
Horowitz, Santos, '13

◆ Formulation of the lattice:

Lattice spacing $2\pi / k_0$ The amplitude of the lattice A_0

A holographic lattice in the probe limit (review)

Kachru-Karch-Yaida ''09, ''10

D3	x	x	x	x			
D5	x				x		xxxx

◆ D3/D5 Introducing probe D5s in $AdS_5 x S^5$

Impurity in a lattice site	Probe branes on an AdS_2 slice

Finite temperature effect

Gravity dual: Entropy of D5 is like $\sqrt{\lambda}NT$

QFT at zero *T*: Statistical Entropy of the fermions ~*N*

Associated with spin degeneracy on sites

non-zero even in zero temperature

Large N Fermi/non−Fermi liquid transition

1st order phase transition :

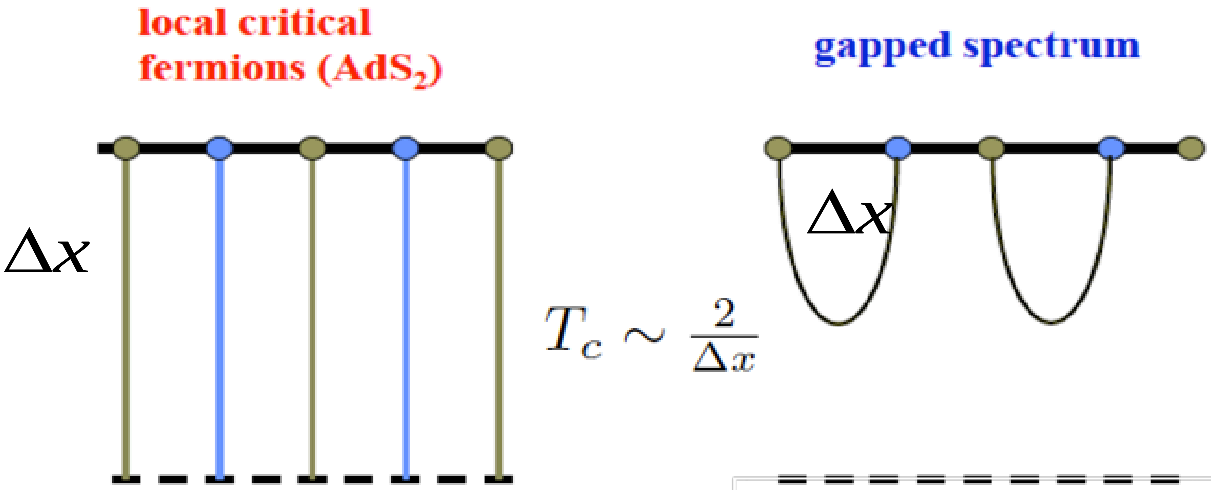


Figure taken from A. Karch's note

After mixing with itinerant fermions, this becomes

non−Fermi/Fermi liquid transition

Hopping effect?

Interesting features of the lattice formulation could be captured in the probe limit.

It is not known how to allow the charge transport of fermions on the probe brane.

◆ It is suppressed contribution in $1/N$ expansion

Applications: the flow of the quantum current on the lattice. c.f. valence bonds

Boson-Hubbard model

Boson-Hubbard model as the effective theory of the cold bosons includes the hopping term.

- Short-range repulsive interactions U on each a site

$$H = -w\sum_{<ij>}(b_i^\dagger b_j + b_j^\dagger b_i) - \mu\sum_j n_j + \frac{U}{2}\sum_j n_j(n_j - 1),$$

hopping term

w : the ratio of the transition

$$n_j = b_j^\dagger b_j$$

$U(1)$ symmetry: $b_i \to e^{i\theta} b_i$

Only two phases:

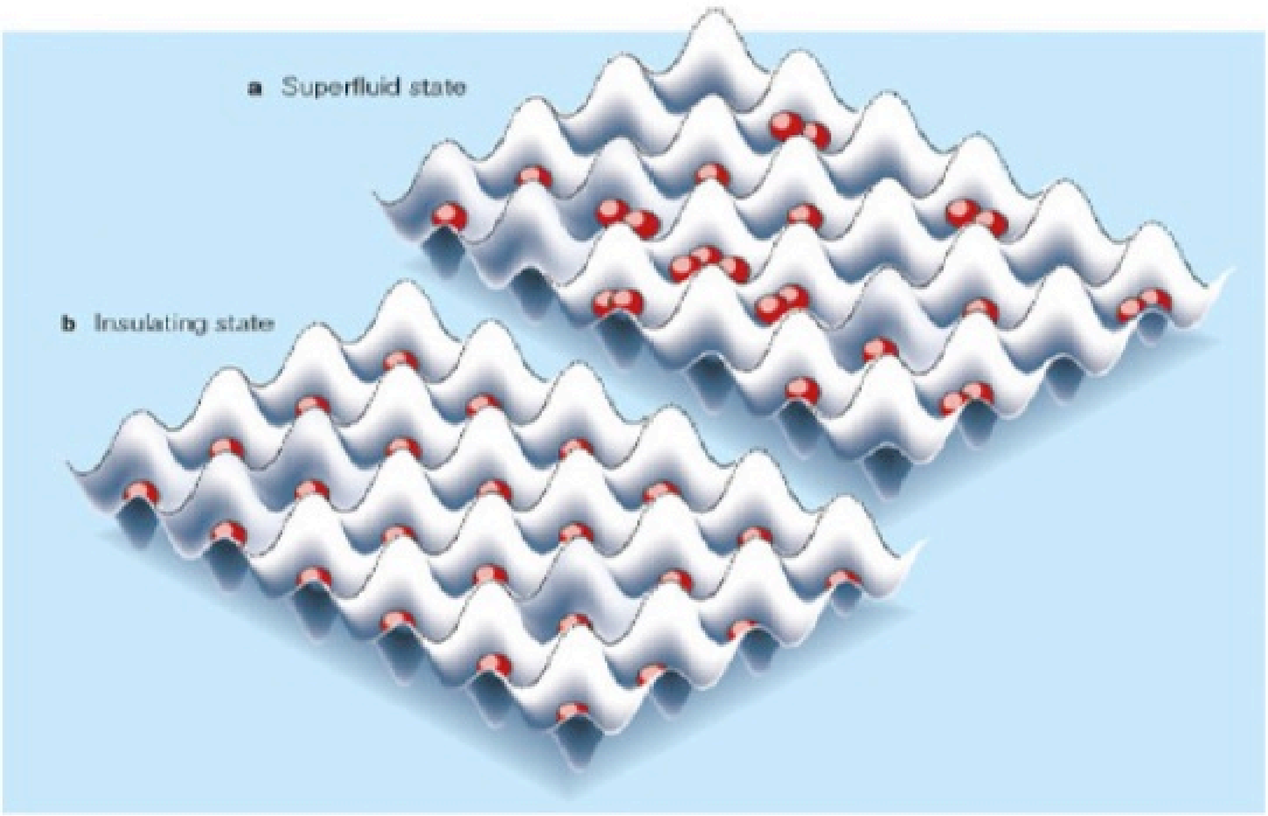
$U/w \gg 1$: Mott insulator phase (localized bosons)

$U/w \ll 1$: Superfluid phase (de localization)

$U(1)$ symmetry is broken in SF

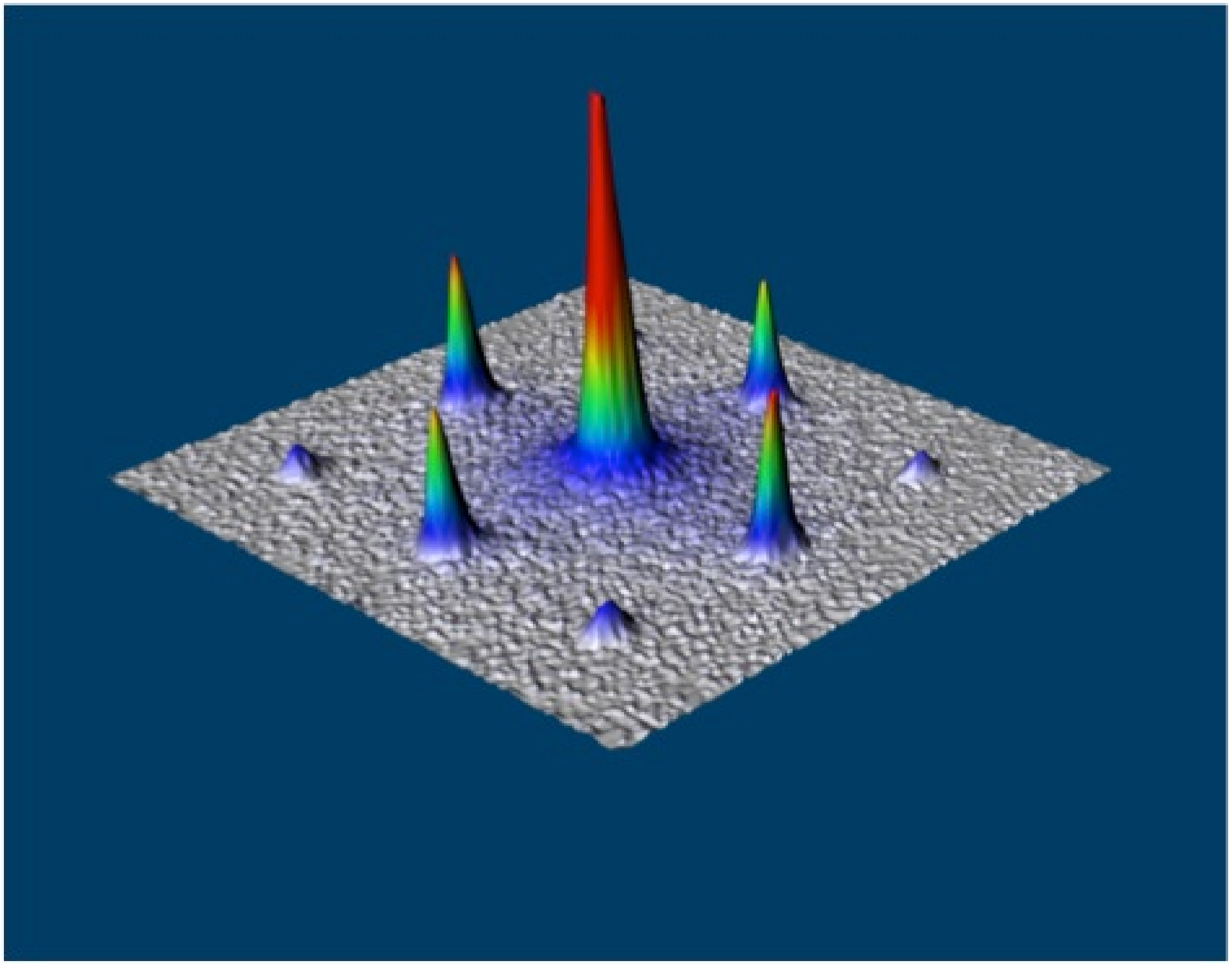
Experimental result: ^{87}Rb cold atoms:

When the kinetic energy $\tilde{w}$ is large, bosons move around the sites (de localization).



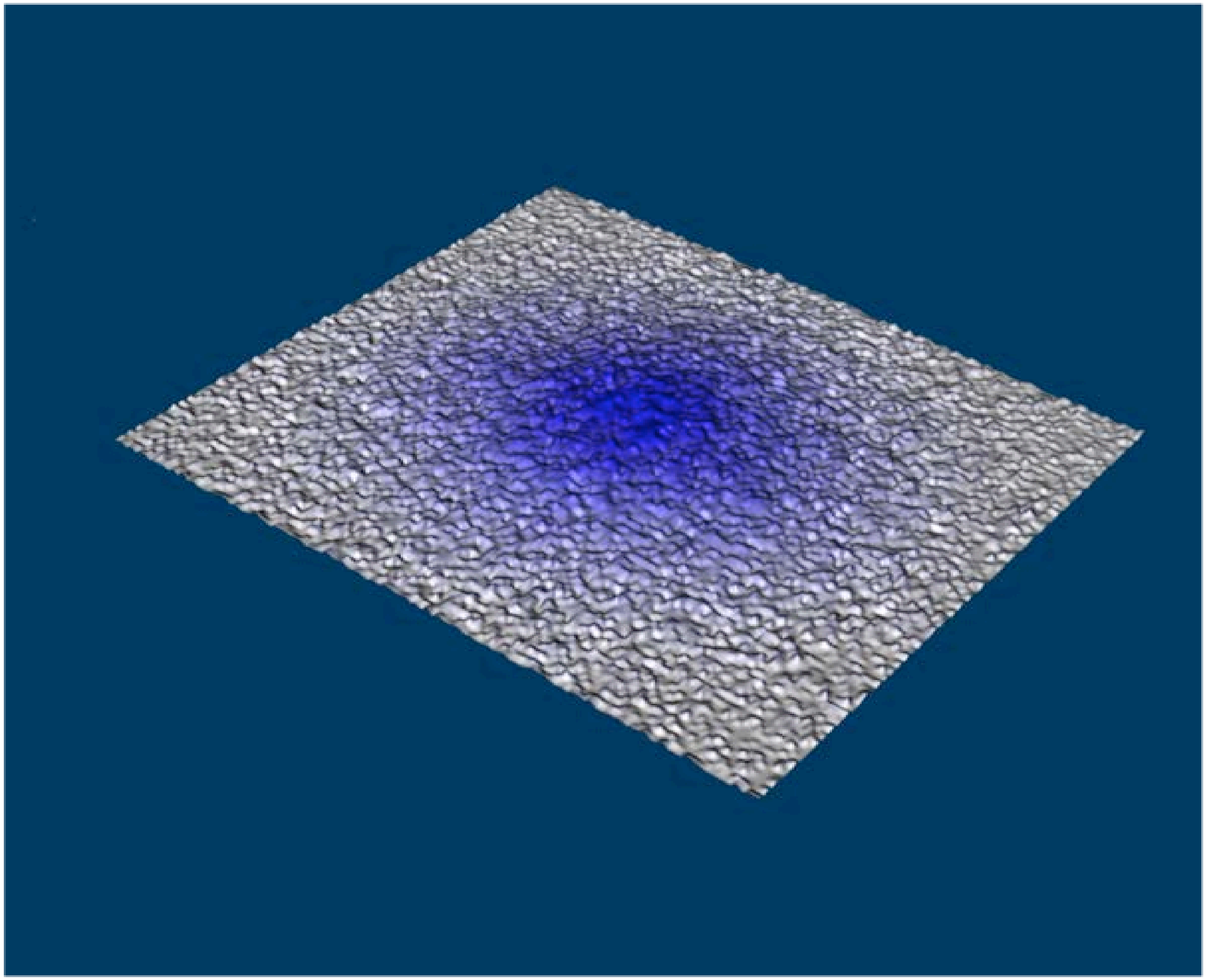
From H.T.C. Stoof, Nature 415, 25 (2002)

Velocity distribution of cold atoms in Superfluid



M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature* **415**, 39 (2002).

Velocity distribution of cold atoms in Mott insulator



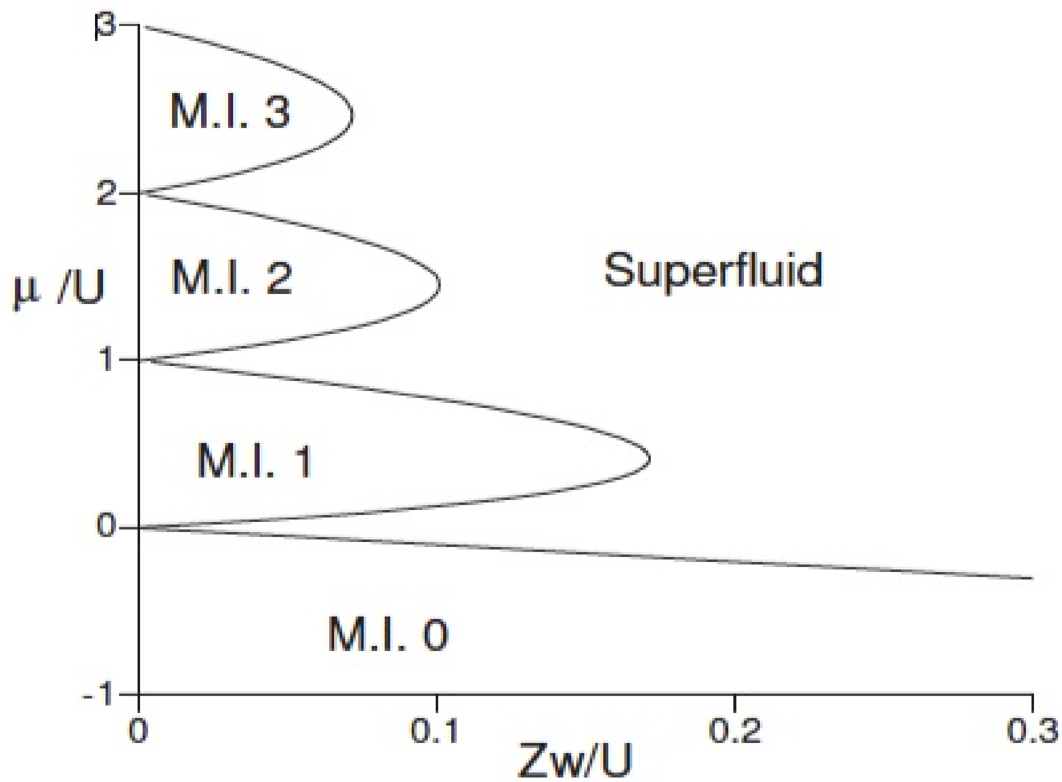
M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature* **415**, 39 (2002).

Lobe-shaped structure

Phase structure of the ground state of the boson-Hubbard model looks like lobe-shaped structure

M.I. means the Mott insulator of equally occupied state.

The amplitude of the lobe decreases as $1/\rho$ when the occupation ρ in M.I. increases.



How to show the phase structure (SF/Mott Transition)

How to show the phase structure in the boson-Hubbard model?

◆ The mean-field approach introducing ψ_B

In the Mott insulator phase, $\psi_B = 0$

◆ $t_{hop} = 0$ state is a precise eigenstate of the total number operator

◆ The ground state at non-zero t_{hop} is not a simple state like $|n_0\rangle$

A holographic construction of large N boson–Hubbard model

To realize the lobe–shaped phase structure of the boson–Hubbard model

However, large N limit is actually needed to cause the phase transition in the finite volume system.

The AdS/CFT correspondence (Summary)
Different from the previous slide!

The large N boson hubbard model side	Gravity side
Occupation number per a site $n_i = b_i^{a\dagger} b_{ia}$	$U(1)^n$ gauge fields A_i
Chemical potential μ_i	$U(1)^n$ gauge fields A_i
Hopping parameter t_{hop}	Bi-fundamental scalar $\phi_{i,j}$
Bi-local condensate $b_i^{a\dagger} b_{ja}$ ($i \neq j$)	Bi-fundamental scalar $\phi_{i,j}$
Coulomb repulsive parameter U	IR cutoff r_h

- Spin indices: $a=1,\cdots,N$

We consider two-site model later ($i,j=1,2$).

A holographic construction of the boson-Hubbard model I ($t_{hop} = 0 \quad \varphi = 0$)

To realize the lobe structure of
the boson-Hubbard model

$U(1)^n$ Abelian theory on the AdS_2 hard wall

The action of the $U(1)^n$ Abelian theory

$$S_{kin} = \sum_k \int d^2x \sqrt{-g} \Big(-\frac{1}{4} F_{(k)\mu\nu} F^{\mu\nu}_{(k)} \Big)$$

◆ $k=1,..,n$: sites of the lattice model

$U(1)^n$ Abelian theory on the AdS_2 hard wall

The action of the $U(1)^n$ Abelian theory

$$S_{kin} = \sum_k \int d^2x\sqrt{-g}\Big(-\frac{1}{4}F_{(k)\mu\nu}F^{\mu\nu}_{(k)}\Big)$$

◆ $k=1,\dots,n$: sites of the lattice model

We use the AdS_2 hard wall with some large IR cutoff r_h

$$ds^2 = -r^2dt^2 + \frac{dr^2}{r^2}$$

in units of the AdS radius 1

Appearance of other possible instabilities at energy scale $> r_h$

Maldacena–Michelson–Strominger ``98

Matching the energy when $t_{hop}=0$.

The free energy of the U(1)^n Abelian theory:

- Free energy (after the analytic continuation to the Euclidean signature)

$$F = -(I_{kin} + I_{cut})/\beta = \sum_k \left(\mu \rho_{(k)} + r_h \frac{\rho_{(k)}^2}{2} \right),$$

- Requiring Dirac quantization of the charges

Matching the energy when $t_{hop}=0$.

The free energy of the $U(1)^n$ Abelian theory:

◆ Free energy (after the analytic continuation to the Euclidean signature)

$$F = -(I_{kin} + I_{cut})/\beta = \sum_k \left(\mu\rho_{(k)} + r_h \frac{\rho_{(k)}^2}{2}\right),$$

◆ Requiring Dirac quantization of the charges

The same as the free energy of the boson-Hubbard model:

$$F_b = \sum_k \left(\mu_b\rho_{(k)} + \frac{1}{2}U\rho_{(k)}(\rho_{(k)} - 1)\right)$$

the replusive interactions between bosons!

$$r_h = U, \quad \mu = \mu_b - \frac{U}{2}$$

via identification

Quantum mechanics require that charges are quantized to be an integer.

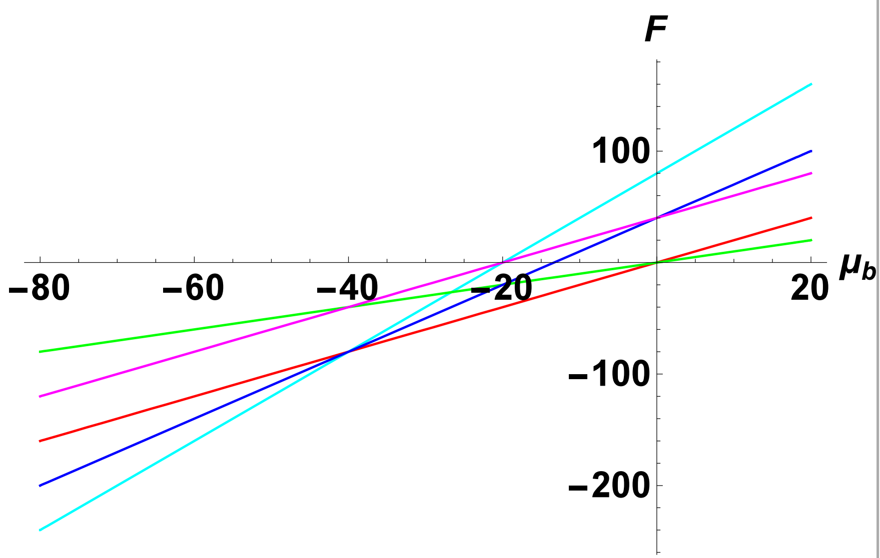
Level–crossing transition

First order phase transition between the Mott insulator phases

Degenerate energies at the critical chemical potential

◆ Interpreted as macroscopic entropy

Figure: two–site model



In the right hand side,

$(\rho_{(1)}, \rho_{(2)}) = (0,0),\ (0,1),\ (1,1),\ (2,0),$

$(1,2)$, and $(2,2)$ from bottom to top.

A holographic construction of the boson-Hubbard model II ($t_{hop} \neq 0$)

To realize the lobe structure of
the boson-Hubbard model

Adding bi-fundamental scalar $t_{hop} \neq 0$

The bi-fundamental matter is dual to the hopping term

- The hopping term can represent the kinetic energy
- Our bi-fundamental can break $U(1)^n \rightarrow U(1)$ subgroup explicitly!

The kinetic term and the IR potential (two-site model):

$$S_{matter} = -\int d^2x\sqrt{-g}|D\phi|^2 - \int_{r=r_h} dt r_h \Lambda(|\phi|^2 + w^2)^2$$

EOM of the total system

$$(r^2\phi')' + \frac{q^2}{r^2}(A_t^{(1)} - A_t^{(2)})^2\phi = 0,$$

$$(A_t^{(l)\prime})' - \frac{2q^2|\phi|^2}{r^2}(A_t^{(l)} - A_t^{(l+1)}) = 0,$$

Homogeneous phase
or
Non-homogeneous phase

homogeneous phase: $A_t^{(1)} = A_t^{(2)}$

- The IR boundary condition: Dirichlet bc or mixed Neumann

  corresponding to Mott insulator phase

  holographic renormalization needed!

  bosons are localized on the site

Homogeneous phase
or
Non-homogeneous phase

homogeneous phase: $A_t^{(1)} = A_t^{(2)}$

- The IR boundary condition: Dirichlet bc or mixed Neumann

corresponding to Mott insulator phase

holographic renormalization needed!

bosons are localized on the site

Non-homogeneous phase: $A_t^{(1)} \neq A_t^{(2)}$

c.f. Axial vector in hard/soft wall AdS/QCD

- The IR boundary condition: Dirichlet bc or mixed Neumann

- corresponding to Superfluid phase (hopping term gives the kinetic energy and de localized bosons)

The lobe–shaped structure

- The phase structure of the two–site model

The lobe–shaped structure

Non–homogeneous phase is favored when $t_{hop}/U >> 1$

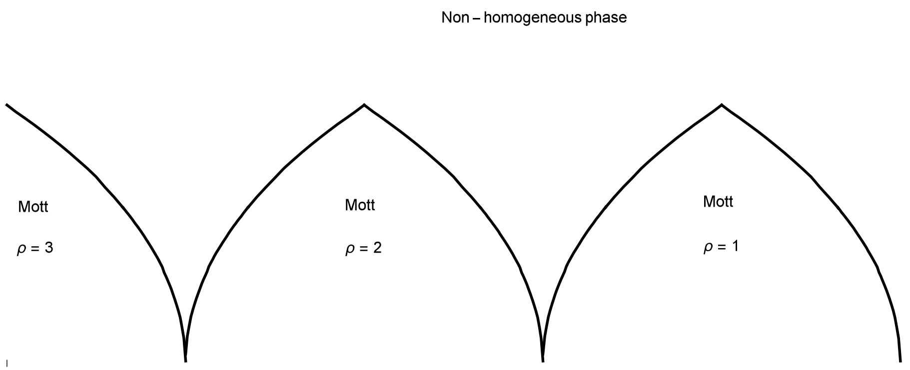
- Order parameters

$$\delta n = n_1 - n_2$$

$$dF / dt_{hop} \approx \left\langle b_i^\dagger b_j \right\rangle$$

Mott insulator is favored when $t_{hop}/U << 1$

Large w decreases the amplitude as $t_{hop} \sim 1/w$



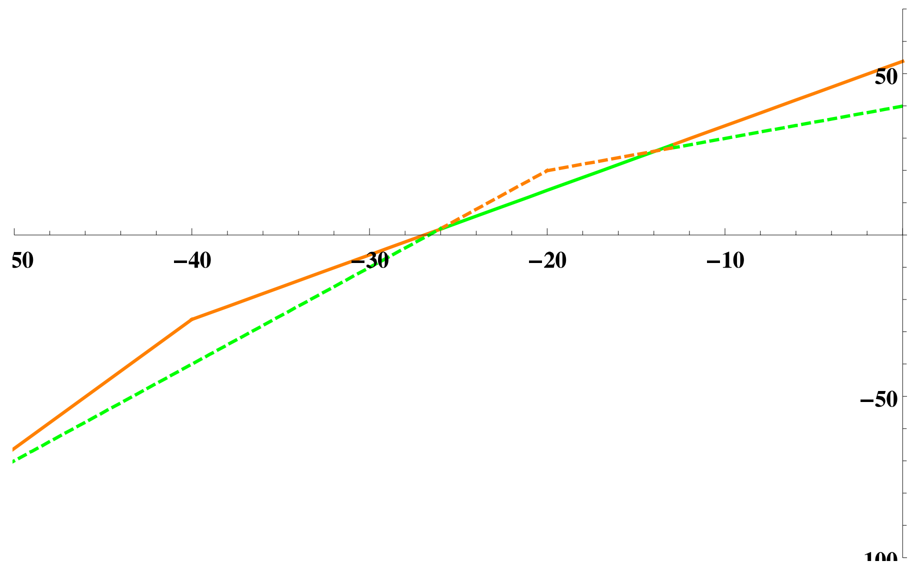
Free energy as the function of the chemical potential

Solid lines: the Mott insulator for t_{hop}=0.4 respectively.

Dashed lines: non–homogeneous phase for t_{hop}=0.4

Phase transition takes place between homogeneous phase and non–homogeneous phase.

- De localization of bosons in non–homogeneous phase



The free energy as the function of the hopping parameter

The difference of the free energy $F - F_{mott}$

◆ Mott insulator $\rho = 2$

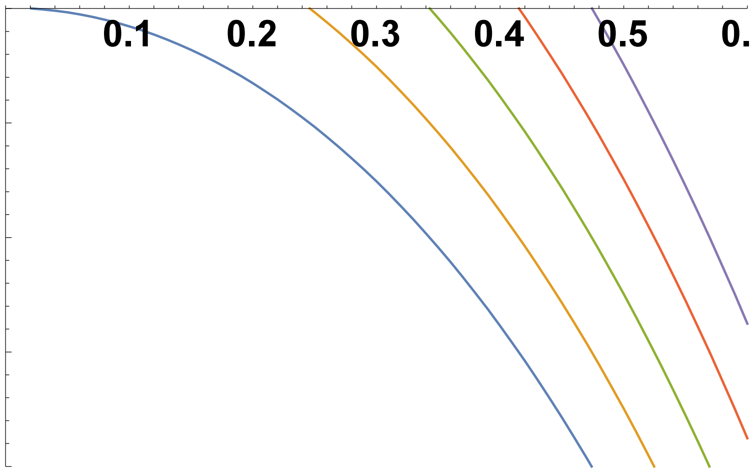
◆ at the critical chemical potential $\mu_b = -U$, non-homogeneous phase is always stable.

From the left to right

$$\mu_b = \mu + r_h/2 = -40, -45, -50, -55, -60,$$

Particle-hole symmetric point

$\mu_b = -1.5\ U$

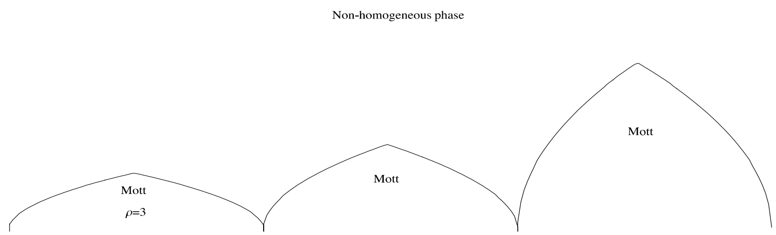


Decreasing the Amplitude of the Lobe

- Generalization to the gauge-invariant IR potential

$$I_{IR}(\phi, F_{rt}) = -\int_{r=r_h} dtr_h \Big(\Lambda_{(1,0)}\,|\phi|^2 + \Lambda_{(2,0)}\,|\phi|^4 + \sum_i \Lambda_{(1,1)}\,|\phi|^2\,F_\mu^{(i)}F^{(i)\mu} + \cdots + \Lambda_{(p,q)}\,|\phi|^{2p}\sum_i (F_\mu^{(i)}F^{(i)\mu})^q\Big),$$

- Realization of 1/ρ behavior of the lobe's amplitude



- Perturbative Spectrum at small hopping (Dirichlet bc at hard wall):

Always gapped in Mott insulator phase

Almost zero mode exists in non-homogeneous phase

Discussion

We realized the lobe shape of the phase structure

of the boson-Hubbard model

◆ The Mott/Superfluid phase transition was of 1st order by using

dF/dt_{hop} and charge difference as the order parameter

A top down model will be obtained from a D3/D5/D7 system

where N D3 are replaced by the AdS_5 soliton

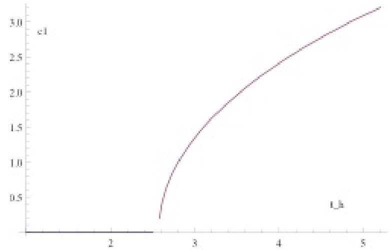
◆ Non-Abelian D5s wrapping $AdS_2 x S^4$ are dual to

the effective theory on the lattice: flavor indices give lattice sites

Future directions

In my talk, U(1) symmetry is not broken in SF phase.

- U(1) fundamental's condensation
- The second order phase transition



Application to the disordered phase called the Bose-glass

- Replica trick is useful $$F = -(1/\beta)\overline{\log Z} = \overline{(Z^n - 1)/n}$$

Application to the Fermi Hubbard model

- Numerical method is needed to solve higher dimensional Hubbard
- Sign problems in Fermi Hubbard model

Thank you!

How to show the phase structure (SF/Mott Transition)

How to show the phase structure in the boson-Hubbard model?

- The mean-field approach introducing ψ_B
- Via the Hubbard-Stratanovich transformation on the coherent state path integral

$$K_3\,|\nabla\Psi_B|^2+\tilde{r}|\Psi_B|^2+\frac{u}{2}|\Psi_B|^4+\cdots$$

Boundary conditions

The solution to the EOM $\partial_\mu F^{\mu\nu}_{(k)} = 0$

$$A_{(k)t} = \mu + \rho_{(k)} r.$$ in the radial gauge

- $\rho_{(k)}$: charge density per a site

The Dirichlet boundary condition at the IR wall: we choose a free parameter which produces interactions

$$A_t\Big|_{r=r_h} = \mu + r_h \rho_{(k)}$$

The UV counter–term

Should cancel the linear divergence of the on–shell action at the boundary *r=R*

- Addition of the counter–term: $I_{cut} = \sum_k \frac{1}{2} \int_{r=R} dt \sqrt{-h} A_{(k)t} A^t_{(k)}$

h: induced metric at the boundary

The UV counter–term

Should cancel the linear divergence of the on–shell action at the boundary $r=R$

- Addition of the counter–term: $$I_{cut} = \sum_k \frac{1}{2} \int_{r=R} dt \sqrt{-h} A_{(k)t} A^t_{(k)}$$

h: induced metric at the boundary

- Not manifestly gauge invariant: Requiring that the gauge transformation does not change the leading coefficient (the charge).

- The large gauge transformation leads to the Dirac quantization:

Charges $\rho_{(k)}$ and the monopole charge are quantized to be an integer.