Seminar at IPMU 2014/10/24

Quantum Entanglement of Local Operators

Masahiro Nozaki

Yukawa Institute for Theoretical Physics (YITP), Kyoto University

1. Based on arXiv:1401.0539v1 [hep-th] (Phys. Rev. Lett. 112, 111602 (2014)) with Tokiro Numasawa, Tadashi Takayanagi
2. Based on arXiv:1405.5875 [hep-th] (accepted in JHEP)
3. Based on arXiv:1405.5946 [hep-th]

(PTEP 2014 (2014) 9, 093B06)

with Pawel Caputa, Tadashi Takayanagi

YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

Introduction

Recently, (Renyi) entanglement entropy ((R)EE) has a center of wide interest in a broad array of theoretical physics.

- It is useful to study the distinctive features of various quantum state in condensed matter physics. (***Quantum Order Parameter***)
- (Renyi) entanglement entropy is expected to be an important quantity which may shed light on the mechanism behind the AdS/CFT correspond .(***Gravity*** $\leftrightarrow$ ***Entanglement***)

Introduction

Recently, (Renyi) entanglement entropy ((R)EE) has a center of wide interest in a broad array of theoretical physics.

- In the lattice gauge theory, it is expected that entanglement entropy is a new order parameter which helps us study QCD more.

- ***But*** entanglement entropy in the gauge theory is ill-defined.

Introduction

Recently, (Renyi) entanglement entropy ((R)EE) has a center of wide interest in a broad array of theoretical physics.

- In the lattice gauge theory, it is expected that entanglement entropy is a new order parameter which helps us study QCD more.

\- ***But*** entanglement entropy in the gauge theory is ill-defined.

It is important to study the properties of (Renyi) entanglement entropy.

Introduction

Recently, (Renyi) entanglement entropy ((R)EE) has a center of wide interest in a broad array of theoretical physics.

- In the lattice gauge theory, it is expected that entanglement entropy is a new order parameter which helps us study QCD more.

- ***But*** entanglement entropy in the gauge theory is ill-defined.

In this work, we investigate the time dependent property of (Renyi) entanglement entropy.

The Definition of (Renyi) Entanglement Entropy

- Definition of Entanglement Entropy

We divide the total Hilbert space into A and B: $H_{tot} = H_A \otimes H_B$.

The reduced density matrix ρ_A is defined by $\rho_A \equiv Tr_B \rho_{tot}$

This means the D O F in B are traced out.

The entanglement entropy is defined by von Neumann entropy S_A.

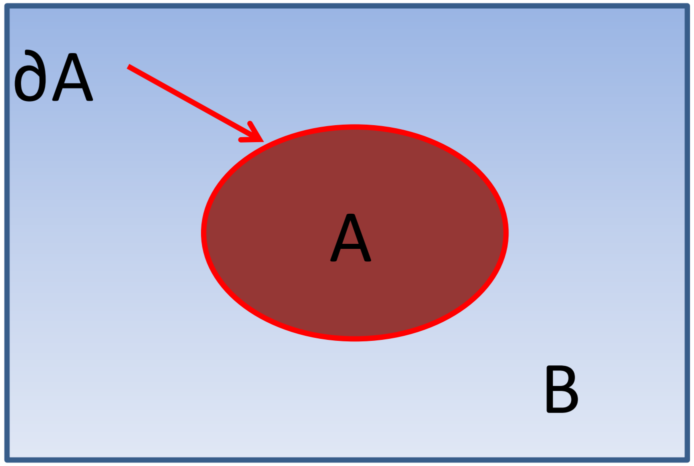
(Renyi) Entanglement Entropy (REE)

$$\boxed{S_A^{(n)} = \frac{\log \mathrm{tr}[\rho_A^n]}{1-n}}$$

n→1

Entanglement Entropy (EE)

$$\boxed{S_A = -\mathrm{tr}_A \rho_A \log \rho_A}$$



on a certain time slice

Example

For a product state: $|\Psi\rangle = \frac{1}{2}\left(|\uparrow\rangle_A + |\downarrow\rangle_A\right) \otimes \left(|\downarrow\rangle_B + |\uparrow\rangle_B\right)$

⇒Reduced density matrix : $\rho_A = \frac{1}{2}\left(|\uparrow\rangle_A + |\downarrow\rangle_A\right)\left(\langle\uparrow|_A + \langle\downarrow|_A\right)$

⇒Entanglement entropy : $S_A = 0$

For an entangled state: $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B\right)$

⇒Reduced density matrix : $\rho_A = \frac{1}{2}\left(|\uparrow\rangle_A \langle\uparrow|_A\right) + \frac{1}{2}\left(|\downarrow\rangle_A \langle\downarrow|_A\right)$

⇒Entanglement Entropy : $S_A = \log 2$

Example

For a product state: $|\Psi\rangle = \frac{1}{2}\left(|\uparrow\rangle_A + |\downarrow\rangle_A\right) \otimes \left(|\downarrow\rangle_B + |\uparrow\rangle_B\right)$

⇒Reduced density matrix : $\rho_A = \frac{1}{2}\left(|\uparrow\rangle_A + |\downarrow\rangle_A\right)\left(\langle\uparrow|_A + \langle\downarrow|_A\right)$

⇒Entanglement entropy : $S_A = 0$

For an entangled state: $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B\right)$

⇒Reduced density matrix : $\rho_A = \frac{1}{2}\left(|\uparrow\rangle_A \langle\uparrow|_A\right) + \frac{1}{2}\left(|\downarrow\rangle_A \langle\downarrow|_A\right)$

⇒Entanglement Entropy : $S_A = \log 2$

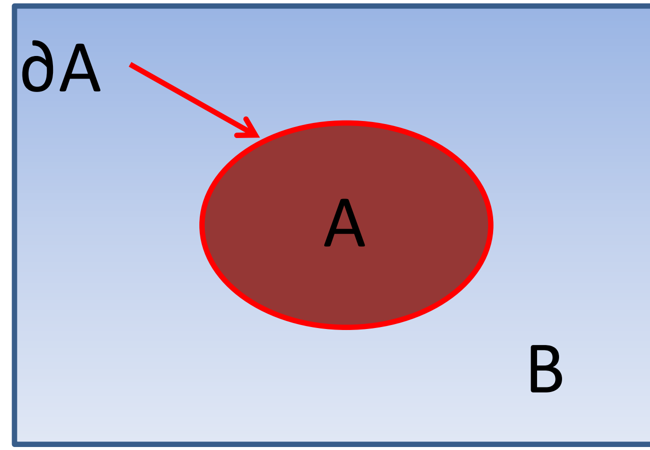
In general ,

Entangled state (not-product state) has the entanglement entropy.
S_A measures the quantum entanglement.

Properties 1

- For a pure state, entanglement entropy satisfies

$$S_A = S_B .$$



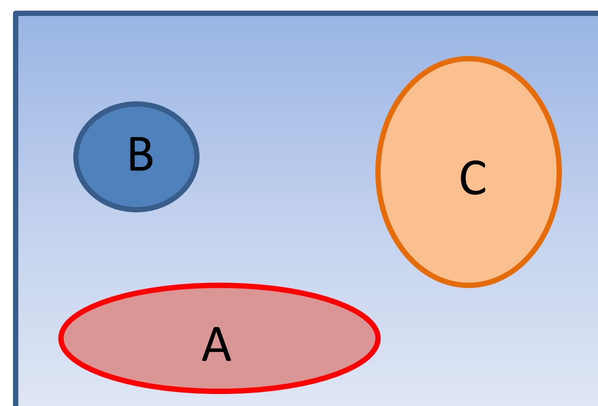
- For a mixed state (thermal state etc.), entanglement entropy satisfies

$$S_A \neq S_B .$$

- Strong Subadditivity

  Entanglement entropy necessarily satisfies

$$\begin{aligned} S_{A+B+C} + S_B &\leq S_{A+B} + S_{B+C} \\ S_A + S_C &\leq S_{A+B} + S_{B+C} . \end{aligned}$$



Entanglement Entropy in QFT

In general, (Renyi) entanglement entropies has ***UV divergence***.

-> We introduce lattice spacing $\boldsymbol{\varepsilon}$.

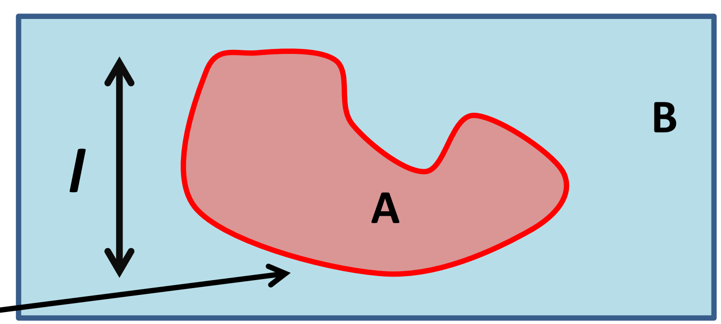
Entanglement Entropy in QFT

In d dimensional CFT,

$$S_A = \underbrace{p_1 \left(\frac{l}{\epsilon}\right)^{d-2}}_{\text{Area law div.}} + p_2 \left(\frac{l}{\epsilon}\right)^{d-4} + \cdots$$

$$+ \begin{cases} p_{d-2}\left(\frac{l}{\epsilon}\right) + p_d, & d : \text{odd} \\ p_{d-3}\left(\frac{l}{\epsilon}\right)^2 + \tilde{c}\log\left(\frac{l}{\epsilon}\right), & d : \text{even}. \end{cases}$$

l: characteristic size of a subsystem



Most strongly entangled

Entanglement Entropy in QFT

In d dimensional CFT,

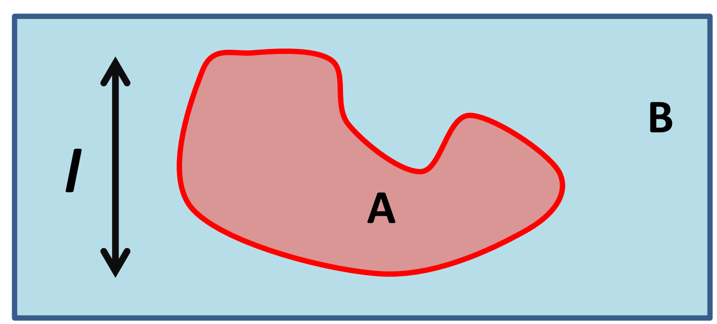
Universal quantities
(do not depend on cutoff)

$$S_A = p_1 \left(\frac{l}{\epsilon}\right)^{d-2} + p_2 \left(\frac{l}{\epsilon}\right)^{d-4} + \cdots$$

Area law div.

$$+ \begin{cases} p_{d-2}\left(\frac{l}{\epsilon}\right) + \underline{p_d}, & d: \text{odd} \\ p_{d-3}\left(\frac{l}{\epsilon}\right)^2 + \underline{\tilde{c}} \log\left(\frac{l}{\epsilon}\right), & d: \text{even.} \end{cases}$$

l: characteristic size of a subsystem



Entanglement Entropy in QFT

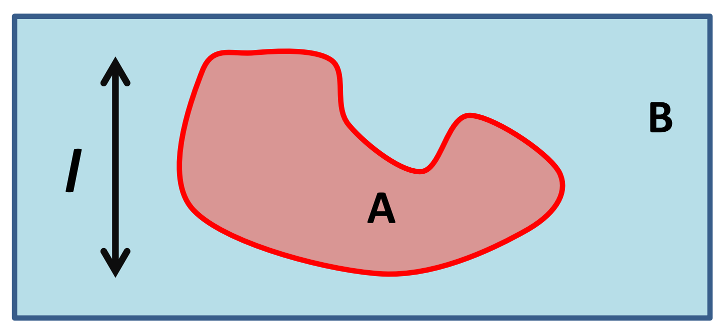
In d dimensional CFT,

$$S_A = \underbrace{p_1 \left(\frac{l}{\epsilon}\right)^{d-2}}_{\text{Area law div.}} + p_2 \left(\frac{l}{\epsilon}\right)^{d-4} + \cdots$$

c is related to central charge

$$+ \begin{cases} p_{d-2}\left(\frac{l}{\epsilon}\right) + \underline{p_d}, & d : \text{odd} \\ p_{d-3}\left(\frac{l}{\epsilon}\right)^2 + \underline{\tilde{c}}\log\left(\frac{l}{\epsilon}\right), & d : \text{even.} \end{cases}$$

***l*: characteristic size of a subsystem**



Our Computation and Result

Contents

- ***Field Theory Side***

1. Based on arXiv:1401.0539v1 [hep-th] (Phys. Rev. Lett. 112, 111602 (2014))

2. Based on arXiv:1405.5875 [hep-th]

A new class of excited states

- We introduce

  a new class of excited states.

New excited states:

$$\boxed{\left|\Psi\right\rangle = \mathcal{N}^{-1}\mathcal{O}(t, x^1)\left|0\right\rangle .}$$

A new class of excited states

- We introduce

  a new class of excited states.

What's new?

New excited states:

$$\boxed{\left|\Psi\right\rangle = \mathcal{N}^{-1}\mathcal{O}(t,x^1)\left|0\right\rangle .}$$

A new class of excited state

Quantum Quench: Prepare the ground state $|\Psi\rangle$ for H_0

$H_0 \longrightarrow H_1$ (change parameter)

$|\Psi\rangle$ is ***not*** the ground state for H_1

We can investigate the ***time evolution*** of (R)EE.

A new class of excited state

Quantum Quench: Prepare the ground state $|\Psi\rangle$ for H_0

$H_0 \longrightarrow H_1$ (change parameter)

$|\Psi\rangle$ is ***not*** the ground state for H_1

A new class of excited state: $|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t, x^1)\,|0\rangle$

A new class of excited state

Quantum Quench: Prepare the ground state $|\Psi\rangle$ for H_0

$H_0 \longrightarrow H_1$ (change parameter)

$|\Psi\rangle$ is ***not*** the ground state for H_1

A new class of excited state: $|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t, x^1)\,|0\rangle$

$\boxed{\text{Excitation is milder than that in } \textit{Quantum Quench}\textbf{.}}$

$\Delta S_A^{(n)}$ is finite even for the size of subsystem is infinite.

A new class of excited state

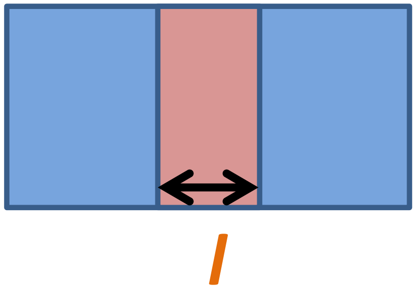
For example,

In Quantum Quench

$|\Psi\rangle$: ground state for H_0

H_0 : Hamiltonian for massive theory → H_1 :Hamiltonian for massless theory

At **late time**, EE is proportional to ***l*** (size of subsystem) . → ***l*** → ∞, EE → ∞



A new class of excited state

For example,

In Quantum Quench

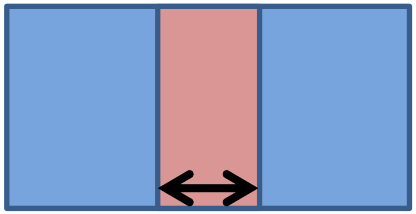
$|\Psi\rangle$: ground state for H_0

H_0 : Hamiltonian for massive theory $\Rightarrow$ H_1 :Hamiltonian for massless theory

At **late time**, EE is proportional to *l* (size of subsystem) . $\Rightarrow$ *l* $\to \infty$, EE $\to \infty$

In *our excited state case*

State: $|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t, x^1)\,|0\rangle$



l

At **late time**, *l* $\to \infty$ $\Rightarrow$ (R)EE approaches ***some constant***.

A new class of excited state

For example,

In Quantum Quench

$|\Psi\rangle$: ground state for H_0

H_0 : Hamiltonian for massive theory ⇨ H_1 :Hamiltonian for massless theory

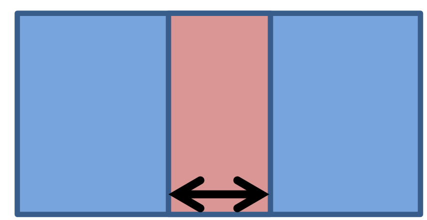
At la

Excitation is milder than that in *Quantum Quench*.

E $\to \infty$

In *our excited state case*

State: $|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t,x^1)\,|0\rangle$



l

At late time, *l* $\to \infty$ ⇨ (R)EE approaches ***some constant***.

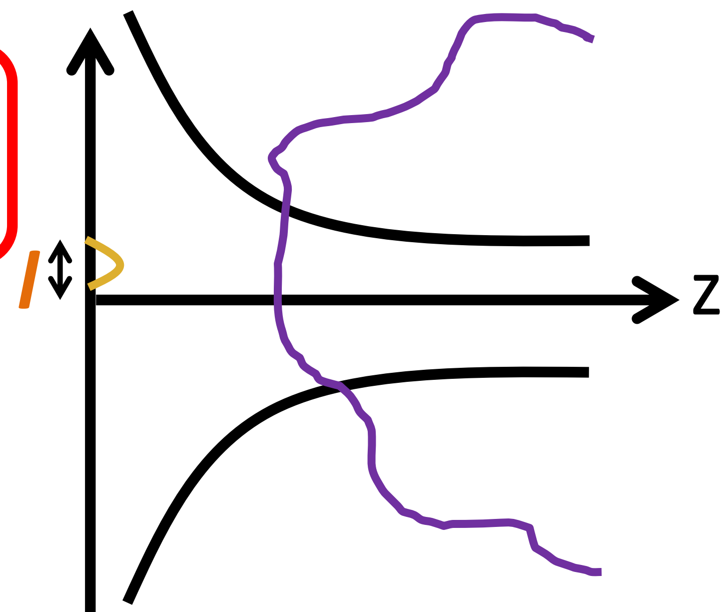
Motivation

Previously, we studied the property of EE for the subsystem whose size (l) is ***very small*** in d+1 CFT.

$$\boxed{l \ll (\text{The Excitation Energy})^{-d},}$$

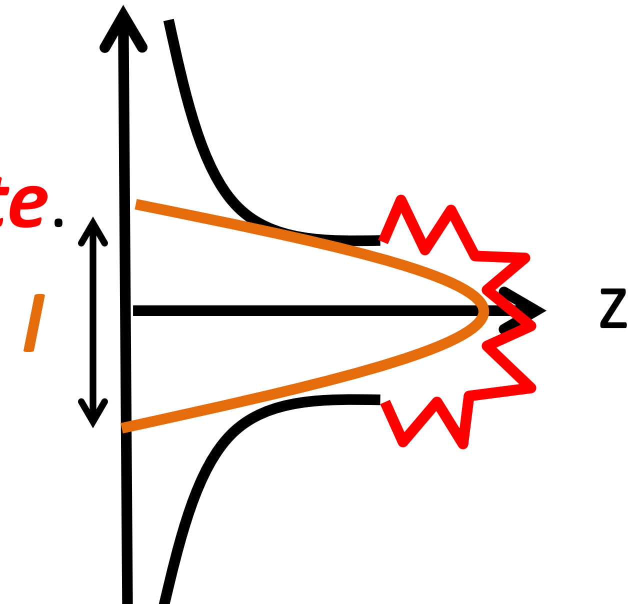
$$\boxed{\Delta E_A = \underline{T_{ent}}\,\Delta S_A}$$

This temperature is universal.



Motivation

We study the property of **(R)EE** for



1. The size of subsystem is ***infinite***.

   A half of the total system:

$$x^1 \geq 0$$

2. A state is defined by acting a local operator on the ground state:

$$|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t, x^1)\,|0\rangle\,.$$

Motivation

We study the property of **(R)EE** for

1. The size of subsystem is ***infinite***.

$$\Delta S_A^{(n)} \Longrightarrow$$

At late time

Some Constants

$$|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t,x^1)\,|0\rangle\,.$$

Motivation

We study the property of **(R)EE** for

1. The size of subsystem is ***infinite***.

$$\Delta S_A^{(n)} \xrightarrow[\text{At late time}]{} \text{Some Constants}$$

Unique Behavior

$$|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t, x^1)\,|0\rangle\,.$$

Results

We compute $\Delta S_A^{(n)}$ for a new class of excited states.

$$|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t,x^1)\,|0\rangle \quad \text{for} \quad \mathcal{O} =: (\partial^m\phi)^k :$$

Results

We compute $\Delta S_A^{(n)}$ for a new class of excited states.

$$|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t,x^1)\,|0\rangle \quad \text{for} \quad \mathcal{O} =: (\partial^m \phi)^k :$$

At late time

(Renyi) Entanglement Entropies of Local Operators

$$\boxed{\Delta S_{A,k}^{(n)f} = \frac{1}{1-n}\log\left(\frac{1}{2^{nk}}\sum_m^k ({}_kC_m)^n\right)}$$

Results

We compute $\Delta S_A^{(n)}$ for a new class of excited states.

$$|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t,x^1)\,|0\rangle \quad \text{for} \quad \mathcal{O} =: (\partial^m \phi)^k :$$

At late time

(Renyi) Entanglement Entropies of Local Operators

$$\Delta S_{A,k}^{(n)f} = \frac{1}{1-n}\log\left(\frac{1}{2^{nk}}\sum_m^k \left({}_kC_m\right)^n\right)$$

They measure the D.O.F of operators and ***characterize*** the operators from the viewpoint of quantum entanglement.

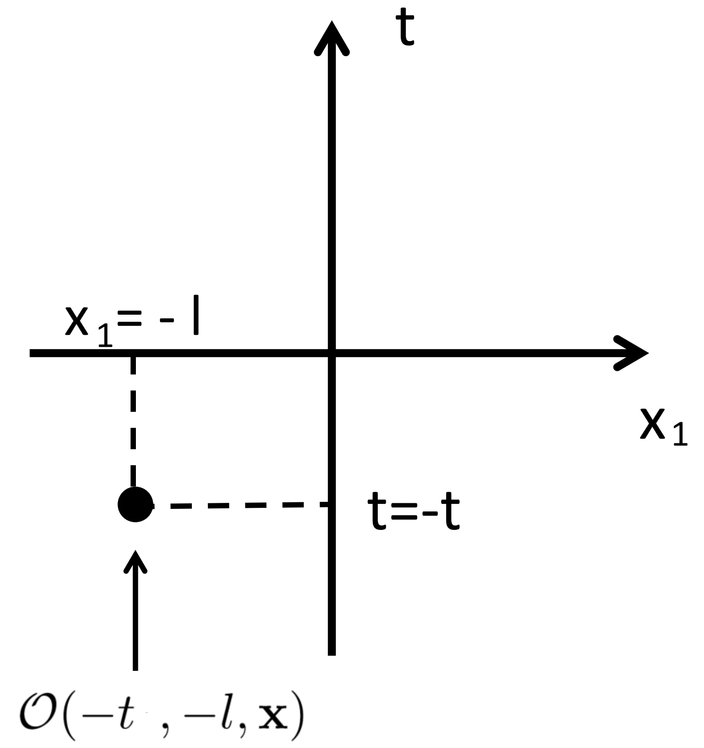
(not conformal dim.)

Setup

We consider d+1 dim. QFT.

We prepare a locally excited state: $|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(-t, -l, \mathbf{x})\,|0\rangle$

$$\mathbf{x} = (x^2, x^3, \cdots, x^d)$$



Setup

We consider d+1 dim. QFT.

We prepare a locally excited state: $|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(-t,-l,\mathbf{x})\,|0\rangle$

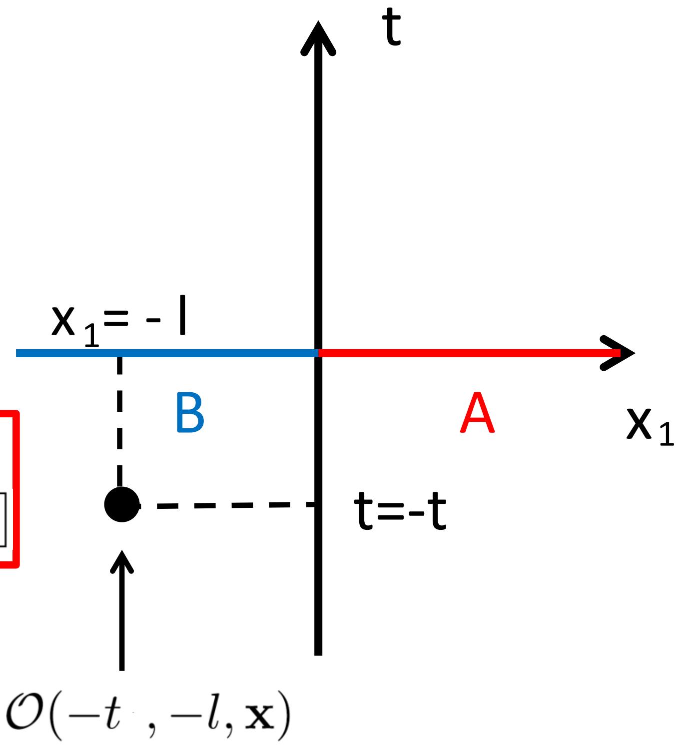
$$\mathbf{x} = (x^2, x^3, \cdots, x^d)$$

We investigate the time evolution of (Renyi) entanglement entropies.

We choose a half space ($x^1 \geq 0$) as subsystem A.

The reduced density matrix
$\rho_A = \mathcal{N}^{-2} tr_B \left[\mathcal{O}(-t_1,-l,\mathbf{x})\,|0\rangle\langle 0|\,\mathcal{O}^{\dagger}(-t_1,-l,\mathbf{x})\right]$

We will compute $S_A^{(n)} = \dfrac{\log \mathrm{tr}[\rho_A^n]}{1-n}$.



Setup

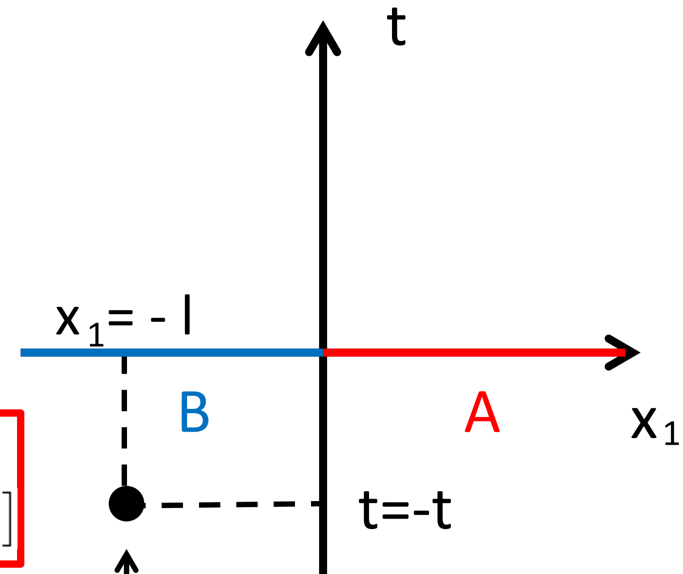
We consider d+1 dim. QFT.

We prepare a locally excited state: $|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(-t,-l,\mathbf{x})\,|0\rangle$

$$\mathbf{x} = (x^2, x^3, \cdots, x^d)$$

We investigate the time evolution of (Renyi) entanglement entropies.

We choose a half space ($x^1 \geq 0$) as subsystem A.



The reduced density matrix
$\rho_A = \mathcal{N}^{-2} tr_B \left[\mathcal{O}(-t_1,-l,\mathbf{x})\,|0\rangle\langle 0|\,\mathcal{O}^\dagger(-t_1,-l,\mathbf{x})\right]$

We will compute $S_A^{(n)} = \frac{\log \mathrm{tr}[\rho_A^n]}{1-n}$.

Replica Method !!

How to compute

1. We compute $\Delta S_A^{(n)} = S_A^{(n)Ex} - S_A^{(n)G}$ by path-integral:

$$\boxed{\begin{aligned}&\Delta S_A^{(n)} = \\ &\frac{1}{1-n}\left(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\right).\end{aligned}}$$

2. After that, we perform an analytic continuation to real time.

Replica Method

In d+1 dim. Euclidean space, the reduced density matrix is given by

$$\hat{\rho}_A = \mathcal{N}^{-2} tr_B \left[\mathcal{O}(\tau_e, -l, \mathbf{x}) \left|0\right\rangle \left\langle 0\right| \mathcal{O}^\dagger(\tau_l, -l, \mathbf{x}) \right]$$

We would like to focus on the time evolution of the (R)EE.

We define $\Delta S_A^{(n)}$ the excess of the (R)EE:

$$\boxed{\Delta S_A^{(n)} = S_A^{(n)Ex} - S_A^{(n)G}},$$

$S_A^{(n)Ex}$: (R)EE for $\hat{\rho}_A$

$S_A^{(n)G}$: (R)EE for the ground state

Replica Method

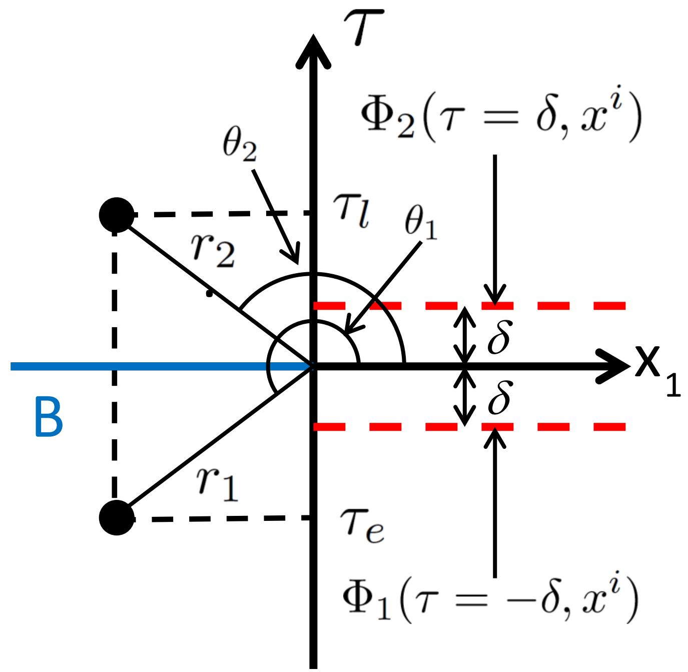
In d+1 dim. Euclidean space, the reduced density matrix is given by

$$\hat{\rho}_A = \mathcal{N}^{-2} tr_B \left[\mathcal{O}(\tau_e, -l, \mathbf{x}) \left|0\right\rangle \left\langle 0\right| \mathcal{O}^\dagger(\tau_l, -l, \mathbf{x}) \right]$$

Configuration on A

$$\langle \Phi_1, \Phi_B | \mathcal{O}(\tau_e, -l, \mathbf{x}) | 0 \rangle$$

Configuration on B

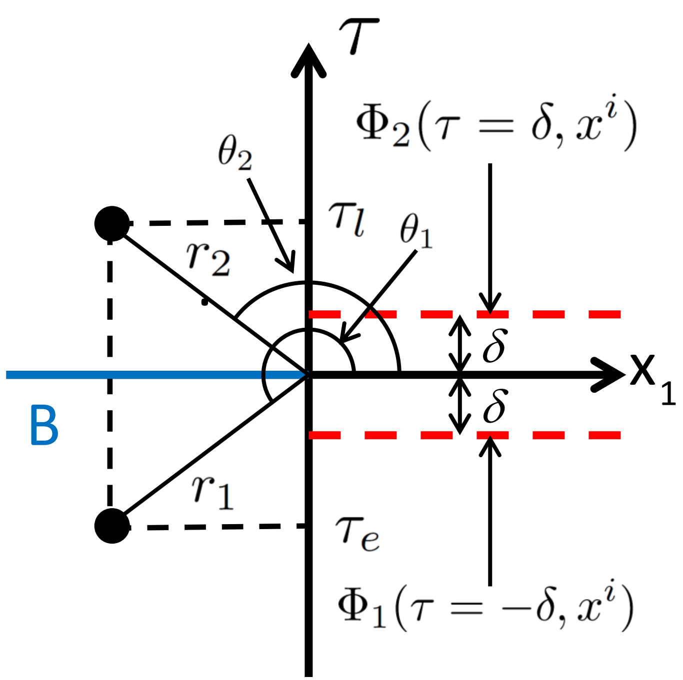


Replica Method

In d+1 dim. Euclidean space, the reduced density matrix is given by

$$\hat{\rho}_A = \mathcal{N}^{-2} tr_B \left[\mathcal{O}(\tau_e, -l, \mathbf{x}) \left|0\right\rangle \left\langle 0\right| \mathcal{O}^\dagger(\tau_l, -l, \mathbf{x}) \right]$$

$$\langle \Phi_1, \Phi_B | \mathcal{O}(\tau_e, -l, \mathbf{x}) | 0 \rangle = \langle \Phi_1, \Phi_B | \, e^{H\tau_e} \mathcal{O}(l, \mathbf{x}) e^{-H\tau_e} \left|0\right\rangle$$



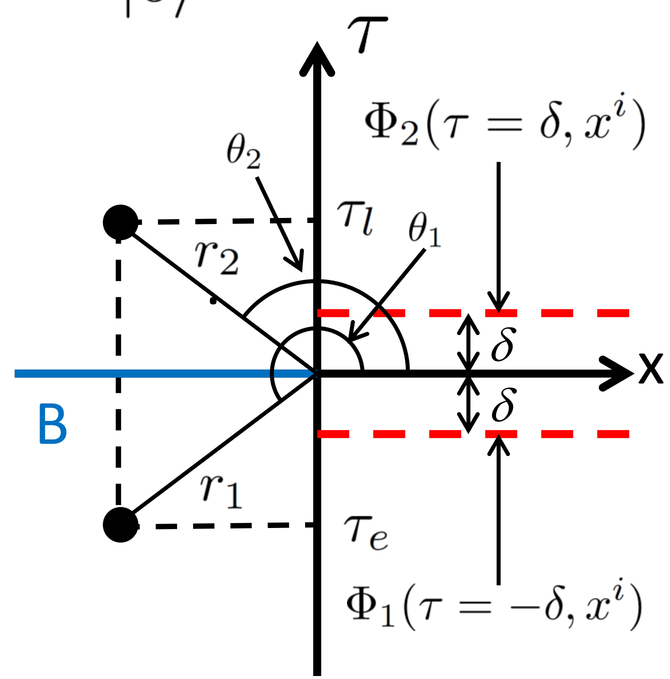
Replica Method

In d+1 dim. Euclidean space, the reduced density matrix is given by

$$\hat{\rho}_A = \mathcal{N}^{-2} tr_B \left[\mathcal{O}(\tau_e, -l, \mathbf{x}) \left|0\right\rangle \left\langle 0\right| \mathcal{O}^\dagger(\tau_l, -l, \mathbf{x}) \right]$$

$$\langle \Phi_1, \Phi_B | \mathcal{O}(\tau_e, -l, \mathbf{x}) | 0 \rangle = \langle \Phi_1, \Phi_B | \, e^{H\tau_e} \mathcal{O}(l, \mathbf{x}) e^{-H\tau_e} \, |0\rangle$$

$$= \langle \Phi_1, \Phi_B | \, e^{-H(0-\tau_e)} \mathcal{O}(l, \mathbf{x}) e^{-H(\tau_e + T)} \, |0\rangle$$



Replica Method

In d+1 dim. Euclidean space, the reduced density matrix is given by

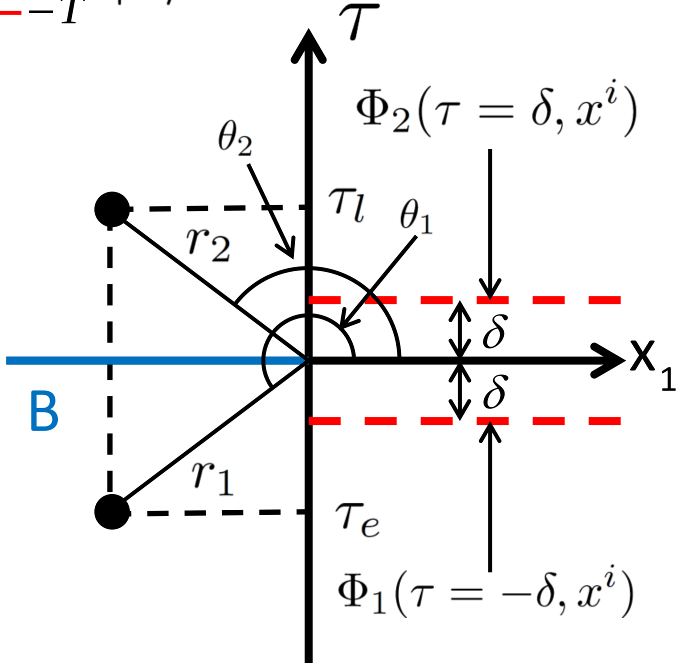
$$\hat{\rho}_A = \mathcal{N}^{-2} tr_B \left[\mathcal{O}(\tau_e, -l, \mathbf{x}) \left|0\right\rangle \left\langle 0\right| \mathcal{O}^\dagger(\tau_l, -l, \mathbf{x}) \right]$$

$$\langle \Phi_1, \Phi_B | \mathcal{O}(\tau_e, -l, \mathbf{x}) | 0 \rangle = \langle \Phi_1, \Phi_B | \, e^{H\tau_e} \mathcal{O}(l, \mathbf{x}) e^{-H\tau_e} \left|0\right\rangle$$

$$= \langle \Phi_1, \Phi_B | \, e^{-H(0-\tau_e)} \mathcal{O}(l, \mathbf{x}) e^{-H(\tau_e + T)} \left|0\right\rangle$$

$0 \leftarrow \tau_e \qquad \tau_e \leftarrow -T$

Introduce a lattice spacing δ



Replica Method

In d+1 dim. Euclidean space, the reduced density matrix is given by

$$\hat{\rho}_A = \mathcal{N}^{-2} tr_B \left[\mathcal{O}(\tau_e, -l, \mathbf{x}) \left|0\right\rangle \left\langle 0\right| \mathcal{O}^\dagger(\tau_l, -l, \mathbf{x}) \right]$$

$$\langle \Phi_1, \Phi_B | \mathcal{O}(\tau_e, -l, \mathbf{x}) | 0 \rangle = \langle \Phi_1, \Phi_B | \, e^{H\tau_e} \mathcal{O}(l, \mathbf{x}) e^{-H\tau_e} \, |0\rangle$$

$$= \langle \Phi_1, \Phi_B | \, \underset{0 \leftarrow \tau_e}{e^{-H(0-\tau_e)}} \mathcal{O}(l, \mathbf{x}) \underset{\tau_e \leftarrow -T}{e^{-H(\tau_e+T)}} \, |0\rangle$$

Introduce a lattice spacing δ

$\Rightarrow$

$$\int_{\Phi(t=-T)}^{\Phi(t=-\delta)} D\Phi \mathcal{O}(\tau_e, -l, \mathbf{x}) \delta \left(\Phi \left(\mathcal{T} = -\delta, x_1 > 0 \right) - \Phi_1 \right) \delta \left(\Phi \left(\mathcal{T} = -\delta, x_1 \leq 0 \right) - \Phi_B \right) e^{-S[\Phi]}$$

$$\langle\Phi_1,\Phi_B|\mathcal{O}(\tau_e,-l,\mathbf{x})|0\rangle$$

$$=\int_{\Phi(t=-T)}^{\Phi(t=-\delta)} D\Phi\mathcal{O}(\tau_e,-l,\mathbf{x})\delta\left(\Phi\left(t=-\delta,x_1>0\right)-\Phi_1\right)\delta\left(\Phi\left(t=-\delta,x_1\leq 0\right)-\Phi_B\right)e^{-S[\Phi]}$$

$$\langle 0|\mathcal{O}^\dagger(\tau_l,-l,\mathbf{x})|\Phi_2,\Phi_B\rangle$$

$$=\int_{\Phi(t=\delta)}^{\Phi(t=T)} D\Phi\mathcal{O}(\tau_l,-l,\mathbf{x})\delta\left(\Phi\left(t=\delta,x_1>0\right)-\Phi_1\right)\delta\left(\Phi\left(t=\delta,x_1\leq 0\right)-\Phi_B\right)e^{-S[\Phi]}$$

$$\hat{\rho}_A=\mathcal{N}^{-2}tr_B\left[\mathcal{O}(\tau_e,-l,\mathbf{x})\left|0\right\rangle\left\langle 0\right|\mathcal{O}^\dagger(\tau_l,-l,\mathbf{x})\right]$$

$$\langle \Phi_1, \Phi_B | \mathcal{O}(\tau_e, -l, \mathbf{x}) | 0 \rangle$$

$$= \int_{\Phi(t=-T)}^{\Phi(t=-\delta)} D\Phi \mathcal{O}(\tau_e, -l, \mathbf{x}) \delta\left(\Phi\left(t=-\delta, x_1 > 0\right) - \Phi_1\right) \delta\left(\Phi\left(t=-\delta, x_1 \leq 0\right) - \Phi_B\right) e^{-S[\Phi]}$$

$$\langle 0 | \mathcal{O}^\dagger(\tau_l, -l, \mathbf{x}) | \Phi_2, \Phi_B \rangle$$

$$= \int_{\Phi(t=\delta)}^{\Phi(t=T)} D\Phi \mathcal{O}(\tau_l, -l, \mathbf{x}) \delta\left(\Phi\left(t=\delta, x_1 > 0\right) - \Phi_1\right) \delta\left(\Phi\left(t=\delta, x_1 \leq 0\right) - \Phi_B\right) e^{-S[\Phi]}$$

$$\hat{\rho}_A = \mathcal{N}^{-2} tr_B \left[\mathcal{O}(\tau_e, -l, \mathbf{x}) |0\rangle \langle 0| \mathcal{O}^\dagger(\tau_l, -l, \mathbf{x})\right]$$

$$\boxed{\begin{aligned} &[\hat{\rho}_A]_{\Phi_1(x^i)\Phi_2(x^i)} \\ &\quad = (Z_1^{EX})^{-1} \int_{\Phi(-\infty, x^i)}^{\Phi(\infty, x^i)} D\Phi\ \mathcal{O}^\dagger(r_2, \theta_2,) \mathcal{O}(r_1, \theta_1) e^{-S[\Phi]} \delta\left(\Phi(-\delta, x^i) - \Phi_1(x^i)\right) \cdot \delta\left(\Phi(\delta, x^i) - \Phi_2(x^i)\right) \\ &\qquad\qquad Z_1^{EX} = \int_{\Phi(-\infty, x^i)}^{\Phi(\infty, x^i)} D\Phi\ \mathcal{O}^\dagger(r_2, \theta_2) \mathcal{O}(r_1, \theta_1) e^{-S(\Phi)} \end{aligned}}$$

Replica Method

In d+1 dim. Euclidean space, the reduced density matrix is given by

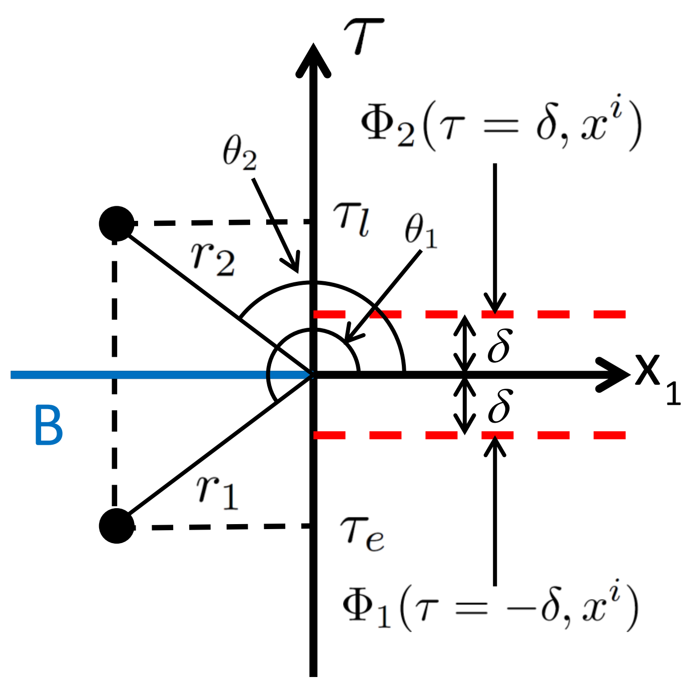
$$\hat{\rho}_A = \mathcal{N}^{-2} tr_B \left[\mathcal{O}(\tau_e, -l, \mathbf{x}) \left|0\right\rangle \left\langle 0\right| \mathcal{O}^\dagger(\tau_l, -l, \mathbf{x}) \right].$$

In the path-integral formalism,

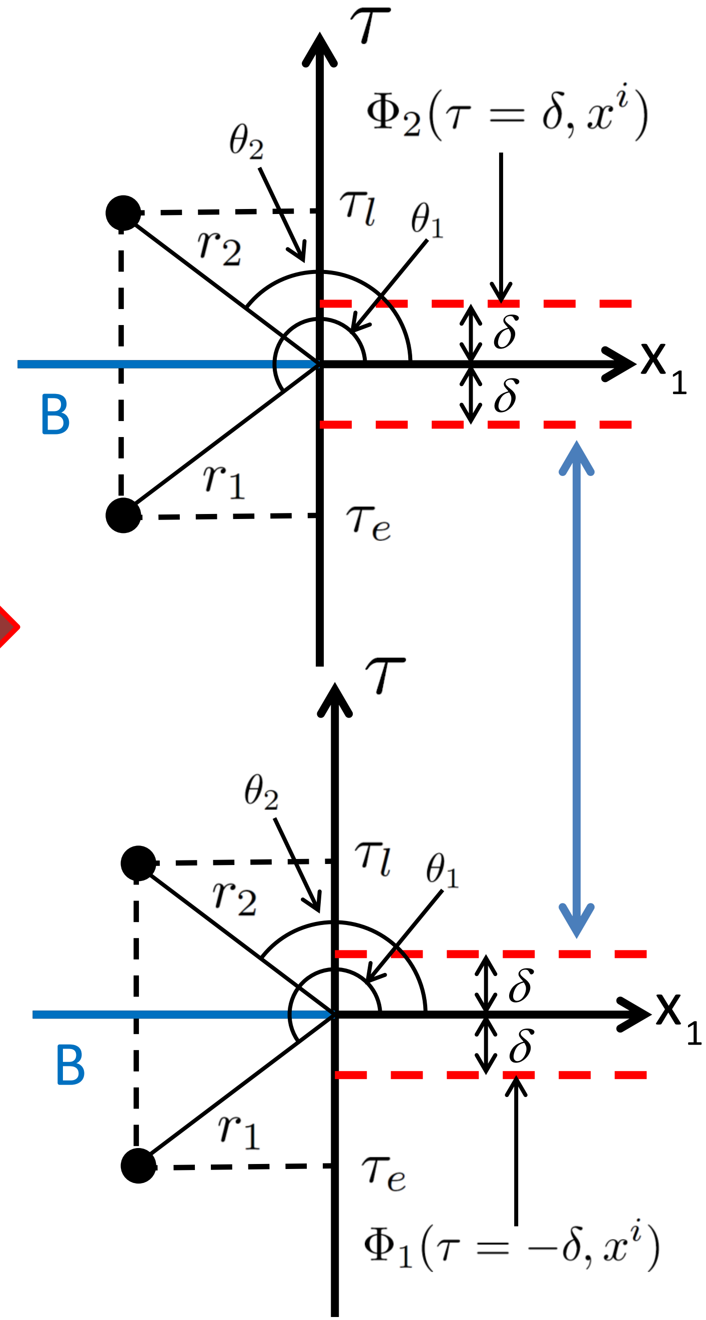
$$[\hat{\rho}_A]_{\Phi_1(x^i)\Phi_2(x^i)}$$
$$= (Z_1^{EX})^{-1} \int_{\Phi(-\infty,x^i)}^{\Phi(\infty,x^i)} D\Phi\ \mathcal{O}^\dagger(r_2,\theta_2,)\mathcal{O}(r_1,\theta_1)e^{-S[\Phi]}\delta\left(\Phi(-\delta,x^i)-\Phi_1(x^i)\right)\cdot\delta\left(\Phi(\delta,x^i)-\Phi_2(x^i)\right)$$

$$Z_1^{EX} = \int_{\Phi(-\infty,x^i)}^{\Phi(\infty,x^i)} D\Phi\ \mathcal{O}^\dagger(r_2,\theta_2)\mathcal{O}(r_1,\theta_1)e^{-S(\Phi)}$$

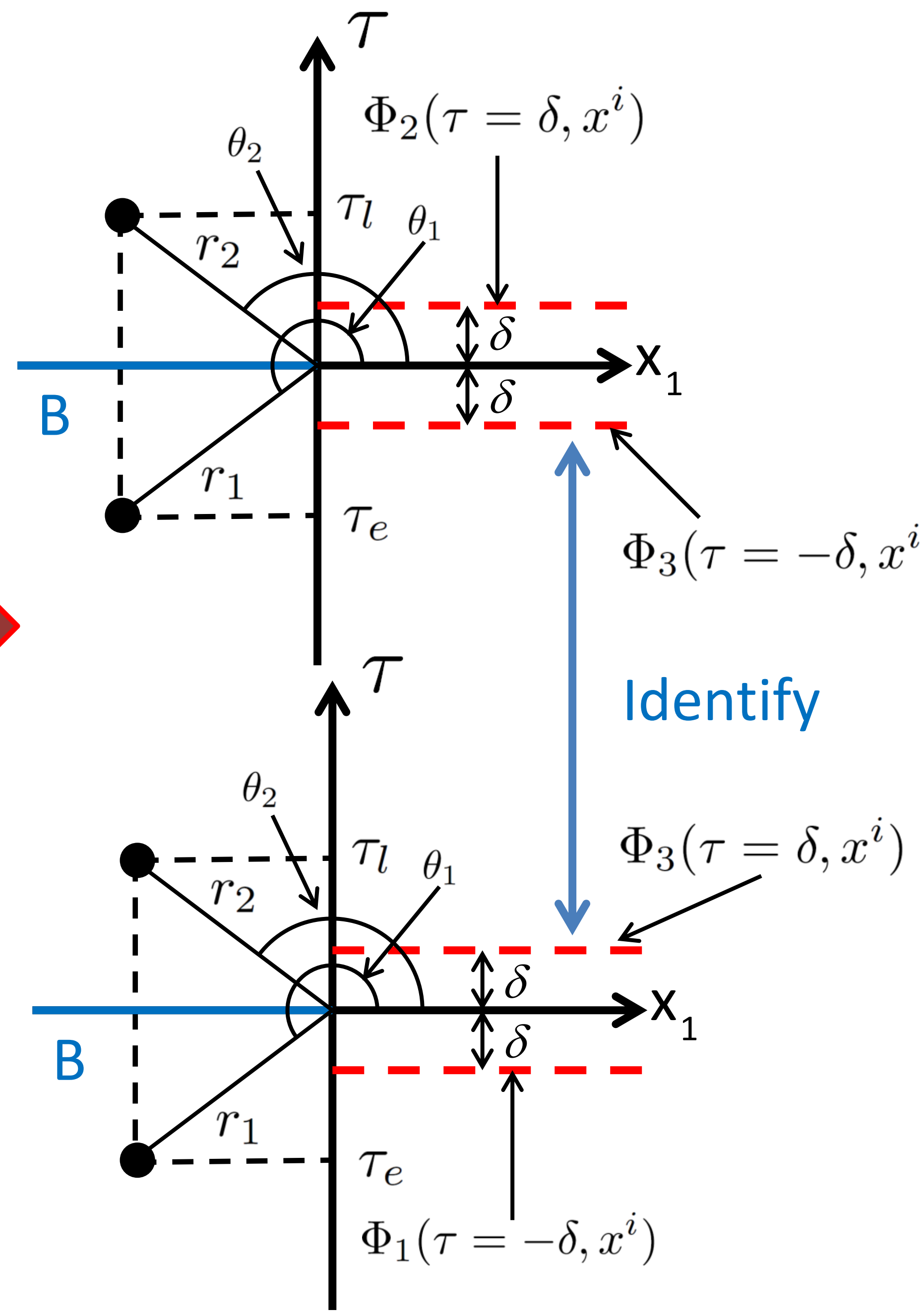
δ : a lattice spacing.



$$\left[\hat{\rho}_A^2\right]_{\Phi_1\Phi_2}$$

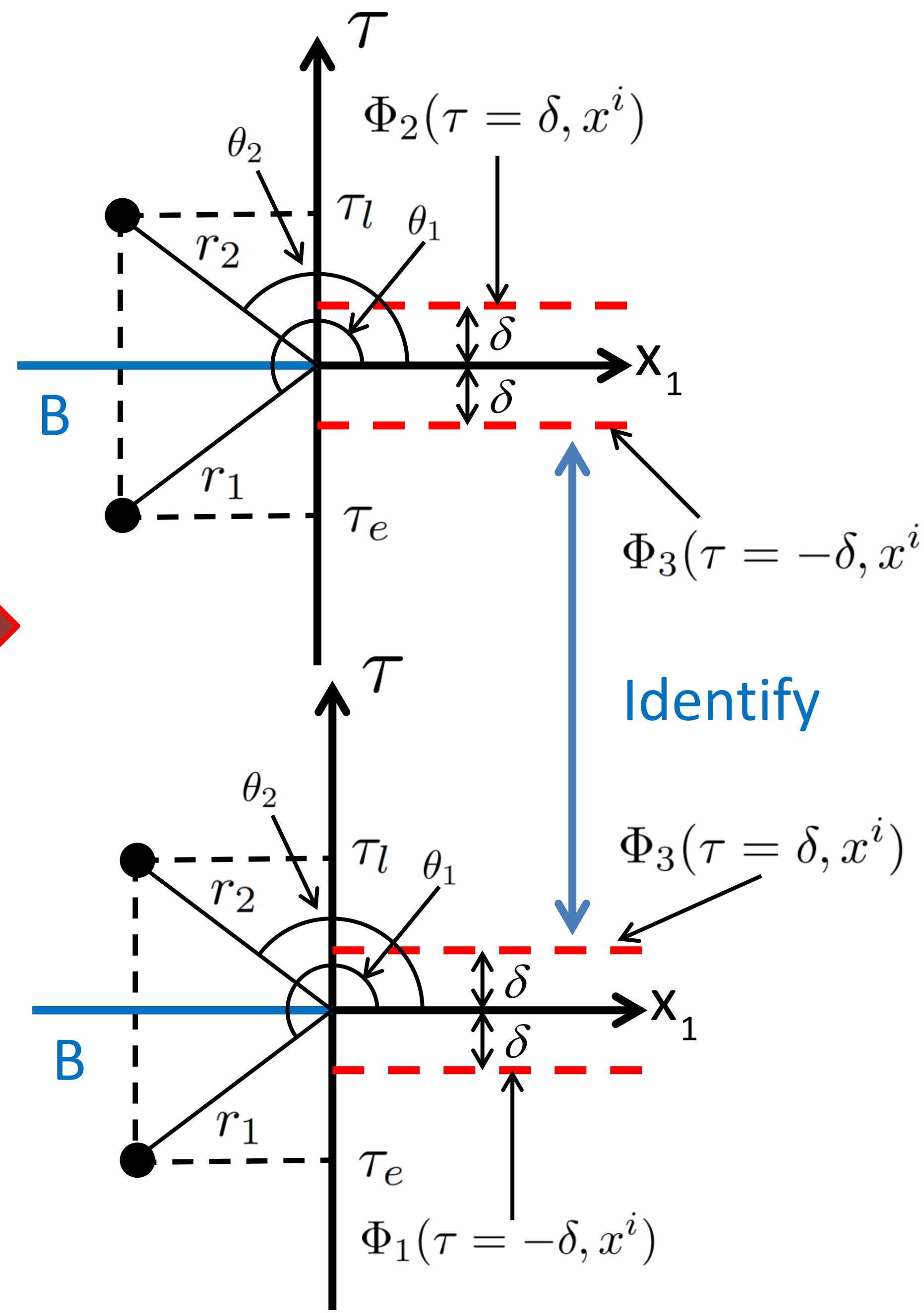


$$[\hat{\rho}_A]_{\Phi_1\Phi_3}\,[\hat{\rho}_A]_{\Phi_3\Phi_2}$$

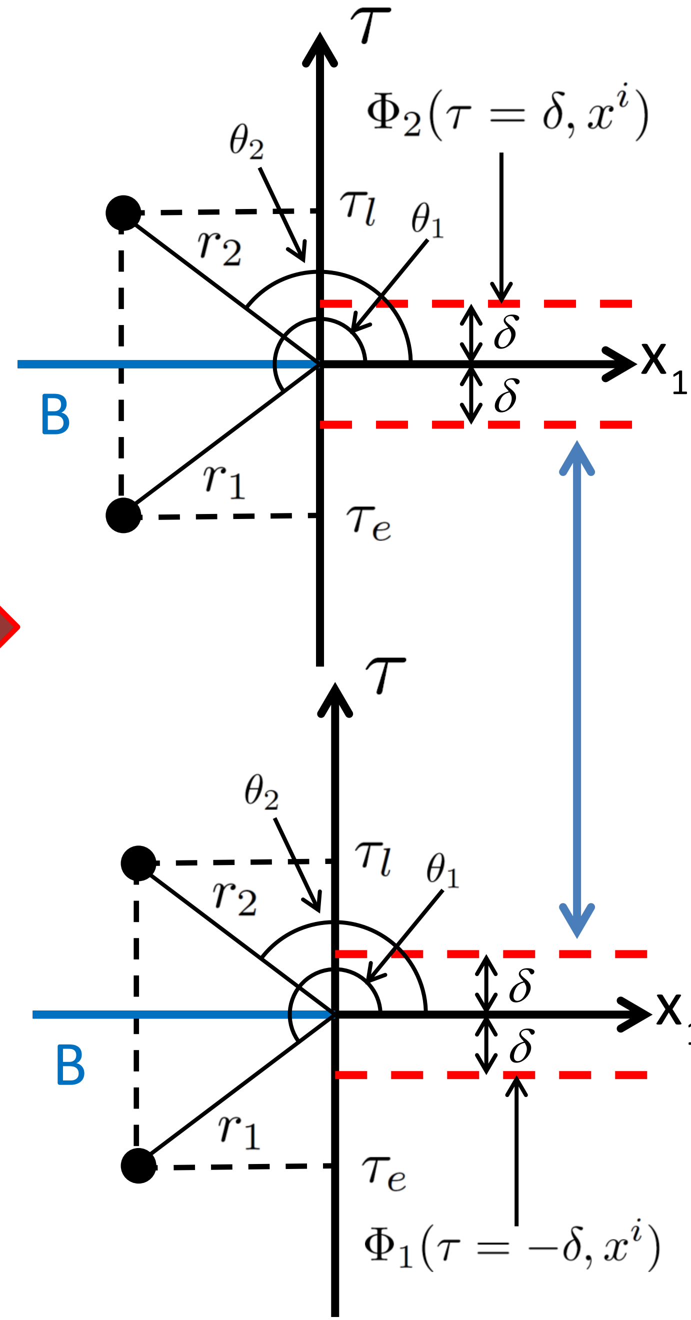


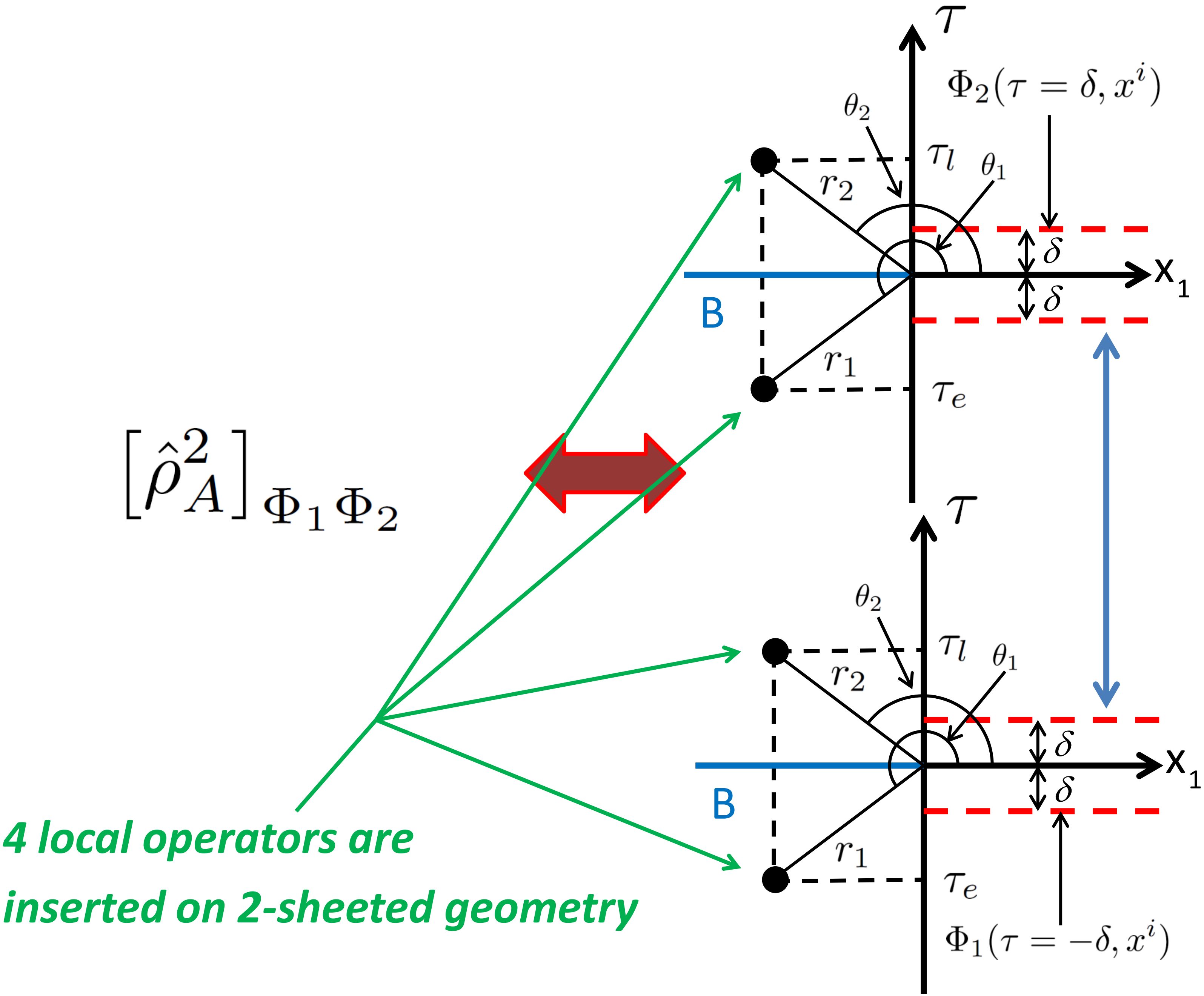
$$\sum_{\Phi_3} [\hat{\rho}_A]_{\Phi_1\Phi_3}\,[\hat{\rho}_A]_{\Phi_3\Phi_2}$$

$$\sum_{\Phi_3} \Longleftrightarrow \int_{\Phi_3}$$



$$\left[\hat{\rho}_A^2\right]_{\Phi_1 \Phi_2}$$





$\left[\hat{\rho}_A^2\right]_{\Phi_1 \Phi_2}$
4 local operators are
inserted on 2-sheeted geometry
τ
$\Phi_2(\tau = \delta, x^i)$
θ_2
τ_l
θ_1
r_2
δ
δ
x_1
B
r_1
τ_e
τ
θ_2
τ_l
θ_1
r_2
δ
δ
x_1
B
r_1
τ_e
$\Phi_1(\tau = -\delta, x^i)$

Replica Method

In d+1 dim. Euclidean space, the reduced density matrix is given by

$$\hat{\rho}_A = \mathcal{N}^{-2} tr_B \left[\mathcal{O}(\tau_e, -l, \mathbf{x}) \left|0\right\rangle \left\langle 0\right| \mathcal{O}^\dagger(\tau_l, -l, \mathbf{x})\right].$$

In the path-integral formalism,

$$[\hat{\rho}_A]_{\Phi_1(x^i)\Phi_2(x^i)}$$
$$= (Z_1^{EX})^{-1} \int_{\Phi(-\infty,x^i)}^{\Phi(\infty,x^i)} D\Phi\ \mathcal{O}^\dagger(r_2,\theta_2,)\mathcal{O}(r_1,\theta_1)e^{-S[\Phi]}\delta\left(\Phi(-\delta,x^i)-\Phi_1(x^i)\right)\cdot\delta\left(\Phi(\delta,x^i)-\Phi_2(x^i)\right)$$

$$Z_1^{EX} = \int_{\Phi(-\infty,x^i)}^{\Phi(\infty,x^i)} D\Phi\ \mathcal{O}^\dagger(r_2,\theta_2)\mathcal{O}(r_1,\theta_1)e^{-S(\Phi)}$$

δ : a lattice spacing.

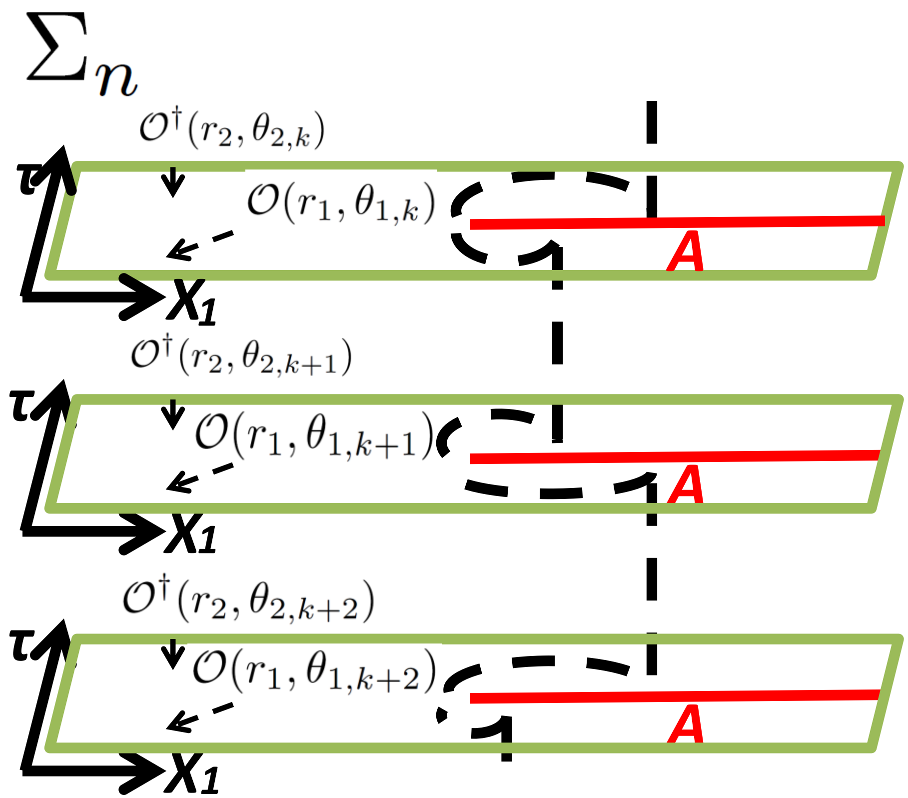
Periodicity along θ : ***2π→2nπ***

$$tr_A\hat{\rho}_A^n = \frac{Z_n^{EX}}{(Z_1^{EX})^n},$$
$$= \left(Z_1^{EX}\right)^{-n} \int_{\Phi(-\infty,x^i)}^{\Phi(\infty,x^i)} D\Phi\ \mathcal{O}^\dagger(r_2,\theta_2)\mathcal{O}(r_1,\theta_1)\cdots\mathcal{O}^\dagger(r_2,\theta_{2,k})\mathcal{O}(r_1,\theta_{1,k})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})e^{-S[\Phi]}$$

$$\boxed{\Delta S_A^{(n)} = S_A^{(n)Ex} - S_A^{(n)G},}$$

$$S_A^{(n)Ex} = \frac{1}{1-n} \log \left[\frac{\int D\Phi \mathcal{O}^\dagger(r_1, \theta_{1,1}) \mathcal{O}(r_2, \theta_{2,1}) \cdots \mathcal{O}^\dagger(r_1, \theta_{1,n}) \mathcal{O}(r_2, \theta_{2,n}) e^{-S_n}}{\left(\int D\Phi \mathcal{O}^\dagger(r_1, \theta_{1,1}) \mathcal{O}(r_2, \theta_{2,1}) e^{-S} \right)^n} \right]$$

$$S_A^{(n)G} = \frac{1}{1-n} \log \left[\frac{Z_n}{Z_1^n} \right]$$



Z_n : The partition function on Σ_n

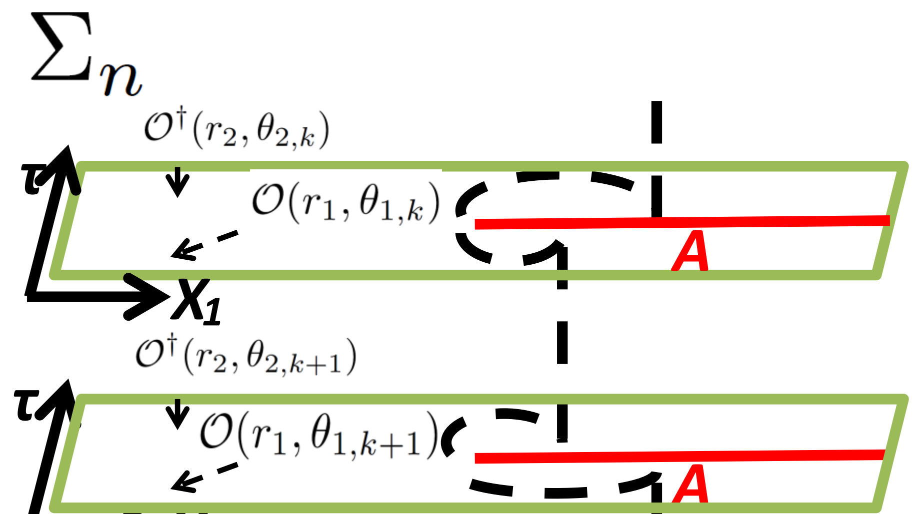
Z_1 : The partition function on Σ_1

Flat Space

$$\boxed{\Delta S_A^{(n)} = S_A^{(n)Ex} - S_A^{(n)G};}$$

$$S_A^{(n)Ex} = \frac{1}{1-n} \log \left[\frac{\int D\Phi \mathcal{O}^\dagger(r_1,\theta_{1,1})\mathcal{O}(r_2,\theta_{2,1}) \cdots \mathcal{O}^\dagger(r_1,\theta_{1,n})\mathcal{O}(r_2,\theta_{2,n}) e^{-S_n}}{\left(\int D\Phi \mathcal{O}^\dagger(r_1,\theta_{1,1})\mathcal{O}(r_2,\theta_{2,1}) e^{-S}\right)^n} \right]$$

$$S_A^{(n)G} = \frac{1}{1-n} \log \left[\frac{Z_n}{Z_1^n} \right]$$

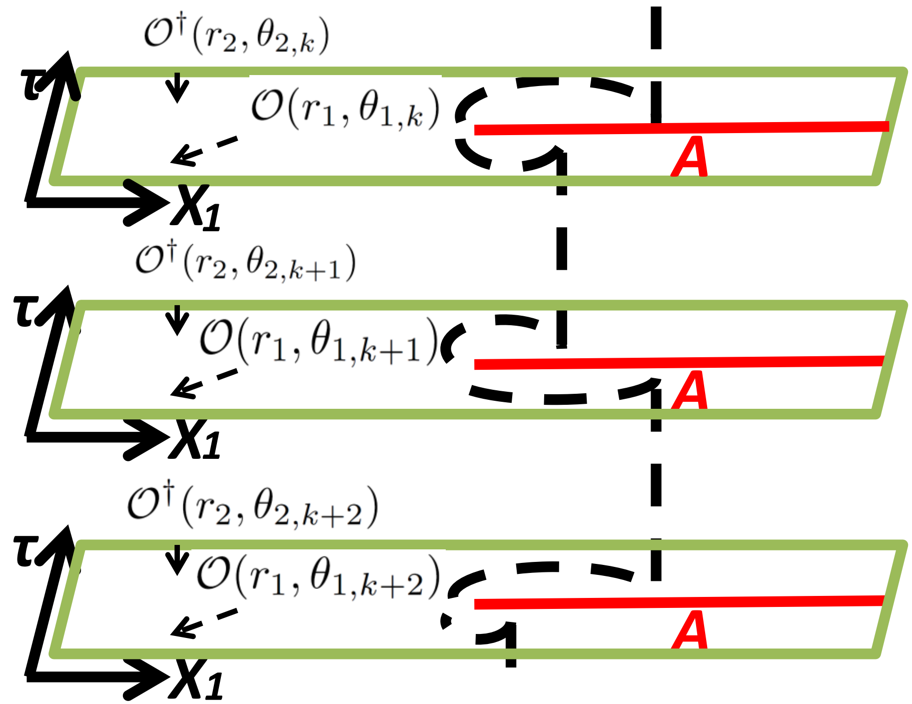


Z_n : The partition function on Σ_n

$$\Delta S_A^{(n)} = \frac{1}{1-n} \log \left[\frac{\int D\Phi \mathcal{O}^\dagger\left(r_1,\theta_{1,1}\right)\mathcal{O}\left(r_2,\theta_{2,1}\right) \cdots \mathcal{O}\left(r_1,\theta_{1,n}\right)^\dagger \mathcal{O}\left(r_2,\theta_{2,n}\right)}{Z_n} \right]$$

$$- \frac{1}{1-n} \log \left[\frac{\left(\int D\Phi \mathcal{O}^\dagger(r_1,\theta_{1,1})\mathcal{O}(r_2,\theta_{2,1})\right)^n}{Z_1^n} \right]$$

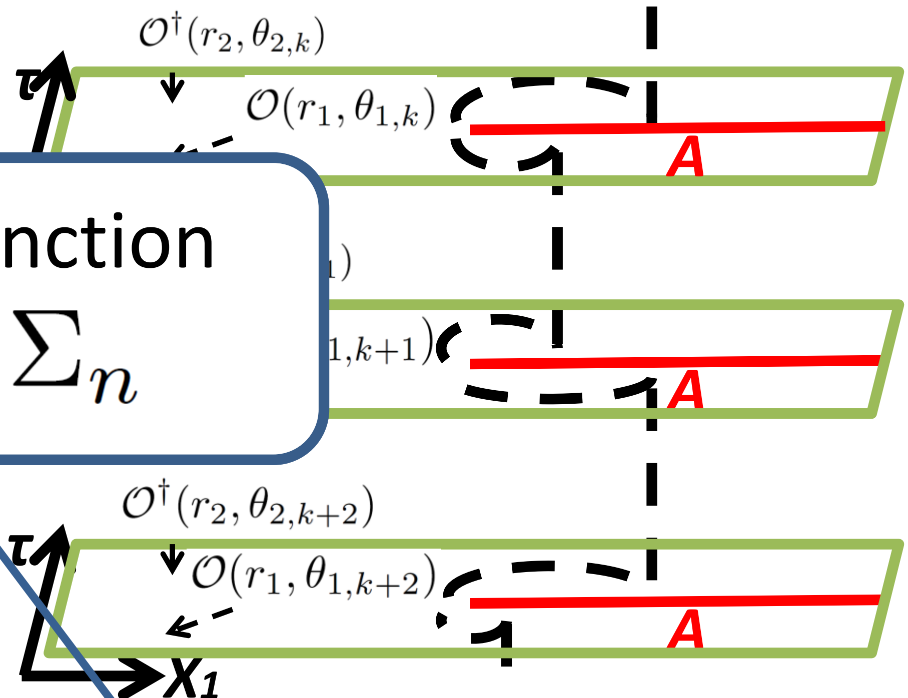
Replica Method



$$\theta_{1,i} = \theta_1 + 2\pi(i-1)$$
$$\theta_{2,i} = \theta_2 + 2\pi(i-1)$$

$$\Delta S_A^{(n)} = \frac{1}{1-n}\left(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\right).$$

Replica Method

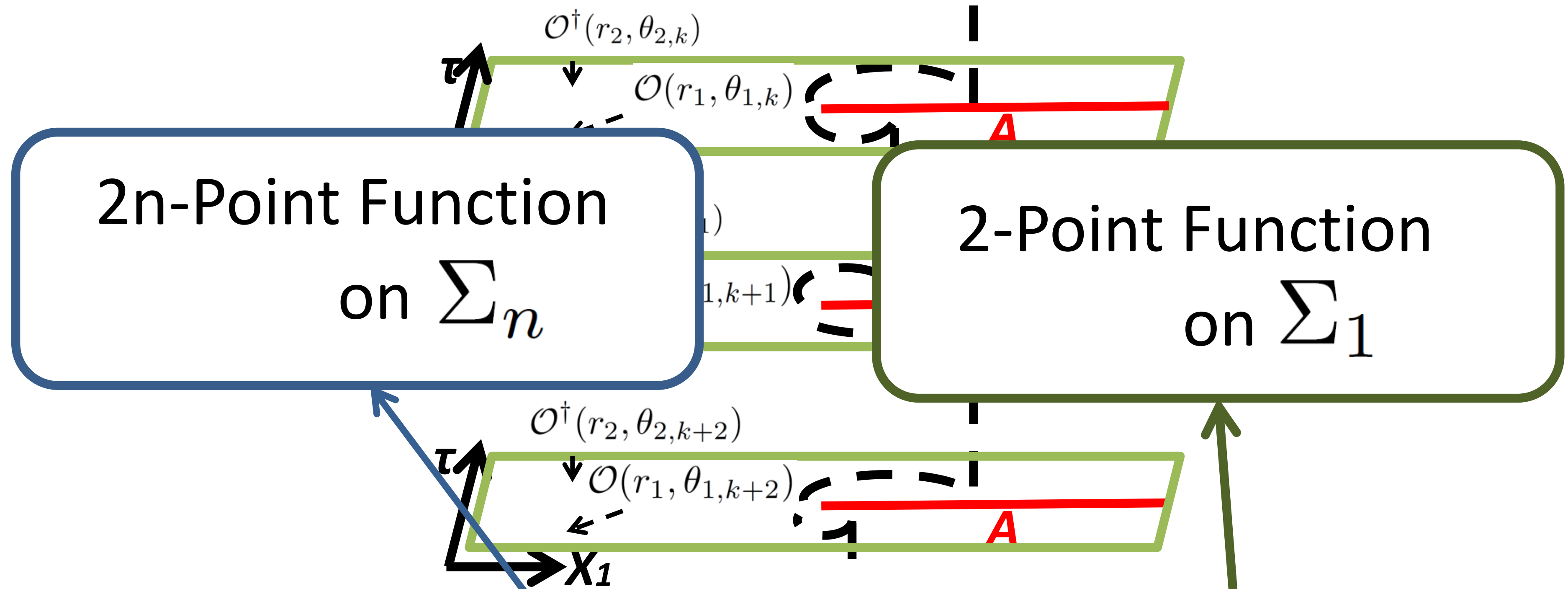


2n-Point Function on Σ_n

$$\Delta S_A^{(n)} = \qquad \theta_{1,i} = \theta_1 + 2\pi(i-1), \quad \theta_{2,i} = \theta_2 + 2\pi(i-1)$$

$$\frac{1}{1-n}\Big(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\Big).$$

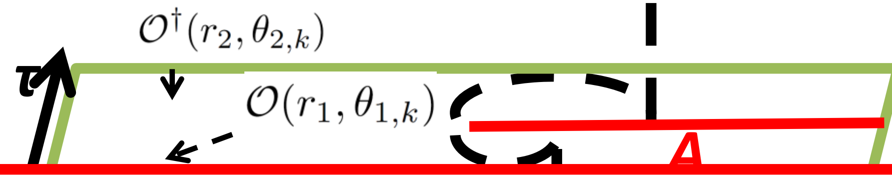
Replica Method



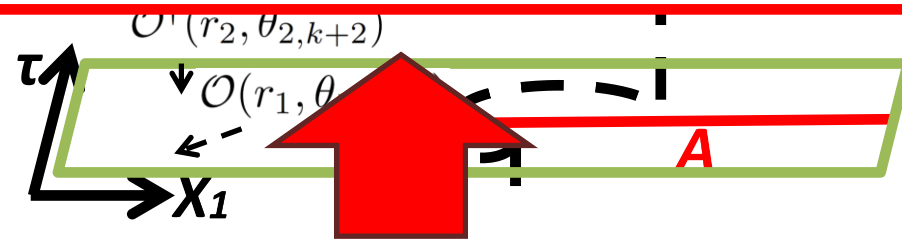
$$\theta_{1,i} = \theta_1 + 2\pi(i-1)$$
$$\theta_{2,i} = \theta_2 + 2\pi(i-1)$$

$$\Delta S_A^{(n)} = \frac{1}{1-n}\left(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\right).$$

Replica Method



This formula holds for any local operators in general QFT in any dimensions.



$$\Delta S_A^{(n)} =$$

$$\theta_{1,i} = \theta_1 + 2\pi(i-1)$$
$$\theta_{2,i} = \theta_2 + 2\pi(i-1)$$

$$\frac{1}{1-n}\left(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\right).$$

Comment on Analytic Continuation

Analytic continuation:

$$\tau_e = -\epsilon - it_1,$$

$$\tau_l = \epsilon - it_1.$$

Comment on Analytic Continuation

Analytic continuation:

$$\tau_e = -\epsilon - it_1,$$

$$\tau_l = \epsilon - it_1.$$

$$\hat{\rho} = \mathcal{N}^{-2} e^{-iHt_1} \underbrace{e^{-\epsilon H}}_{} \mathcal{O}(-l, \mathbf{x}_1) \left|0\right\rangle \left\langle 0\right| \mathcal{O}^{\dagger}(-l, \mathbf{x}_1) e^{-\epsilon H} e^{iHt_1}$$

Suppress high energy mode!

Comment on Analytic Continuation

Analytic continuation:
$$\tau_e = -\epsilon - it_1,$$
$$\tau_l = \epsilon - it_1.$$

$$\hat{\rho} = \mathcal{N}^{-2} e^{-iHt_1} \underline{e^{-\epsilon H}} \mathcal{O}(-l, \mathbf{x}_1) |0\rangle \langle 0| \mathcal{O}^\dagger(-l, \mathbf{x}_1) e^{-\epsilon H} e^{iHt_1}$$

Suppress high energy mode!

For example :2d CFT, $\mathcal{O}$ =a primary operator subsystem= finite interval

Internal energy in A $\sim \dfrac{\Delta_O}{\epsilon}$

Comment on Analytic Continuation

Analytic continuation:

$$\tau_e = -\epsilon - it_1,$$

$$\tau_l = \epsilon - it_1.$$

$$\hat{\rho} = \mathcal{N}^{-2} e^{-iHt_1} \underbrace{e^{-\epsilon H}} \mathcal{O}(-l, \mathbf{x}_1) \left|0\right\rangle \left\langle 0\right| \mathcal{O}^\dagger(-l, \mathbf{x}_1) e^{-\epsilon H} e^{iHt_1}$$

Suppress high energy mode!

For example :2d CFT, $\mathcal{O}$ =a primary operator subsystem= finite interval

Internal energy in A $\sim \dfrac{\Delta_O}{\epsilon}$ ***Conformal dim.***

Motivation

We study the properties of REE

for locally excited state

when the size of subsystem is infinite.

***Researches*:**

1. The time dependence of REE

2. The late time value of REE

Example

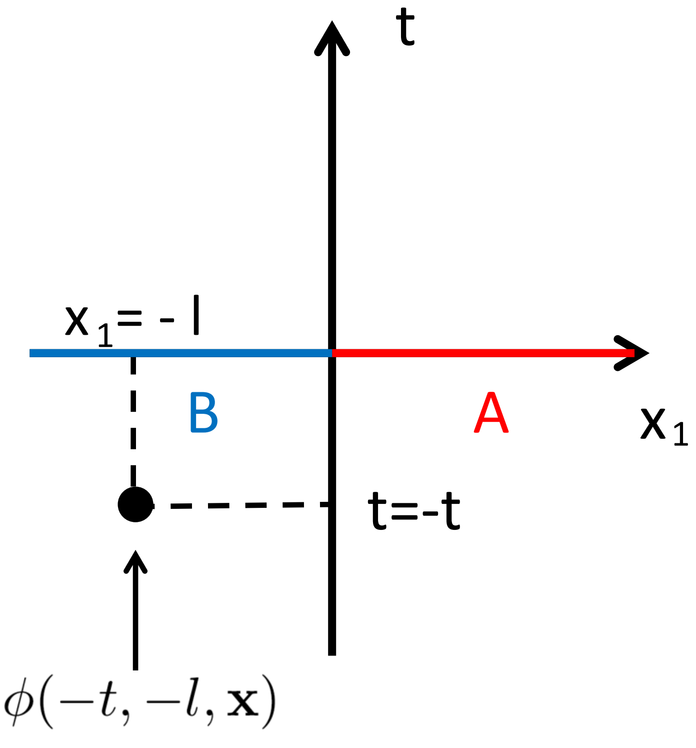
We consider ***free massless scalar*** field theory in ***d+1 dim.***

Especially, we focus on that in ***4 dim.***

We act a local operator $\phi(-t,-l,\mathbf{x})$ on the ground state:

$$|\Psi\rangle = \mathcal{N}^{-1}\phi(-t,-l,\mathbf{x})\,|0\rangle\ .$$

We measure the (Renyi) entanglement entropies at t=0.



⇨ ***Time evolution!!***

Example

Let's compute $\Delta S_A^{(2)}$ for $|\Psi\rangle = \mathcal{N}^{-1}\phi(-t,-l,\mathbf{x})\,|0\rangle$ in 4-dimensional free massless scalar field theory.

$$\boxed{\Delta S_A^{(2)} = -\log\left[\frac{\langle\phi(r_1,\theta_1)\phi(r_2,\theta_2)\phi(r_1,\theta_1+2\pi)\phi(r_2,\theta_2+2\pi)\rangle_{\Sigma_2}}{\langle\phi(r_1,\theta_1)\phi(r_2,\theta_2)\rangle_{\Sigma_1}^2}\right]}$$

Green function:

$$\langle\phi\left(r,\theta,\mathbf{x}\right)\phi\left(s,\theta',\mathbf{x}\right)\rangle = \frac{1}{8\pi^2\left(r+s\right)\left(r+s-2\sqrt{rs}\cos\left(\frac{\theta-\theta'}{2}\right)\right)}$$

Example

$$\Delta S_A^{(2)} = -\log\left[\frac{\langle\phi(r_1,\theta_1)\phi(r_2,\theta_2)\phi(r_1,\theta_1+2\pi)\phi(r_2,\theta_2+2\pi)\rangle_{\Sigma_2}}{\langle\phi(r_1,\theta_1)\phi(r_2,\theta_2)\rangle_{\Sigma_1}^2}\right]$$

Green function:

$$\langle\phi\left(r,\theta,\mathbf{x}\right)\phi\left(s,\theta',\mathbf{x}\right)\rangle = \frac{1}{8\pi^2\left(r+s\right)\left(r+s-2\sqrt{rs}\cos\left(\frac{\theta-\theta'}{2}\right)\right)}$$

We compute $\Delta S_A^{(2)}$ by using Green function.

After that, we perform analytic continuation to real time.

Example

$$\Delta S_A^{(2)} = -\log\left[\frac{\langle \phi(r_1,\theta_1)\phi(r_2,\theta_2)\phi(r_1,\theta_1+2\pi)\phi(r_2,\theta_2+2\pi)\rangle_{\Sigma_2}}{\langle \phi(r_1,\theta_1)\phi(r_2,\theta_2)\rangle_{\Sigma_1}^2}\right]$$

Green f

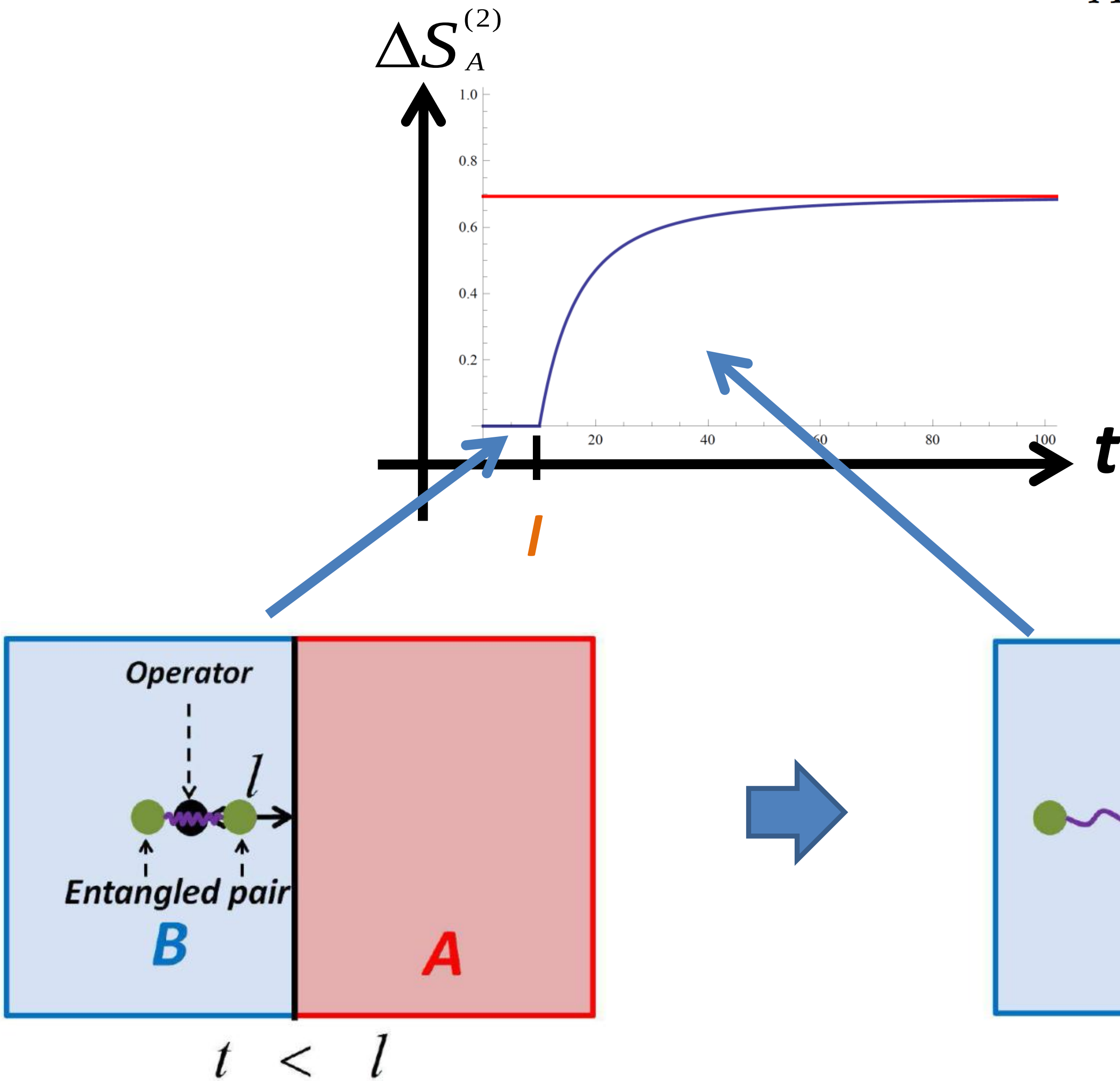
$\langle \phi(r,\theta,\mathbf{x})$ … $\cos\left(\frac{\theta-\theta'}{2}\right)\Big)$

$$\Delta S_A^{(2)} = \log\left[\frac{2t^2}{t^2+l^2}\right]$$

We compute $\Delta S_A^{(2)}$ by using Green function.

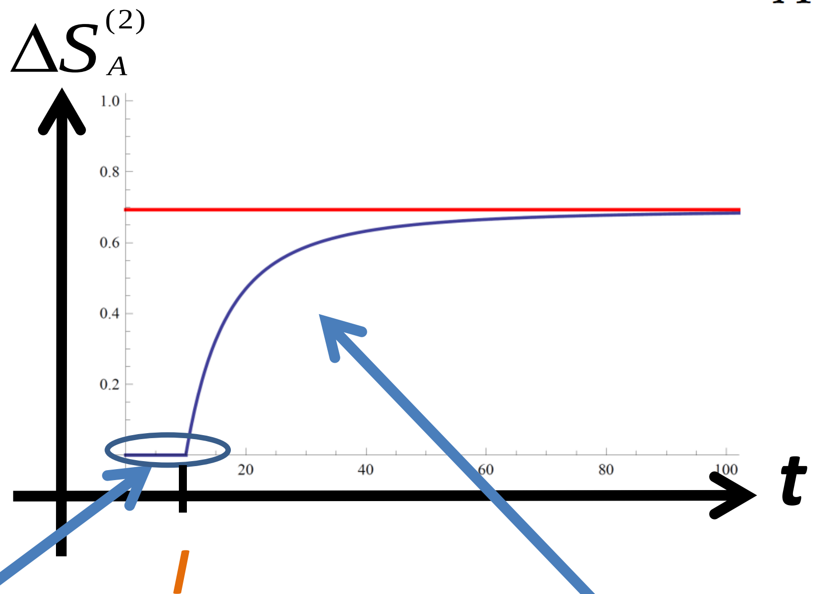
After that, we perform analytic continuation to real time.

Time Evolution of $\Delta S_A^{(2)}$

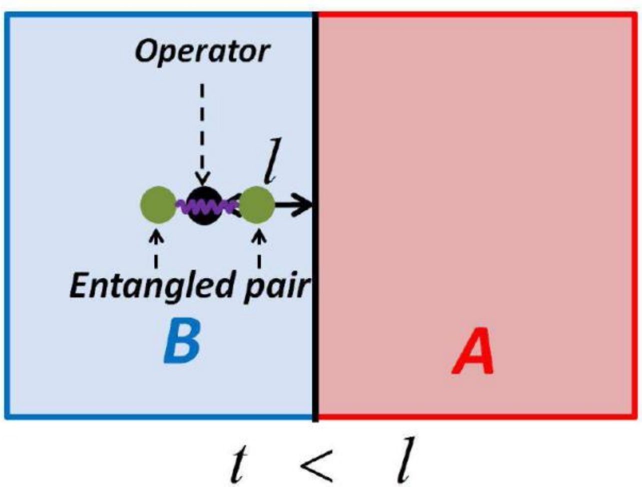


$$\Delta S_A^{(2)} = \log\left[\frac{2t^2}{t^2 + l^2}\right]$$

Time Evolution of $\Delta S_A^{(2)}$



$$\Delta S_A^{(2)} = \log\left[\frac{2t^2}{t^2 + l^2}\right]$$

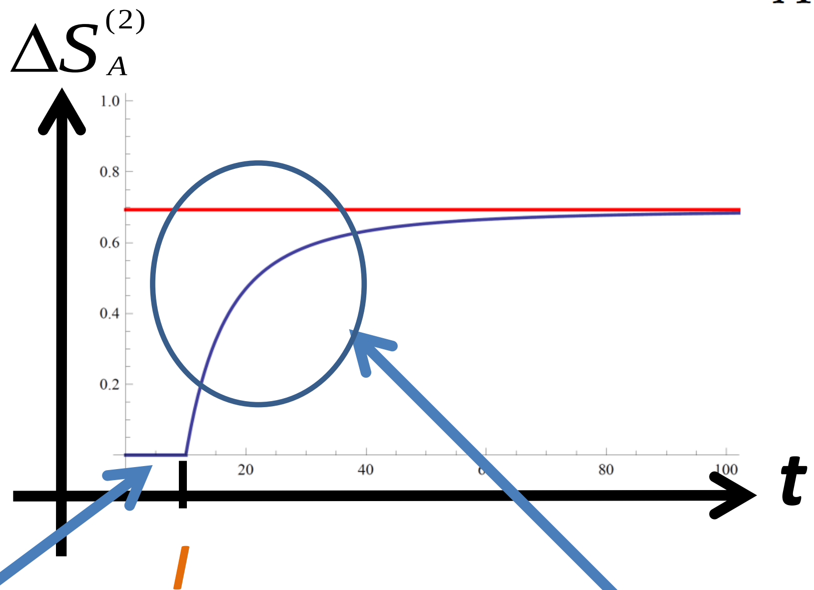


$t < l$

An entangled pair appears.

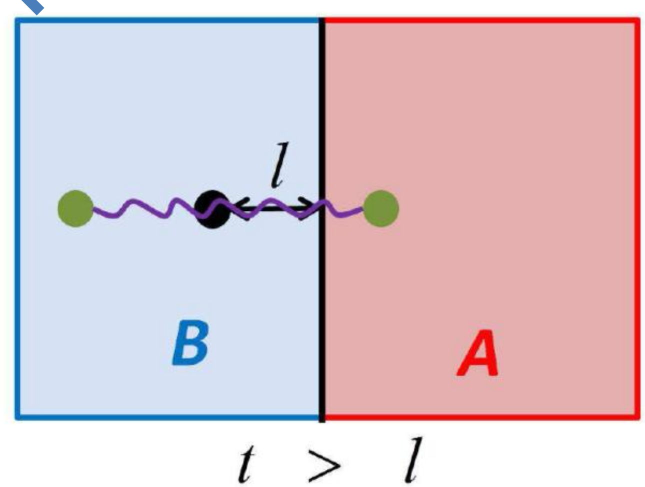
Each of pair is included in the region B.

Time Evolution of $\Delta S_A^{(2)}$

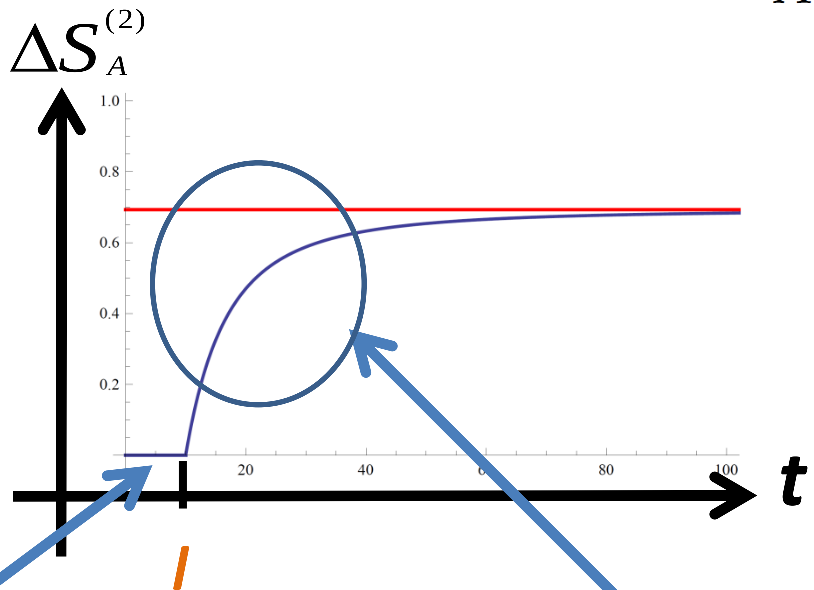


$t \geq l$

In this region, two quanta is included in A and B respectively .



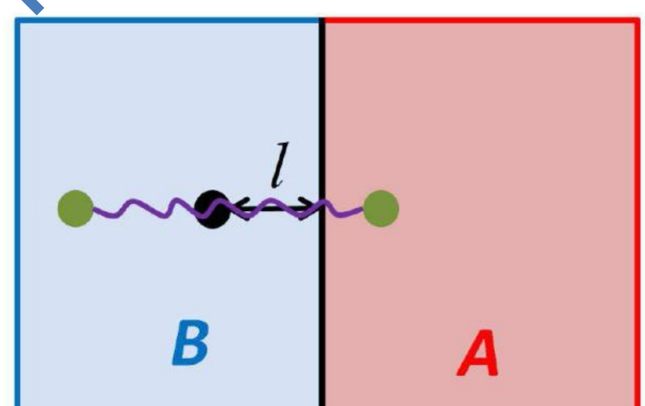
Time Evolution of $\Delta S_A^{(2)}$



$$\Delta S_A^{(2)} = \log\left[\frac{2t^2}{t^2 + l^2}\right]$$

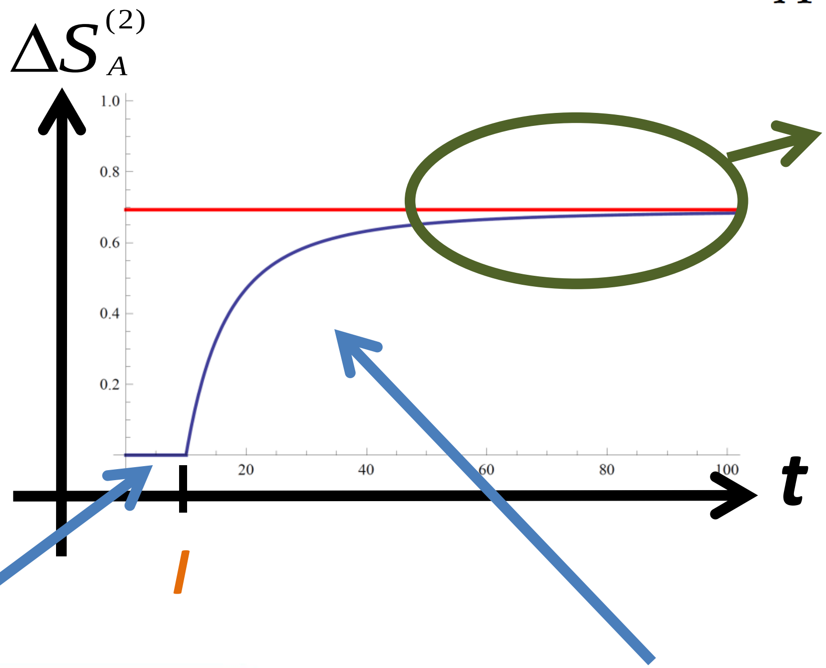
$t \geq l$

In this region, two quanta is included in A and B respectively .



Entanglement between quanta can contribute to $\Delta S_A^{(2)}$.

Time Evolution of $\Delta S_A^{(2)}$



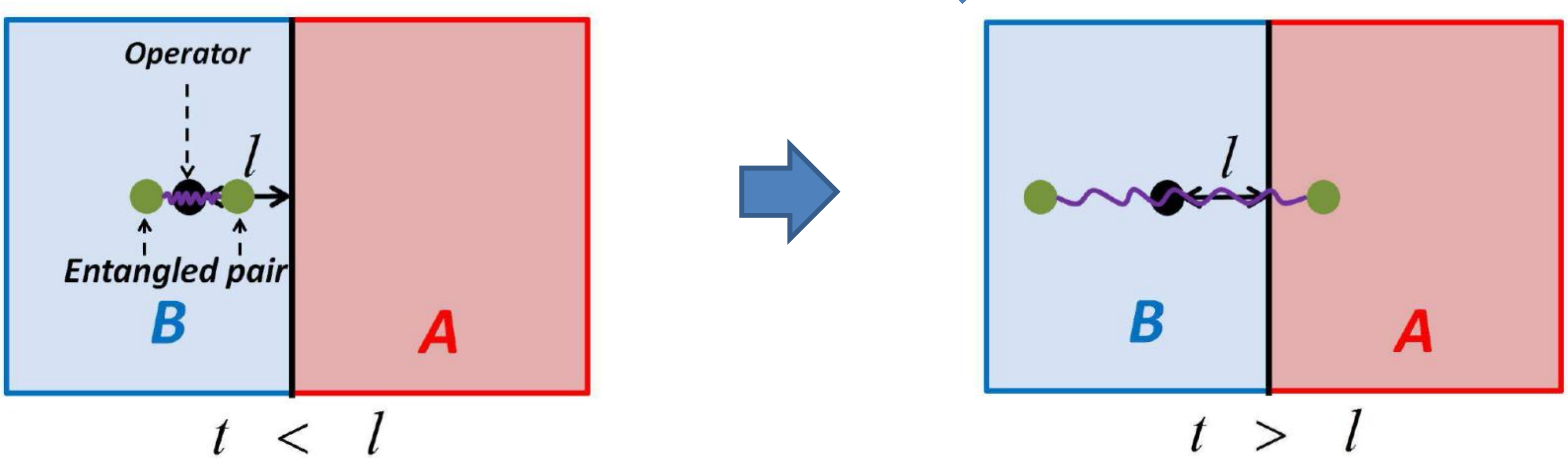
Subsystem

= a half of the

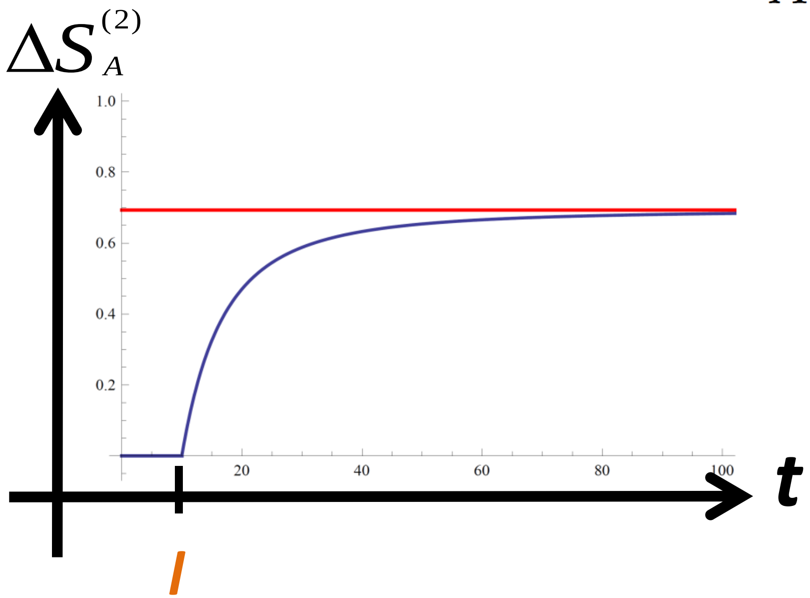
total space

$\Delta S_A^{(2)}$ approaches

Constant!!



Time Evolution of $\Delta S_A^{(2)}$



$\Delta S_A^{(2)}$ for $|\Psi\rangle = \mathcal{N}^{-1}\phi(-t,-l,\mathbf{x})\,|0\rangle$ approaches ***constants! !***

(log2)

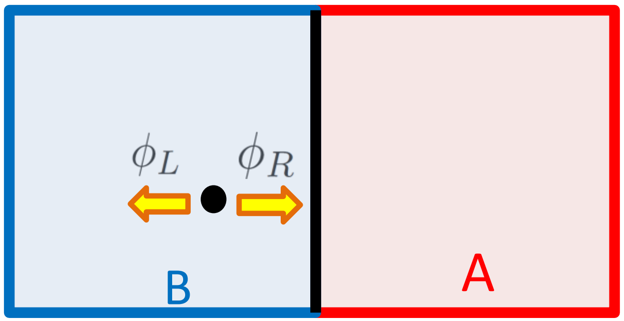
We call them ***the (Renyi) entanglement entropies of operators*****.**

Entangled Pair Interpretation

We derive $\Delta S^{(n)}_{A,k}$ for $|\Psi\rangle = \mathcal{N}^{-1} : \phi^k(-t,-l,\mathbf{x}) : |0\rangle$ from the entangled pair interpretation.

We decompose ϕ into the left moving mode and the right moving mode,

$$\boxed{\phi = \phi_L + \phi_R}$$



Generalize

In two dimensional CFT, we decompose ϕ into the left moving mode and right moving mode,

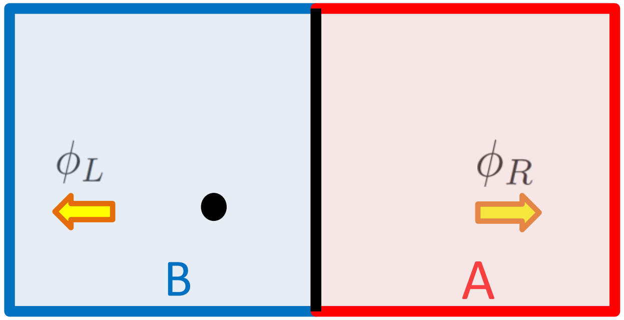
$$\phi(z,\bar{z}) = \phi_L(z) + \phi_R(\bar{z})$$

Entangled Pair Interpretation

We derive $\Delta S_{A,k}^{(n)}$ for $|\Psi\rangle = \mathcal{N}^{-1} : \phi^k(-t,-l,\mathbf{x}) : |0\rangle$ from the entangled pair interpretation.

We decompose ϕ into the left moving mode and the right moving mode,

$$\boxed{\phi = \phi_L + \phi_R}$$



At late time, the d o f in the region B can be identified with the d o f of left moving mode.

Entangled Pair Interpretation

Under this decomposition: $\boxed{\phi = \phi_L + \phi_R}$

$$\boxed{|\Psi\rangle = \mathcal{N}^{-1} : \phi^k(-t, -l, \mathbf{x}) : |0\rangle}$$

Tracing out the d.o.f in B $\Rightarrow$

$$\rho_A^f = 2^{-k}({}_kC_0\ ,\ {}_kC_1\ ,\ \cdots\ ,\ {}_kC_k)$$

Entangled Pair Interpretation

Under this decomposition: $\boxed{\phi = \phi_L + \phi_R}$

$$\boxed{|\Psi\rangle = \frac{1}{2^{\frac{k}{2}}} \sum_{m=0}^{k} \sqrt{{}_kC_m}\, |m\rangle_A \otimes |k-m\rangle_B \,.}$$

Tracing out

the d.o.f in B

$$\rho_A^f = 2^{-k}\left({}_kC_0 \ ,\ {}_kC_1 \ ,\ \cdots \ ,\ {}_kC_k\right)$$

Entangled Pair Interpretation

Under this decomposition: $\boxed{\phi = \phi_L + \phi_R}$

$|\Psi$

$$\Delta S_A^{(n)f} = \frac{1}{1-n} \log \left(\frac{1}{2^{nk}} \sum_{j=0}^{k} \left({}_kC_j\right)^n \right).$$

$$\Delta S_A = k \cdot \log 2 - \frac{1}{2^k} \sum_{j=0}^{k} {}_kC_j \log {}_kC_j\,.$$

Tr

the d.o.f in B ρ_A $=$ $({}_kC_0$, ${}_kC_1$, $\ldots$, ${}_kC_k)$

Entangled Pair Interpretation

Under this decomposition: $\boxed{\phi = \phi_L + \phi_R}$

$|\Psi$

$$\Delta S_A^{(n)f} = \frac{1}{1-n}\log\left(\frac{1}{2^{nk}}\sum_{j=0}^{k}\left({}_kC_j\right)^n\right).$$

$$\Delta S_A = k\cdot\log 2 - \frac{1}{2^k}\sum_{j=0}^{k}{}_kC_j\log{}_kC_j.$$

Tr

They agree with the results which we obtain by the Replica trick (See My paper!!).

Comments on Result

We defined ***the (Renyi) entanglement entropies of operators*** by the late time values of $\Delta S_A^{(n)}$.

The (Renyi) entanglement entropies of $:\phi^k:$ is given by

$$\boxed{\begin{aligned} \Delta S_A^{(n)f} &= \frac{1}{1-n}\log\left(\frac{1}{2^{nk}}\sum_{j=0}^{k}\left({}_kC_j\right)^n\right). \\ \Delta S_A &= k\cdot\log 2 - \frac{1}{2^k}\sum_{j=0}^{k} {}_kC_j \log {}_kC_j . \end{aligned}}$$

Generalize Results

We defined ***the (Renyi) entanglement entropies of operators*** by the late time values of $\Delta S_A^{(n)}$.

The (Renyi) entanglement entropies of specific operators ($: (\partial^m \phi)^k :$) which are composed of single species operator are given by

$$\Delta S_A^{(n)f} = \frac{1}{1-n} \log \left(\frac{1}{2^{nk}} \sum_{j=0}^{k} ({}_kC_j)^n \right).$$

$$\Delta S_A = k \cdot \log 2 - \frac{1}{2^k} \sum_{j=0}^{k} {}_kC_j \log {}_kC_j.$$

Generalize Results

We defined ***the (Renyi) entanglement entropies of operators*** by the late time values of $\Delta S_A^{(n)}$.

The (Renyi) entanglement entropies of specific operators ($:(\partial^m\phi)^k:$) which are composed of single species operator are given by

$$\Delta S_A^{(n)f} = \frac{1}{1-n}\log\left(\frac{1}{2^{nk}}\sum_{j=0}^{k}\left({}_kC_j\right)^n\right).$$

$$\Delta S_A = k\cdot\log 2 - \frac{1}{2^k}\sum_{j=0}^{k} {}_kC_j \log {}_kC_j .$$

for ***any dimension***.

They characterize the local operators from the viewpoint of quantum entanglement!!

Generalize Results

We defined ***the (Renyi) entanglement entropies of operators*** by the late time values of $\Delta S_A^{(n)}$.

The (Renyi) entanglement entropies of specific operators ($:(\partial^m\phi)^k:$) which are composed of single species operator are given by

$$\Delta S_A^{(n)f} = \frac{1}{1-n}\log\left(\frac{1}{2^{nk}}\sum_{j=0}^{k}\left({}_kC_j\right)^n\right).$$

$$\Delta S_A = k\cdot\log 2 - \frac{1}{2^k}\sum_{j=0}^{k}{}_kC_j\log{}_kC_j.$$

for ***any dimension.***

Large k, $\Delta S_A^{(n)} \sim \frac{1}{2}\log k$

Other type operators

An excited state is defined by acting a local operator $:\phi\partial_r\phi:$ on the ground state.

$:\phi\partial_r\phi:$ is constructed of multispecies operators ϕ and $\partial_r\phi$.

The second (Renyi) entanglement entropy of $:\phi\partial_r\phi:$ is given by

	Dimension of Spacetime	Entanglement of Operators
Second Rényi Entropy	4	$\log\left[\frac{100}{37}\right]$
	6	$\log\left[\frac{324}{121}\right]$
	8	$\log\left[\frac{676}{253}\right]$

Other type operators

An excited state is defined by acting a local operator $:\phi\partial_r\phi:$ on the ground state.

$:\phi\partial_r\phi:$ is constructed of multispecies operators ϕ and $\partial_r\phi$.

The second (Renyi) entanglement entropy of $:\phi\partial_r\phi:$ is given by

	Dimension of Spacetime	Entanglement of Operators
Second Rényi Entropy	4	$\log\left[\frac{100}{37}\right]$
	6	$\log\left[\frac{324}{121}\right]$
	8	$\log\left[\frac{676}{253}\right]$

➡ $:\phi\partial_r\phi:$ depends on spacetime dimension.

➡ Another formula for $:\phi\partial_r\phi:$?

Sum rule

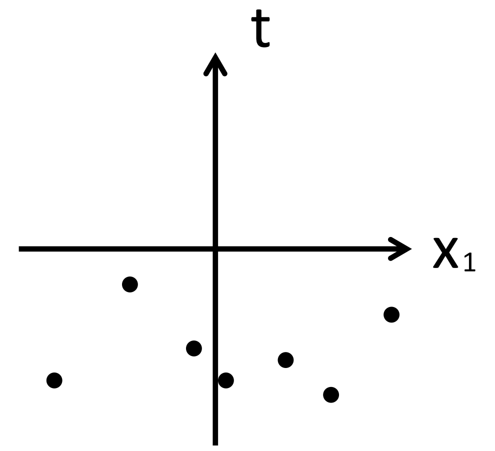
We acts various local operators on the ground state.

A locally excited state is given by $|\psi\rangle = \mathcal{N}^{-1}\mathcal{T}\prod_{i}^{k}\mathcal{O}^i(t^i, x^{1,i})\,|0\rangle$.

Time ordering Op.

The late time of $\Delta S_A^{(n)}$ is given by

$$\boxed{\Delta S_A^{(n)} = \sum_{i=1}^{k} \Delta S_A^{(n)i}}$$

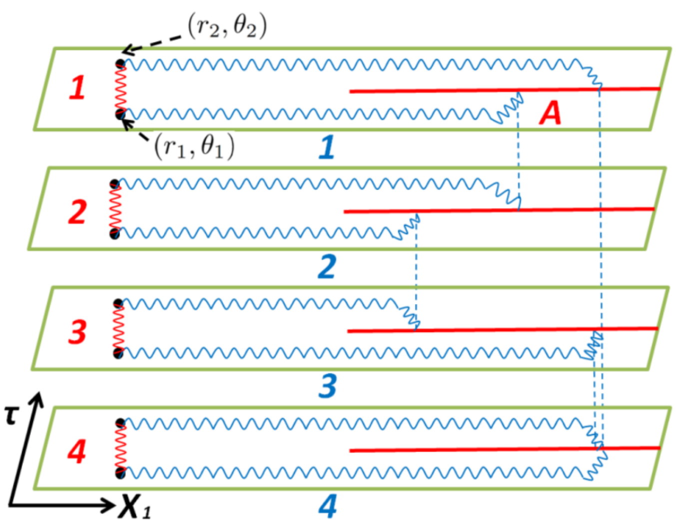


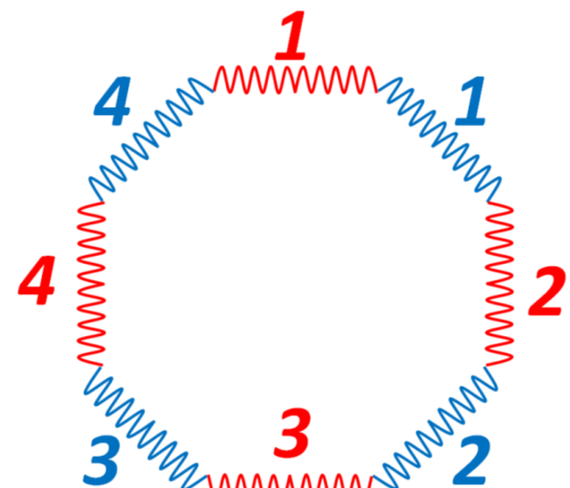
Local operators

They are given by the sum of the REE for the state defined by acting each operators $\mathcal{O}^i(t^1, x^{1,i})$ on the ground state.

Comment on propagators

- After performing analytic continuation, we take ε →0 limit. Only two diagrams can contribute to $\Delta S_A^{(n)}$.



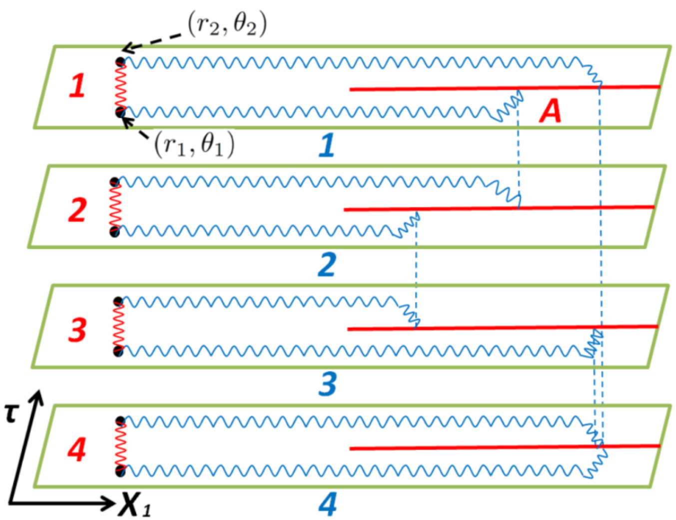


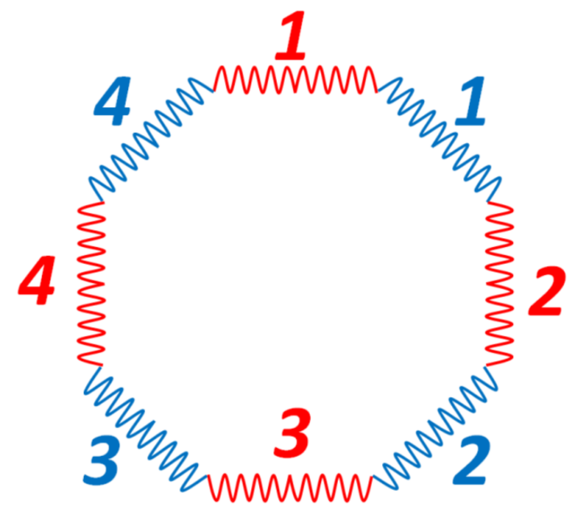
One diagram ⇔ product of red lines

One diagram ⇔ product of blue lines

Comment on propagators

- After performing analytic continuation, we take ε →0 limit. We also take late time limit (t>>l).





Red wavy lines and Blue wavy lines are called ***Dominant Propagators*** $D^{(n)}$.

$$\frac{D^{(n)}}{D^{(1)}} = \frac{\text{the number of sheets}}{\text{the number of propagators on the circle}} = \frac{n}{2n} = \frac{1}{2}.$$

Summary (Up To Here)

Field Theory Side

- We defined the (Renyi) entanglement entropies of local operators.

  -They characterize local operators from the viewpoint of quantum entanglement.

- These entropies of the operators (constructed of single-species operator) are given by the those of binomial distribution.

  -The results we obtain in terms of entangled pair agree with the results we obtain by replica method.

- They obey the sum rule.

Contents

- ***AdS/CFT Correspondence***

1. Motivation (from now on)

2. Large *N* limit in Field Theory Side

3. Holographic computation

4. The Result and Possibility in Gravity

3. Based on arXiv:1405.5946 [hep-th]

Motivation

How REE for locally excited state behave in large N strongly coupled theories ?

Field Theory Side

We consider large N free U(N) gauge theory.

We act $Tr(\mathcal{Z}^J) = Tr(\phi_1 + i\phi_2)^J$ on the ground state.

$$J = 2, \qquad \Delta S_R^{(n)} = \frac{1}{1-n} \log\left(2^{1-2n} + \frac{1}{2^n N^{2(n-1)}}\right).$$

Field Theory Side

We consider large N free U(N) gauge theory.

We act $Tr(\mathcal{Z}^J) = Tr(\phi_1 + i\phi_2)^J$ on the ground state.

$$J = 2, \qquad \Delta S_R^{(n)} = \frac{1}{1-n} \log\left(2^{1-2n} + \frac{1}{2^n N^{2(n-1)}}\right).$$

$$\Big\Downarrow \quad N \to \infty$$

$$\Delta S_R^{(n\geq 2)} \simeq \frac{2n-1}{n-1} \cdot \log 2 \quad \xRightarrow{n \to 1} \quad \textit{Diverge!!}$$

Field Theory Side

We consider large N free U(N) gauge theory.

We act $Tr(\mathcal{Z}^J) = Tr(\phi_1 + i\phi_2)^J$ on the ground state.

$$J = 2, \qquad \Delta S_R^{(n)} = \frac{1}{1-n} \log \left(2^{1-2n} + \frac{1}{2^n N^{2(n-1)}} \right).$$

$$\Big\Downarrow \; n \to 1$$

$$\Delta S_R^{(1)} = \log \left(2\sqrt{2} N \right) \xrightarrow{N \to \infty} \textbf{\textit{Finite!!}}$$

Field Theory Side

We consider large N free U(N) gauge theory.

We act $Tr(\mathcal{Z}^J) = Tr(\phi_1 + i\phi_2)^J$ on the ground state.

$N \to \infty \Rightarrow n \to 1$: ***Incorrect Limit***

$n \to 1 \Rightarrow N \to \infty$: ***Correct limit***

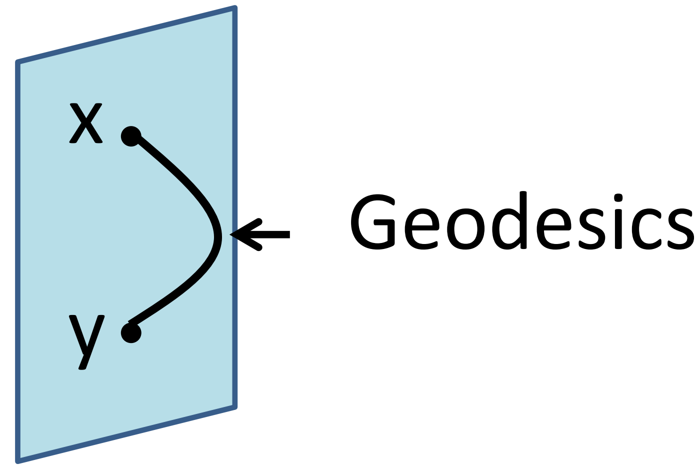
$\Downarrow n \to 1$

$$\Delta S_R^{(1)} = \log\left(2\sqrt{2}N\right) \xRightarrow{N \to \infty} \textbf{\textit{Finite!!}}$$

Comment on AdS/CFT correspondence

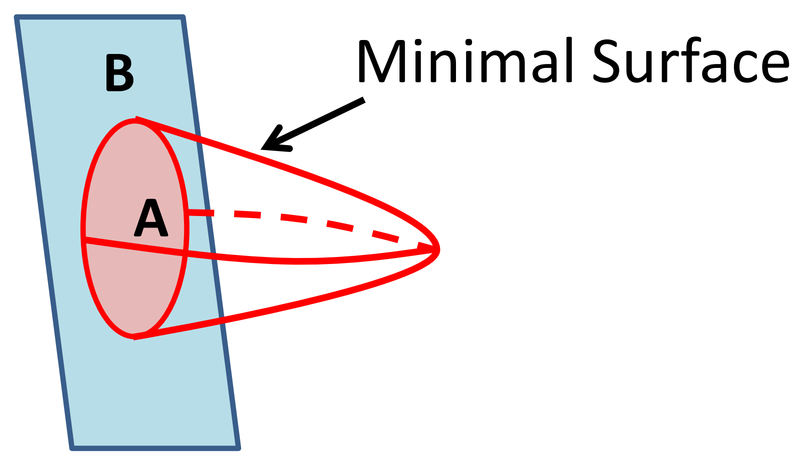
Field theory | **Gravity**

$$\langle \mathcal{O}(x) O(y) \rangle$$



Entanglement Entropy

(Wilson loop)



Comment on the Result In AdS/CFT

We compute. $S_A^{(n)} = \frac{\log \mathrm{tr}[\rho_A^n]}{1-n}$ for $|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t, x^i)\,|0\rangle$.

Field Theory

n-th (Renyi)
Entanglement Entropies in CFT_d

$n \to 1,$

N :Finite

Entanglement Entropies in CFT_d

Comment on the Result In AdS/CFT

$\boxed{\text{We compute. } S_A^{(n)} = \frac{\log \mathrm{tr}[\rho_A^n]}{1-n} \text{ for } |\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t,x^i)\,|0\rangle\,.}$

Field Theory

n-th (Renyi)
Entanglement Entropies in CFT_d

$n \to 1,$
N :Finite

$n \to 1,$
$N \to \infty$

Entanglement Entropies in CFT_d ⇒ Nontrivial Background

Comment on the Result In AdS/CFT

We compute. $S_A^{(n)} = \frac{\log \mathrm{tr}[\rho_A^n]}{1-n}$ for $|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t,x^i)\,|0\rangle$.

Field Theory

n-th (Renyi) Entanglement Entropies in CFT_d

$n \neq 1,$ $N \to \infty$

Gravity

Topological Black Hole in AdS_{d+1}

$n \to 1,$ N :Finite

Entanglement Entropies in CFT_d

$n \to 1,$ $N \to \infty$

Nontrivial Background

Gravity Side

We compute

$$\Delta S_A^{(n)} = \frac{1}{1-n}\left(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\right).$$

on topological black hole by using geodesic approximation in d+1 dim.

$$ds^2 = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2 + r^2 e^{-2\phi} dx_i^2 \qquad f(r) = -1 - \frac{\mu}{r^{d-2}} + \frac{r^2}{R^2}$$

Periodicity: $\beta = 2\pi n R$ ⟷ Replica number: n

Boundary of topological black hole ⟷ Σ_n

Topological Black Hole

$$ds^2 = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2 + r^2 e^{-2\phi} dx_i^2$$

$$f(r) = -1 - \frac{\mu}{r^{d-2}} + \frac{r^2}{R^2}$$

$$r \longrightarrow \infty$$

Asymptotic region

$$ds^2 = \frac{R^2}{r^2} d\tau^2 + R^2 \frac{dr^2}{r^2} + r^2 d\phi^2 + r^2 e^{-2\phi} dx_i^2$$

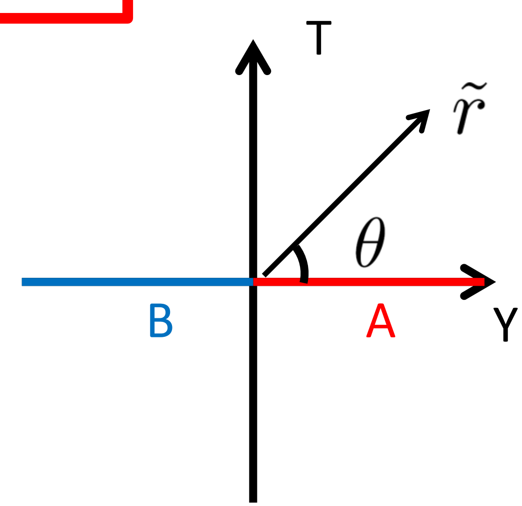
Topological Black Hole

$$ds^2 = \frac{R^2}{r^2}d\tau^2 + R^2\frac{dr^2}{r^2} + r^2 d\phi^2 + r^2 e^{-2\phi}dx_i^2$$

$$ds^2 = R^2\frac{dZ^2 + dT^2 + dY^2 + dx_i^2}{Z^2}$$

Map:
$$\begin{aligned} Y - iT &= \sqrt{1-\frac{R^2}{r^2}}e^{\phi - i\tau/R} \\ Y + iT &= \sqrt{1-\frac{R^2}{r^2}}e^{\phi + i\tau/R} \\ Z &= \frac{R}{r}e^{\phi}. \end{aligned}$$

$$\boxed{\begin{aligned} \tilde{r} &= e^{\phi} \\ \theta &= \frac{\tau}{R} \end{aligned}}$$



Topological Black Hole

$$ds^2 = \frac{R^2}{r^2} d\tau^2 + R^2 \frac{dr^2}{r^2} + r^2 d\phi^2 + r^2 e^{-2\phi} dx_i^2 \quad \Rightarrow \quad ds^2 = R^2 \frac{dZ^2 + dT^2 + dY^2 + dx_i^2}{Z^2}$$

Map:

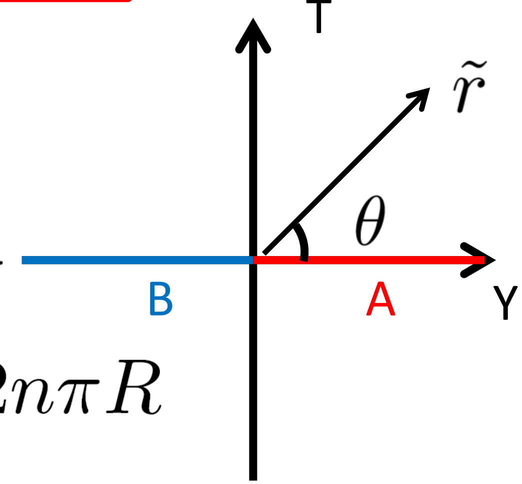
$$\begin{aligned} Y - iT &= \sqrt{1 - \frac{R^2}{r^2}} e^{\phi - i\tau/R} \\ Y + iT &= \sqrt{1 - \frac{R^2}{r^2}} e^{\phi + i\tau/R} \\ Z &= \frac{R}{r} e^{\phi}. \end{aligned}$$

$$\boxed{\begin{aligned} \tilde{r} &= e^{\phi} \\ \theta &= \frac{\tau}{R} \end{aligned}}$$

$$\Sigma_1 \longrightarrow \Sigma_n$$

$$\theta \sim \theta + 2\pi \rightarrow \theta \sim \theta + 2n\pi$$

$$\tau \sim \tau + 2\pi R \rightarrow \tau \sim \tau + 2n\pi R$$



Topological Black Hole

$$ds^2 = \frac{R^2}{r^2}d\tau^2 + R^2\frac{dr^2}{r^2} + r^2 d\phi^2 + r^2 e^{-2\phi}dx_i^2 \Rightarrow ds^2 = R^2\frac{dZ^2 + dT^2 + dY^2 + dx_i^2}{Z^2}$$

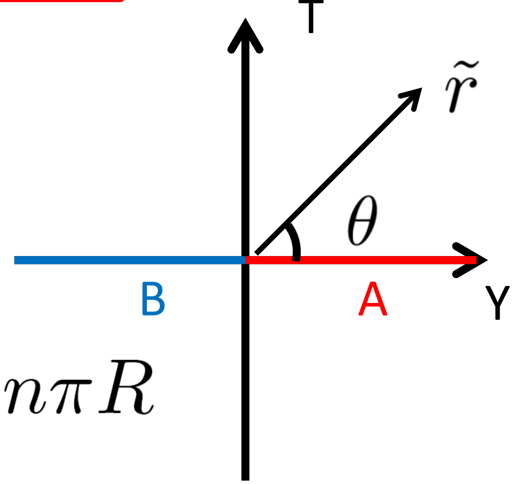
$$\boxed{\tau \sim \tau + 2n\pi R}$$

$$\boxed{\begin{aligned}\tilde{r} &= e^{\phi} \\ \theta &= \frac{\tau}{R}\end{aligned}}$$

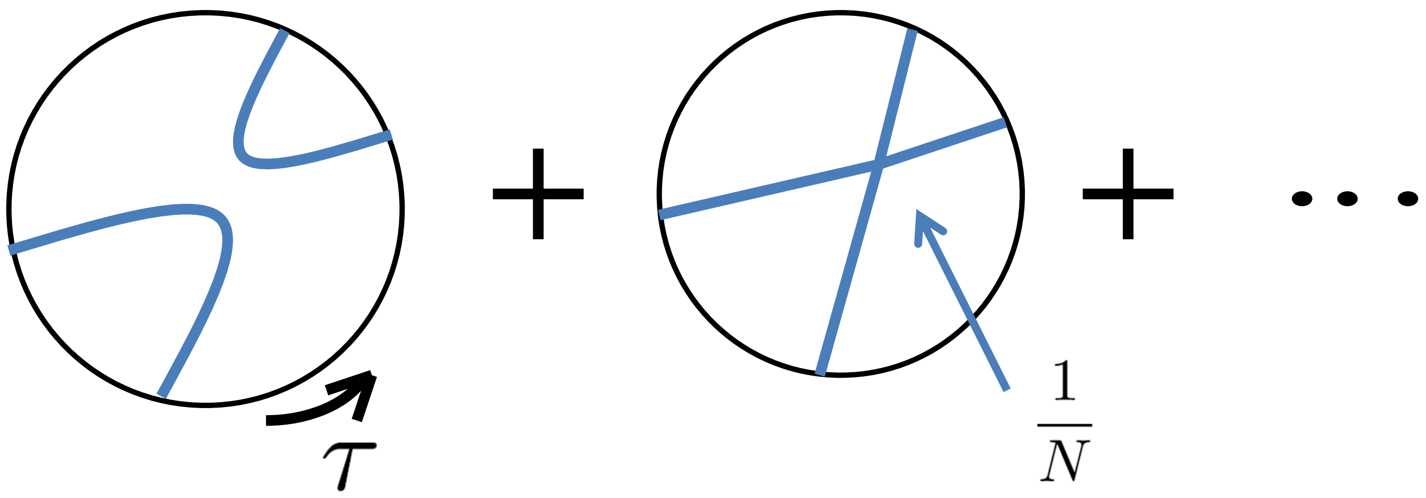
$$\Sigma_1 \longrightarrow \Sigma_n$$

$$\theta \sim \theta + 2\pi \rightarrow \theta \sim \theta + 2n\pi$$

$$\tau \sim \tau + 2\pi R \rightarrow \tau \sim \tau + 2n\pi R$$

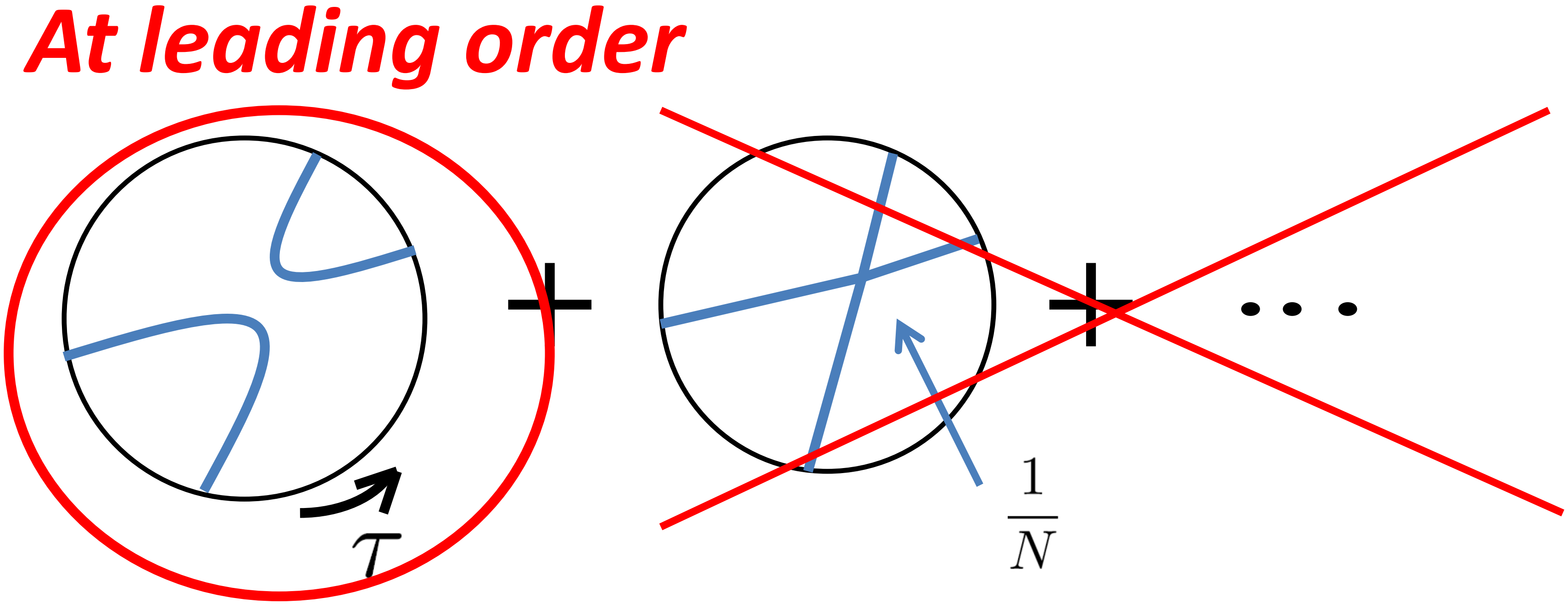


Large N limit



can contribute to correlation function.

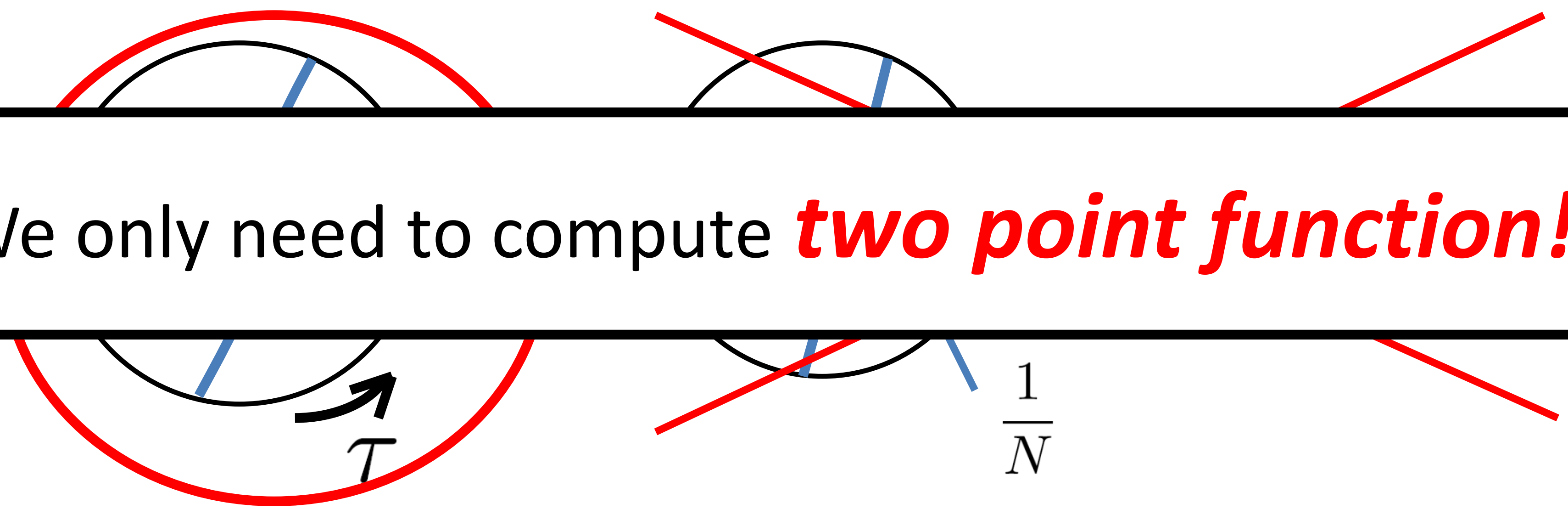
Large N limit



can contribute to correlation function.

Large N limit

At leading order



We only need to compute ***two point function!!***

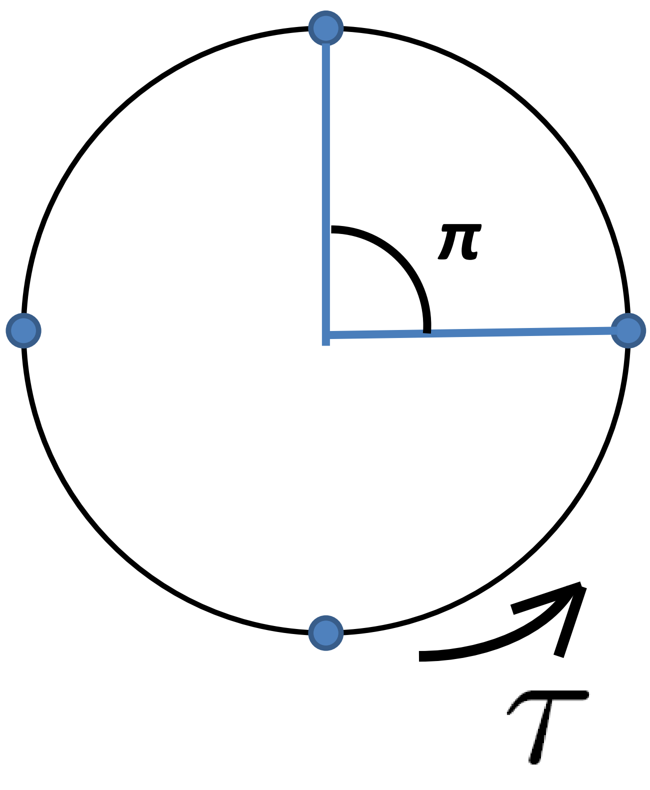
can contribute to correlation function.

At Late time

At late time ($t>>l>>\varepsilon$),

operators are inserted symmetrically:

n=2

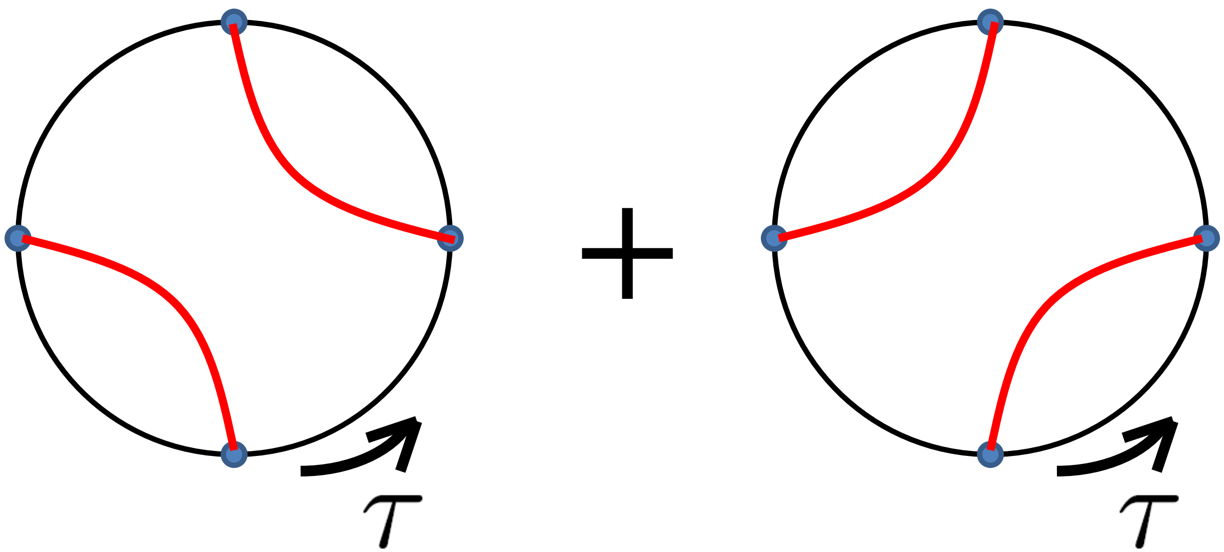


At Late time

At late time (t>>l>>ε),

operators are inserted symmetrically:

n=2



Only two diagrams

can contribute to ***correlation function*** !!.

Gravity Side

We compute

$$\Delta S_A^{(n)} = \frac{1}{1-n}\left(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\right).$$

on topological black hole by using geodesic approximation in d+1 dim.

$$ds^2 = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2 + r^2 e^{-2\phi} dx_i^2 \qquad f(r) = -1 - \frac{\mu}{r^{d-2}} + \frac{r^2}{R^2}$$

Periodicity: $\beta = 2\pi n R \Longleftrightarrow$ Replica number: n

At the late time,

$$\boxed{\Delta S_A^{(n)} \sim \frac{4n\Delta}{d(n-1)}\log t.}$$

Gravity Side

We compute

$$\Delta S_A^{(n)} = \frac{1}{1-n}\left(\log \left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\right).$$

on topological black hole by using geodesic approximation in d+1 dim.

At the late time,

Conformal dim. of inserted operator

$$\boxed{\Delta S_A^{(n)} \sim \frac{4n\Delta}{d(n-1)}\log t.}$$

$\Downarrow$

We can not take n → 1 limit.

Gravity Side

We compute

$$\Delta S_A^{(n)} = \frac{1}{1-n}\left(\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,n})\mathcal{O}(r_1,\theta_{1,n})\cdots\mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_{1,1})\right\rangle_{\Sigma_n} - n\log\left\langle \mathcal{O}^\dagger(r_2,\theta_{2,1})\mathcal{O}(r_1,\theta_1)\right\rangle_{\Sigma_1}\right).$$

on topological black hole by using geodesic approximation in d+1 dim.

At the late time,

Conformal dim. of inserted operator

$$\boxed{\Delta S_A^{(n)} \sim \frac{4n\Delta}{d(n-1)}\log t.}$$

Large N limit,

$\Delta S_A^{(n)}$ can not approach constant.

Comment on the Result In AdS/CFT

We compute. $S_A^{(n)} = \frac{\log \mathrm{tr}[\rho_A^n]}{1-n}$ for $|\Psi\rangle = \mathcal{N}^{-1}\mathcal{O}(t,x^i)\,|0\rangle$.

Field Theory

Gravity

n-th (Renyi) Entanglement Entropies in $\mathrm{CFT_d}$

$n \neq 1,$ $N \to \infty$

Topological Black Hole in $\mathrm{AdS}_{\mathrm{d+1}}$

$n \to 1,$ N :Finite

$n \to 1,$ $N \to \infty$

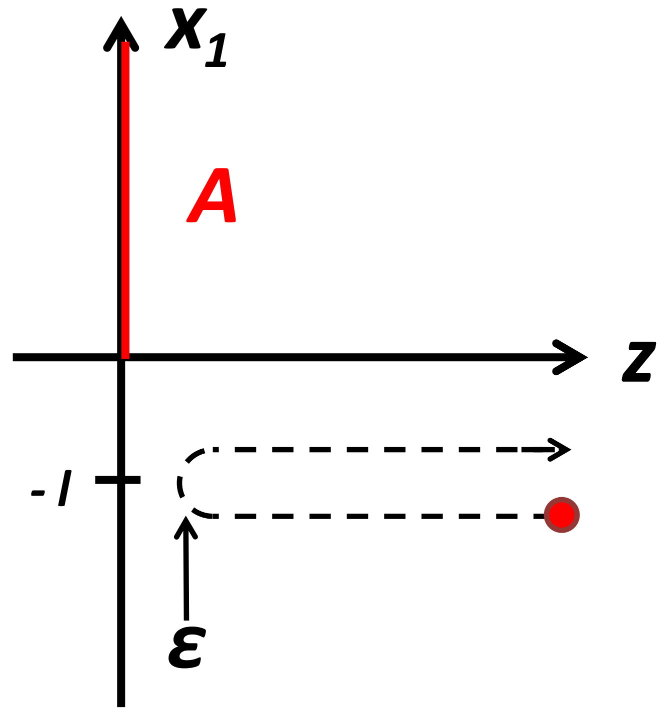
$n \to 1,$ $N \to \infty$

Entanglement Entropies in $\mathrm{CFT_d}$

Nontrivial Background

Entanglement Entropy

Back ground : Falling particle in AdS



We compute holographic entanglement entropy.

At late time,

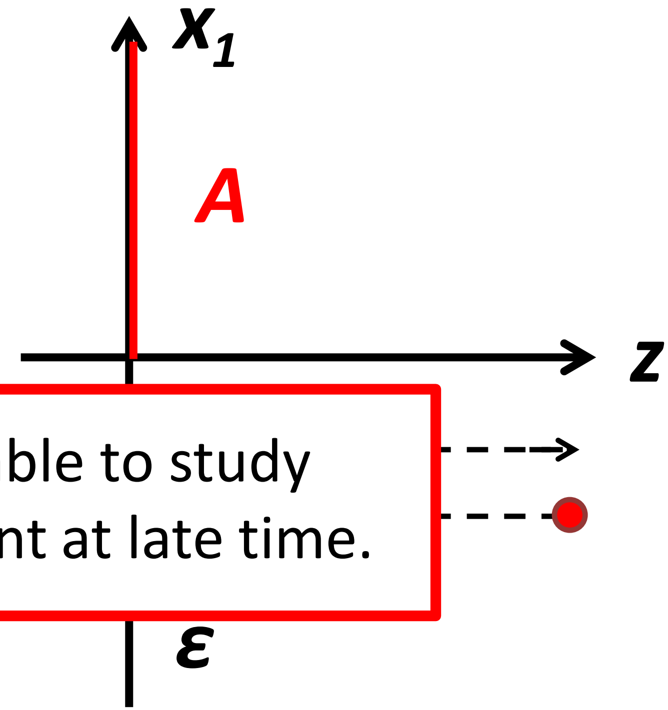
$$\Delta S_A^{(1)} \simeq \frac{c}{6} \log \frac{t}{\epsilon} + \frac{c}{6} \log \frac{\Delta}{c}$$

Large N limit,

$\Delta S_A^{(n)}$ can not approach constant even if n=1.

Entanglement Entropy

Back ground : Falling particle in AdS



We con

e

In large N expansion, we are ***not*** able to study whether it approach some constant at late time.

At late time,

$$\Delta S_A^{(1)} \simeq \frac{c}{6}\log\frac{t}{\epsilon} + \frac{c}{6}\log\frac{\Delta}{c}$$

Large N limit,

$\Delta S_A^{(n)}$ can not approach constant even if n=1.

Possibility

If $\Delta S_A^{(n)}$ approaches some constant, it is expected that $\Delta S_A^{(n)}$ is given by

$$\boxed{\Delta S_A^{(n)} \simeq \frac{1}{1-n} \log \left[\frac{1}{D_n} + \mu_n \left(\frac{p}{t} \right)^{\nu_n} \right]}.$$

μ_n, a_n, b_n :Parameters are dependent of n.

The constant value is expected to come from the non-perturbative effect:

$$\boxed{D_n \sim e^{b_n \cdot N^{a_n}}.}$$

Possibility

If $\Delta S_A^{(n)}$ approaches some constant, it is expected that $\Delta S_A^{(n)}$ is given by

$$\boxed{\Delta S_A^{(n)} \simeq \frac{1}{1-n} \log \left[\frac{1}{D_n} + \mu_n \left(\frac{p}{t} \right)^{\nu_n} \right]}.$$

To find whether $\Delta S_A^{(n)}$ approaches constant or diverge, we need the information *beyond* the large N limit.

The … ative effect:

$$\boxed{D_n \sim e^{b_n \cdot N^{a_n}}.}$$

Summary

Field Theory Side

- We defined the (Renyi) entanglement entropies of local operators.

  -They characterize local operators from the viewpoint of quantum entanglement.

- These entropies of the operators (constructed of single-species operator) are given by the those of binomial distribution.

  -The results we obtain in terms of entangled pair agree with the results we obtain by replica method.

- They obey the sum rule.

Summary

AdS/CFT correspondence

- To take n → 1 limit does not commute with taking N → ∞ limit.

  - After taking large N limit, we can not take n → 1 (EE for excited state diverge.).

- In AdS/CFT correspondence, $\Delta S_A^{(n)}$ does not approach some constants.

  - In large N limit, we are not able to study

    whether it can approach some constant or it diverges.

  - In large N expansion, the leading contribution of $\Delta S_A^{(n)}$ is proportional to the conformal dim. of operators which are acted on the ground state.

Future Problems

- The formula for the operators constructed of multi-species operators: $:(\partial_r^m \phi)^k \phi^l:$ (generally depend on the spacetime dimension).

- The (Renyi) entanglement entropies of operators in the interacting field theory . (also massive and charged Renyi.)

- Beyond large N, we investigate $\Delta S_A^{(n)}$.

  - approach constant?

  - diverge?