Resurgence in quantum field theory: dealing with the Devil's invention

Part 1

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arXiv:1308.0127, 1403.1277, 1410.0388, …

Talk at IPMU, November 4 2014

The dark side of perturbation theory

In QFTs with small coupling λ, observables computable as

$$\mathcal{O}(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + \cdots$$

ask an undergrad

graduate student

postdoc, faculty

computers?

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But in interesting QFTs like QCD, $c_n \sim n!$ for large n

Dyson 1952

Perturbation theory yields divergent series!

That's strange. If perturbative expansions are divergent, then why do they work so well?

Historically, this caused a lot of confusion…

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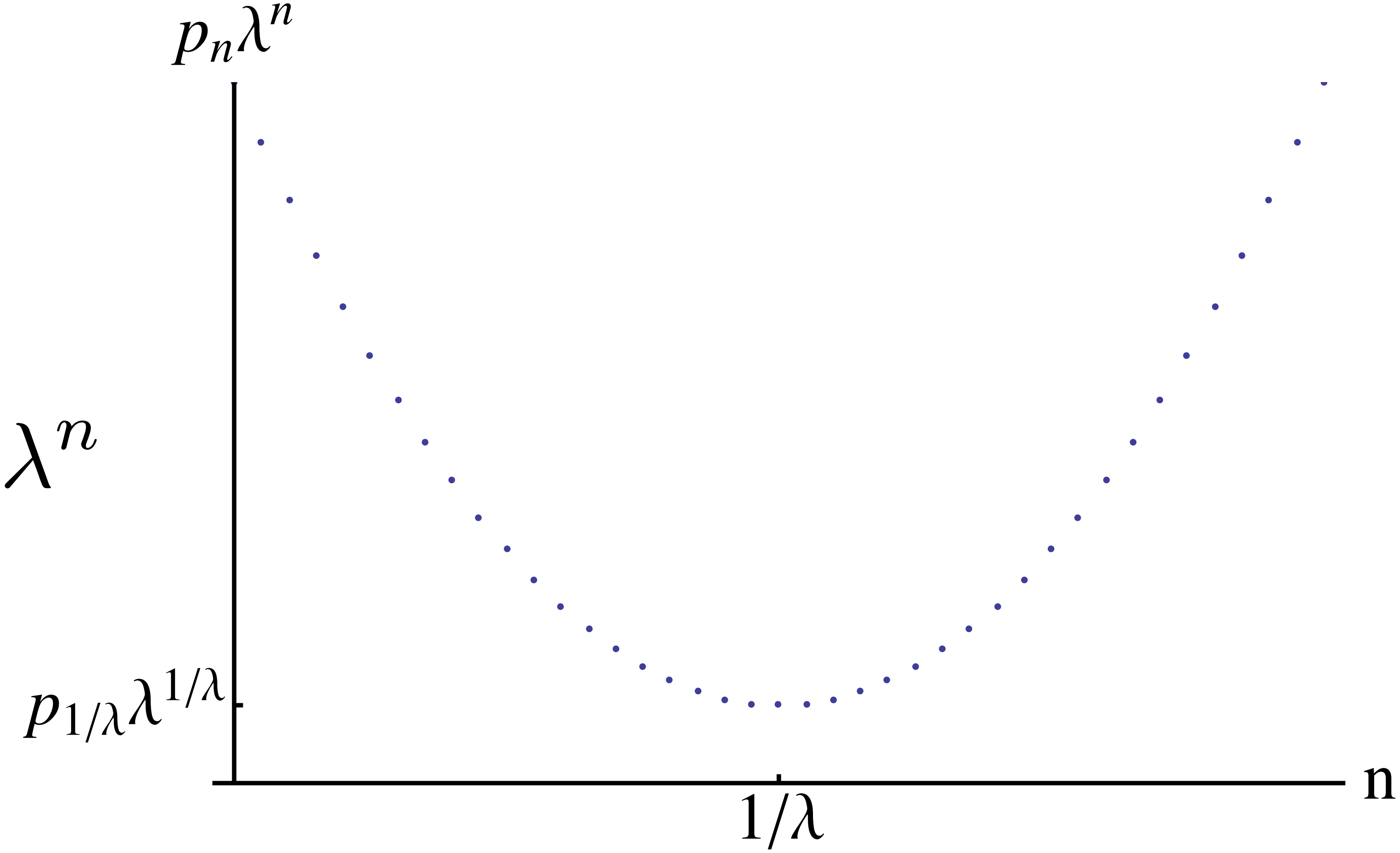
N. H. Abel, 1802-1829

Doh!

Traditional view on asymptotic series

$$\sum_{n=1}^{\infty} p_n \lambda^n$$



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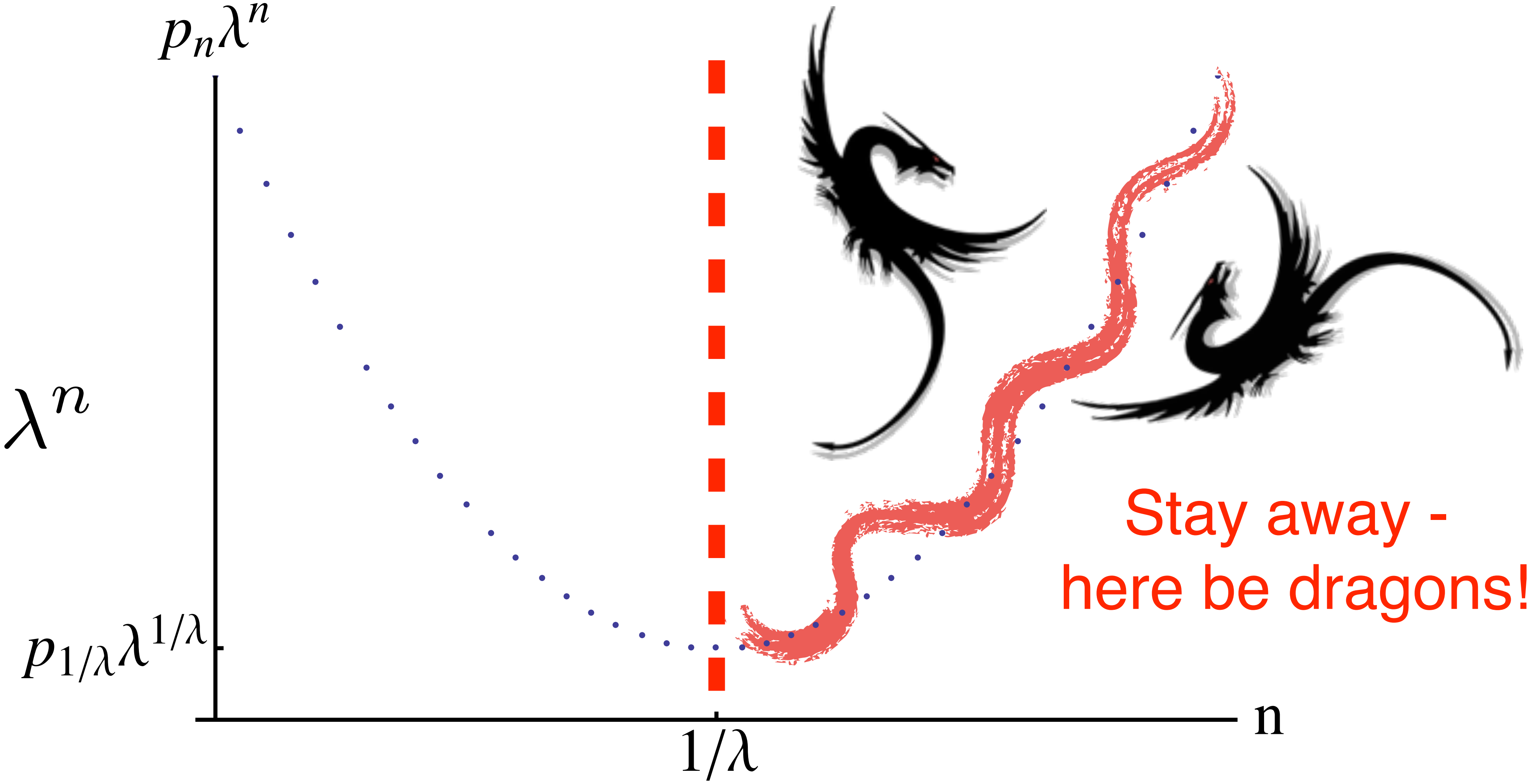

Poincare
1886

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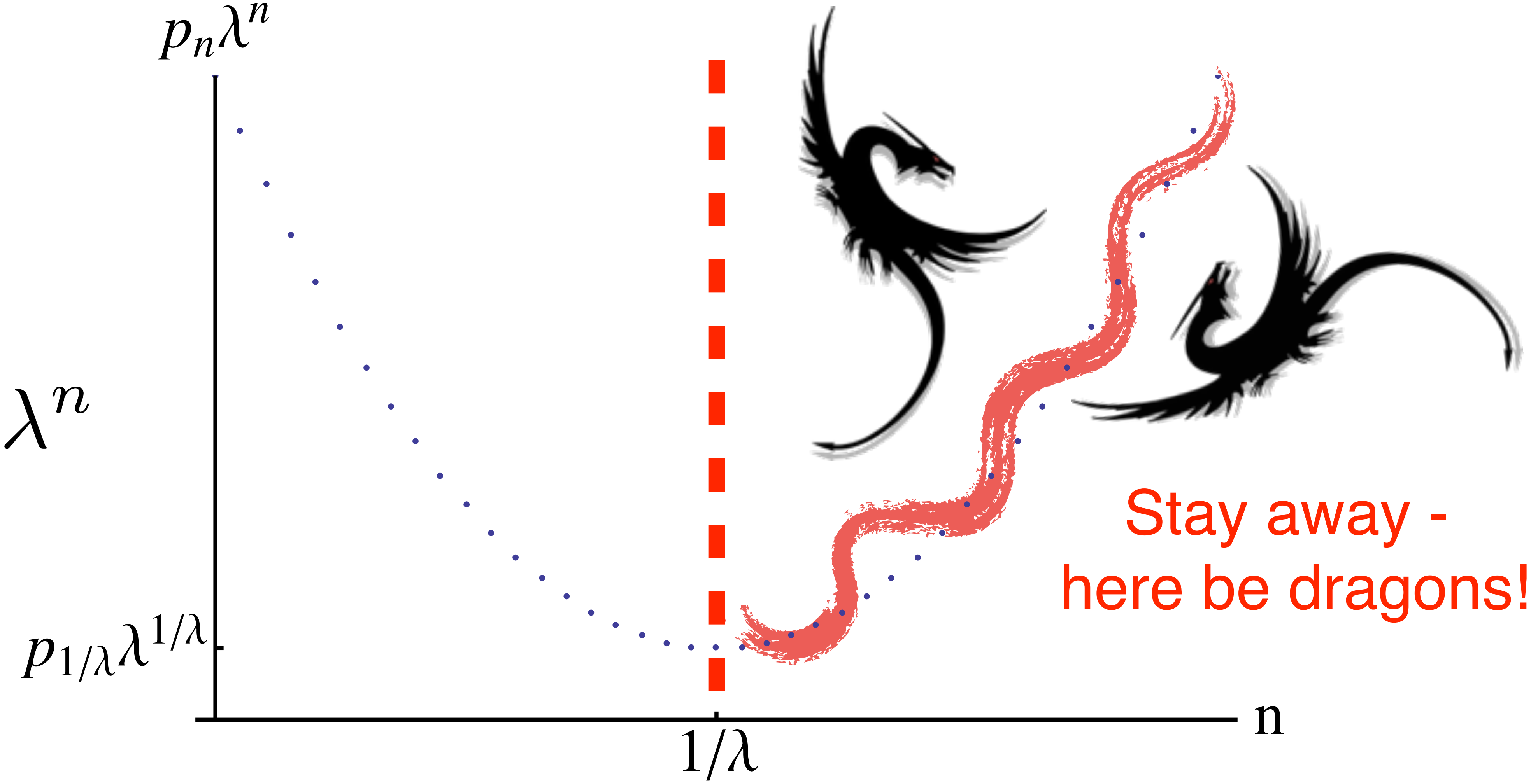


Poincare
1886

Can argue that `mistake' made is order $e^{-1/\lambda}$

Exponentially small - so is it uninteresting?

$e^{-1/\lambda}$ is precisely scale of non-perturbative effects in e.g. QCD

In asymptotically-free theories, at least, non-perturbative effects drive the most interesting part of the physics!

A more systematic approach is called for…

Resurgence theory in a toy example

$$\mathcal{Z}(\lambda) = \frac{1}{\sqrt{\lambda}} \int_{-\pi/2}^{\pi/2} dx\, e^{-\frac{1}{2\lambda}\sin(x)^2}$$

0d prototype for a QFT partition function. Two saddle-points: x = 0, x= π/2

If λ is small, approximate Z(λ) by perturbation expansion around x=0

$$\mathcal{Z}(\lambda) \stackrel{?}{\simeq} \sqrt{2\pi}\left[1 + \frac{\lambda}{2} + \frac{9\lambda^2}{8} + \frac{75\lambda^3}{16} + \cdots\right]$$

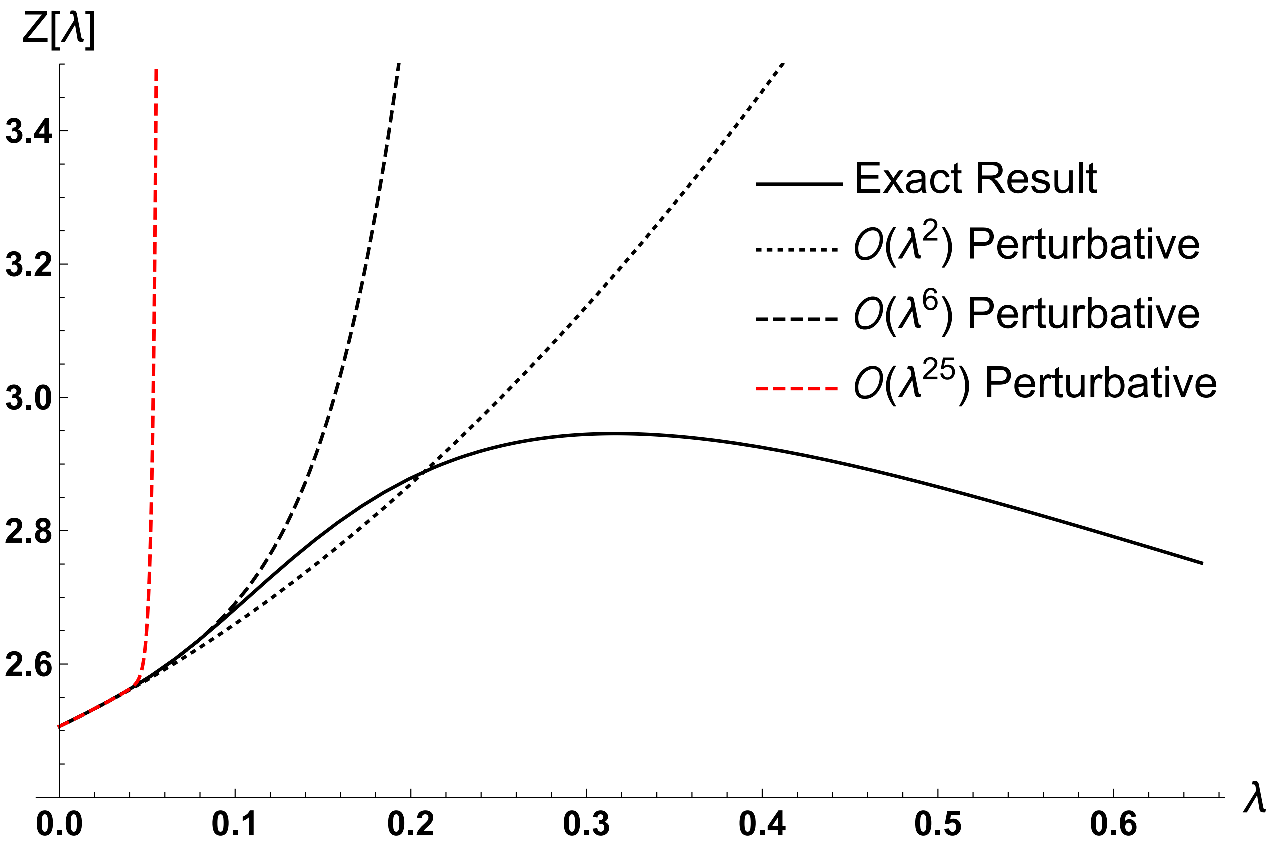
$$p_n \sim n! \text{ for } n \gg 1$$

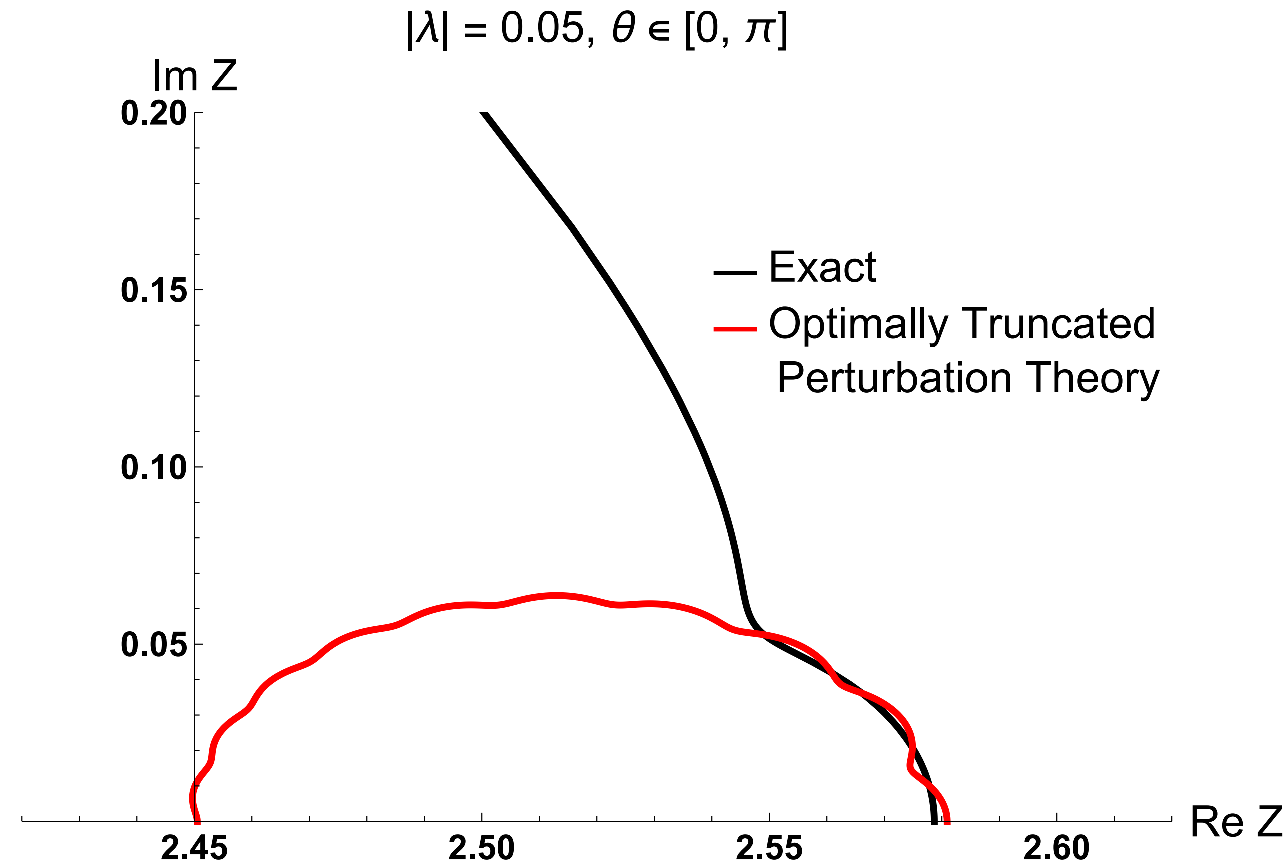
With order of expansion fixed, and |λ| tiny,
subsequent terms become smaller and smaller.

If λ is not small, series will (obviously) not approximate Z(λ).

Might naively expect perturbation theory to
be good for small |λ|, for any arg(λ)

Two failures of perturbation theory





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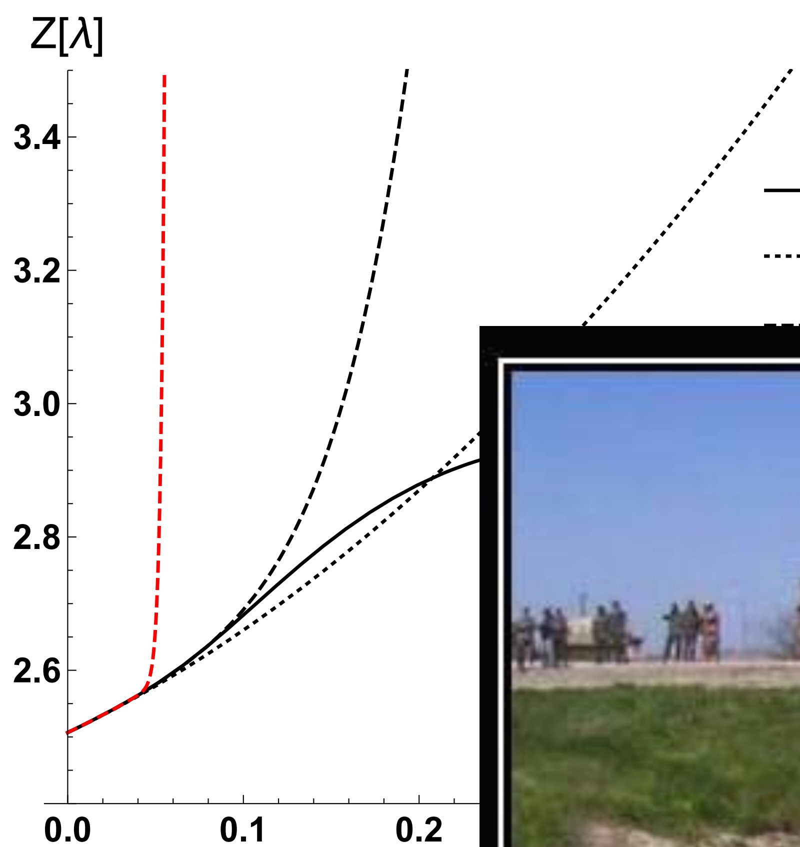

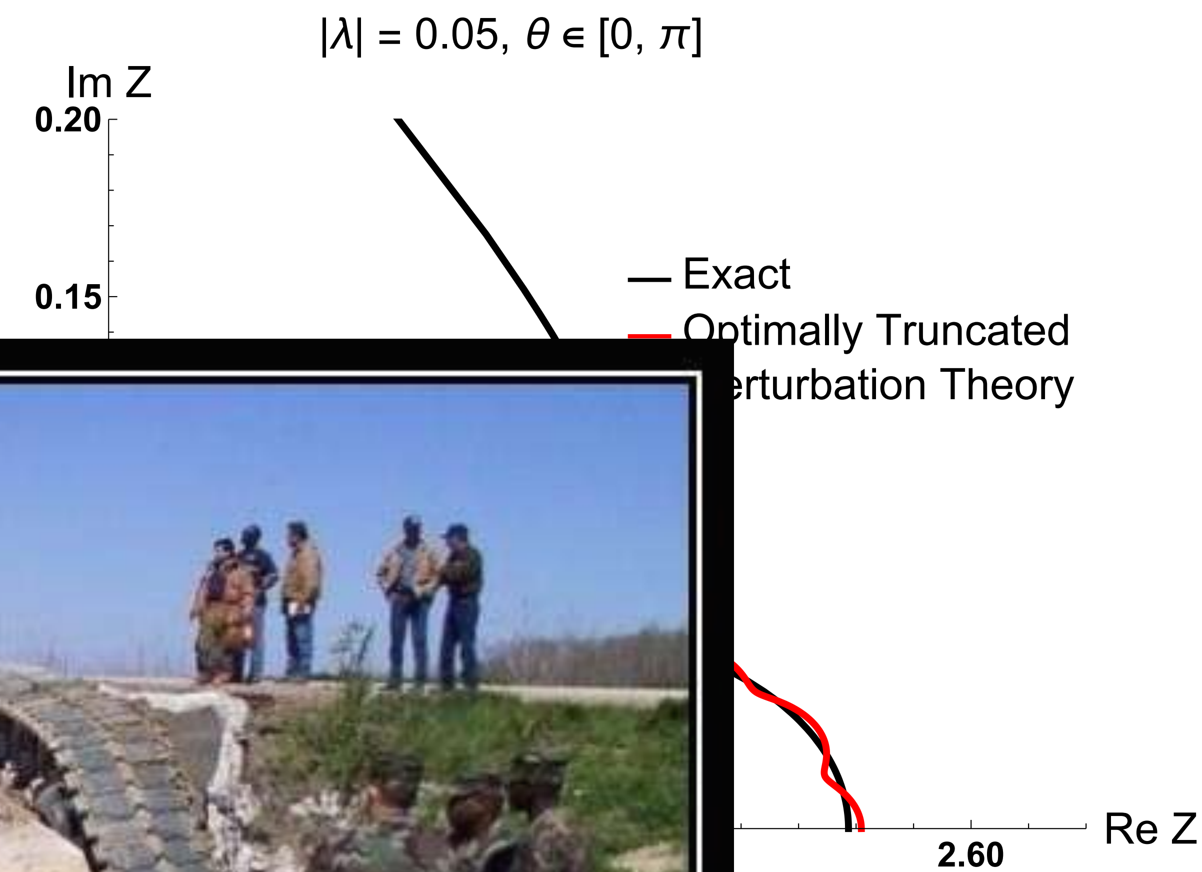

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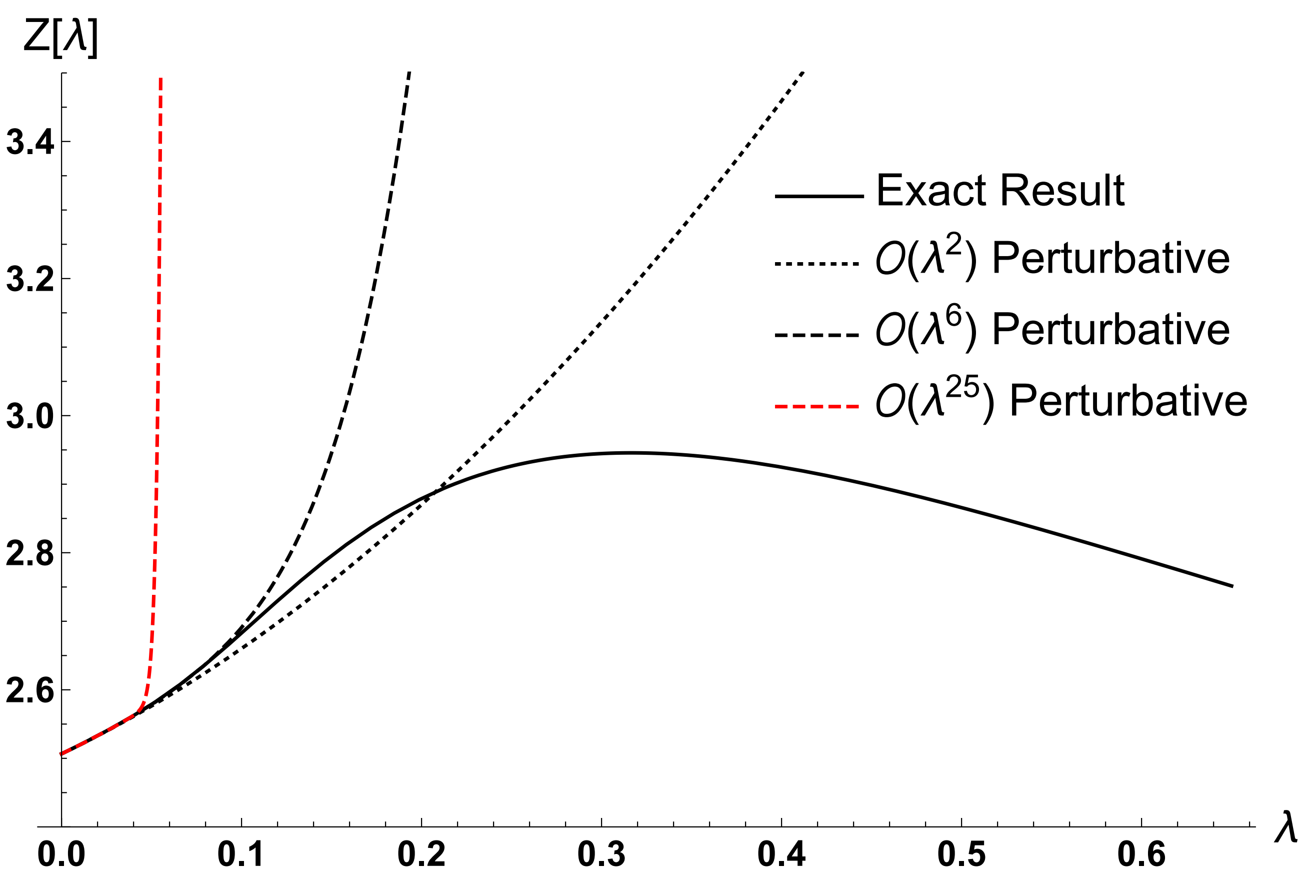
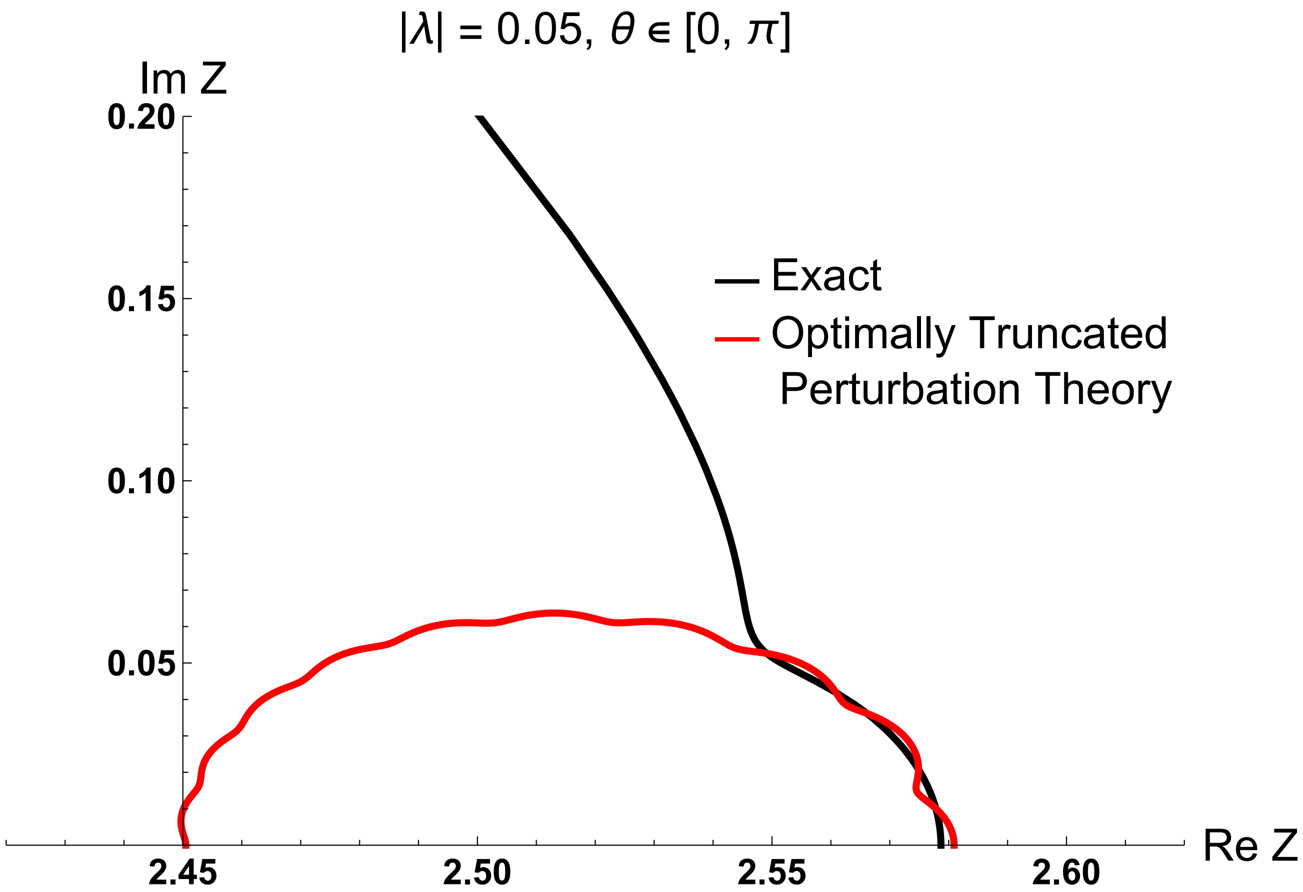




Failure for large λ

Failure for |λ|<< 1, arg λ > 0

May naively seem like two different issues…

But they actually have the same origin, and the same cure - proper inclusion of NP saddle contributions via resurgence theory

The reason for the problems

The integral has two saddle-points; we ignored the non-perturbative one!

$$\mathcal{Z}(\lambda) \stackrel{?}{=} \sum_n p_{n,P}\lambda^n + e^{-\frac{1}{2\lambda}} \sum_n p_{n,NP}\lambda^n$$

Both series above are divergent asymptotic series.

Traditional perspective would be that result still
has irreducible error of order exp[-1/(2λ)]

Resurgence theory: P and NP series are correlated, and errors cancel.

$$p_{n,P} \longrightarrow \frac{(n-1)!}{\sqrt{\pi}\left(\frac{1}{2}\right)^n}\left(p_{0,NP} + \frac{p_{1,NP}}{(n-1)} + \frac{p_{2,NP}}{(n-1)(n-2)} + \cdots\right)$$

Origin of 'resurgence' term: exact data on low-order behavior
of NP saddle encoded in large-order behavior of P saddle

Borel summation: a language for resurgence

original formal series
$$\mathcal{O} = \sum_{n=1}^{\infty} p_n \lambda^n, \quad p_n \sim n!$$

`Borel transform'
$$B[\mathcal{O}](t) \equiv \sum_{n=1}^{\infty} \frac{p_n}{(n-1)!} t^{n-1}$$

BO(t) defines function analytic within finite radius around t=0

Borel sum
$$\mathcal{SO}(\lambda) = \frac{1}{\lambda} \int_0^{\infty} dt\, e^{-t/\lambda} B[\mathcal{O}](t)$$

SO(λ) has same power expansion as O(λ)

Should think about *SO*(λ) as a useful representation of data in formal series with $|p_n| \leq n!\, c^n$

But the integral — and hence sum — doesn't always exist!

Borel summation: a language for resurgence

Borel sum: $$\mathcal{S}\mathcal{O}(\lambda) = \frac{1}{\lambda}\int_0^\infty dt\, e^{-t/\lambda} B[\mathcal{O}](t)$$

Working case: $$E(\lambda) = \sum_{n=0}^{\infty} (-1)^n \, n! \, \lambda^{n+1} \;\Rightarrow\; B[E(\lambda)] = \frac{1}{1+t}$$

No pole on R+ contour, Borel integral exists, resummation unambiguous

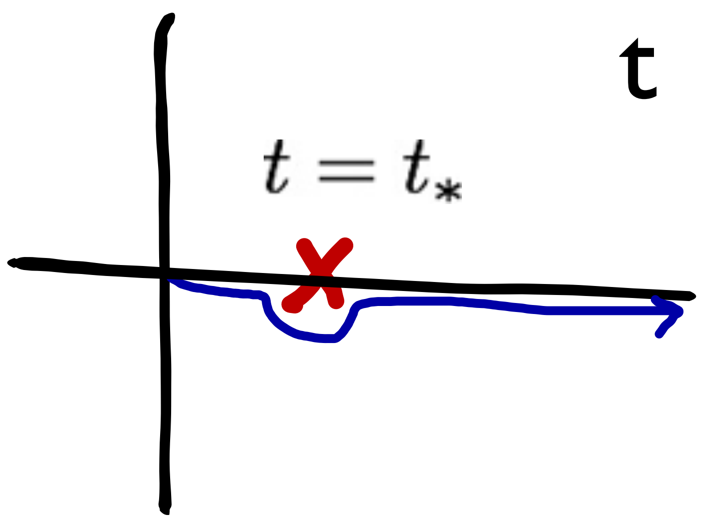
Failing case: $$E(\lambda) = \sum_{n=0}^{\infty} (+1)^n \, n! \, \lambda^{n+1} \;\Rightarrow\; B[E(\lambda)] = \frac{1}{1-t}$$ singularity on R+ !

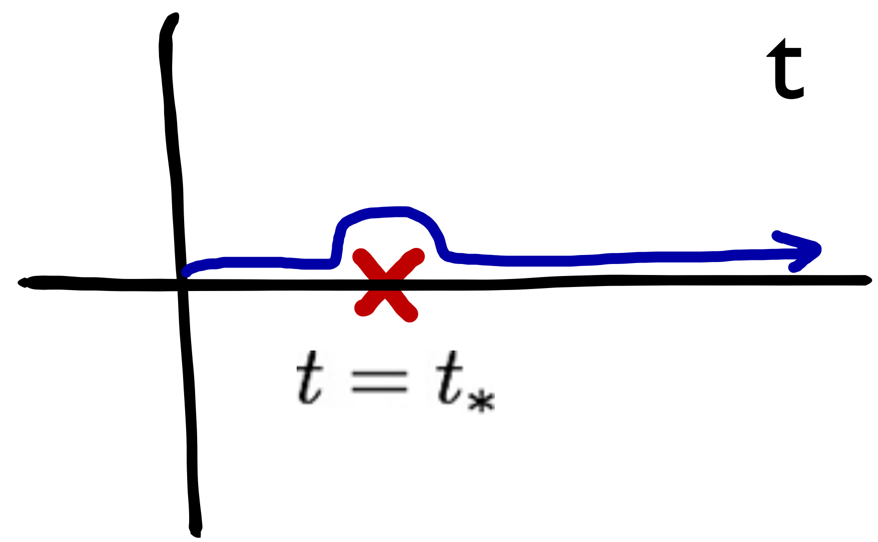
Singularity on R+ contour, Borel sum does not exist.

Typical situation in series coming from QFT, and 0d toy example

Ambiguities in non-Borel summable series

Can deform contour, above or below real axis.





Amounts to analytic continuation of path integral $\lambda \to \lambda(1 \pm i\epsilon)$

Imaginary non-perturbative (NP) ambiguity in resummation, depending on direction of continuation

$$\mathcal{S}_{\pm}\mathcal{O}(\lambda) = \mathrm{Re}\left[\tilde{\mathcal{O}}(\lambda)\right] \pm 2\pi i\, e^{-t_*/\lambda}$$

Form of ambiguity points to the guilty party:

Contribution from NP saddle with action $S = t_*/\lambda$

Dingle, Berry, Howls…

Ambiguity of perturbative series in toy example

$$\mathcal{S}_\pm \Phi_P(\lambda) = \mathrm{Re}\, \mathcal{S}_\pm \Phi_P(\lambda) \pm 2i e^{-1/2g} \mathcal{S}\Phi_{NP}(\lambda)$$

Perturbative fluctuations
around P and NP saddle-points

Ambiguity in sum of perturbative series weighed by action of NP saddle x = π/2, times series around NP saddle

Note the power of the equation: all data about fluctuations around NP saddle encoded in P fluctuations, and vice versa.

$$p_{n,P} \longrightarrow \frac{(n-1)!}{\sqrt{\pi}\left(\frac{1}{2}\right)^n} \left(p_{0,NP} + \frac{p_{1,NP}}{(n-1)} + \frac{p_{2,NP}}{(n-1)(n-2)} + \cdots \right)$$

Resurgence theory in toy example

Motivates *transseries* as an exact, unambiguous representation of Z(λ):

Ecalle 1980s, …

$$\mathcal{Z}(\lambda) = \begin{cases} \mathcal{S}_\theta \Phi_{\mathrm{P}}(\lambda) - ie^{-\frac{1}{2\lambda}} \mathcal{S}_\theta \Phi_{\mathrm{NP}}(\lambda), & \theta \in (0, \pi) \\ \mathcal{S}_\theta \Phi_{\mathrm{P}}(\lambda) + ie^{-\frac{1}{2\lambda}} \mathcal{S}_\theta \Phi_{\mathrm{NP}}(\lambda), & \theta \in (-\pi, 0) \end{cases}$$

Berry, Howls 1991; Witten 2010; …

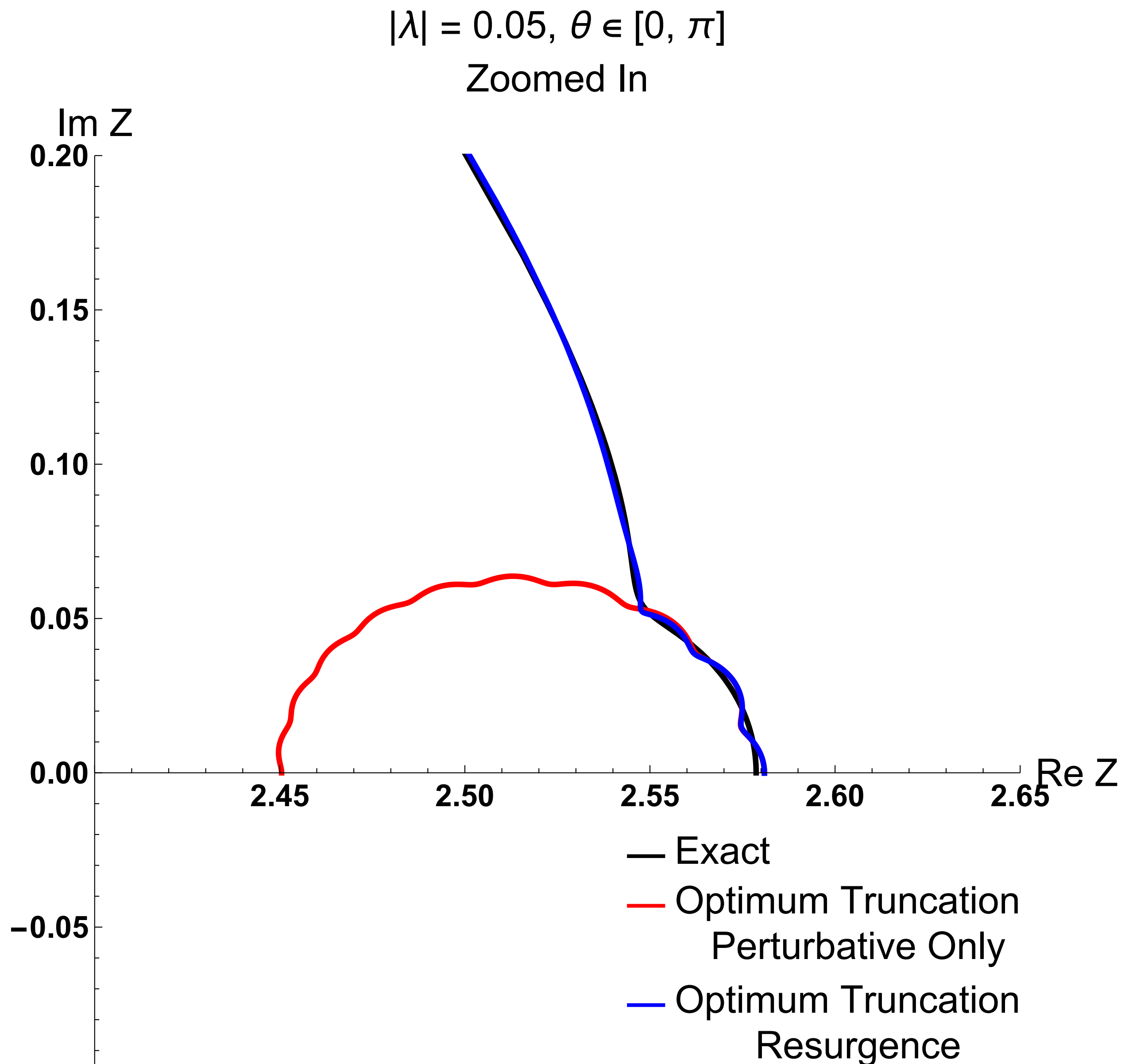
Sign flip of NP part is a Stokes phenomenon; related to choices of integration contours in complexified 'path integral'

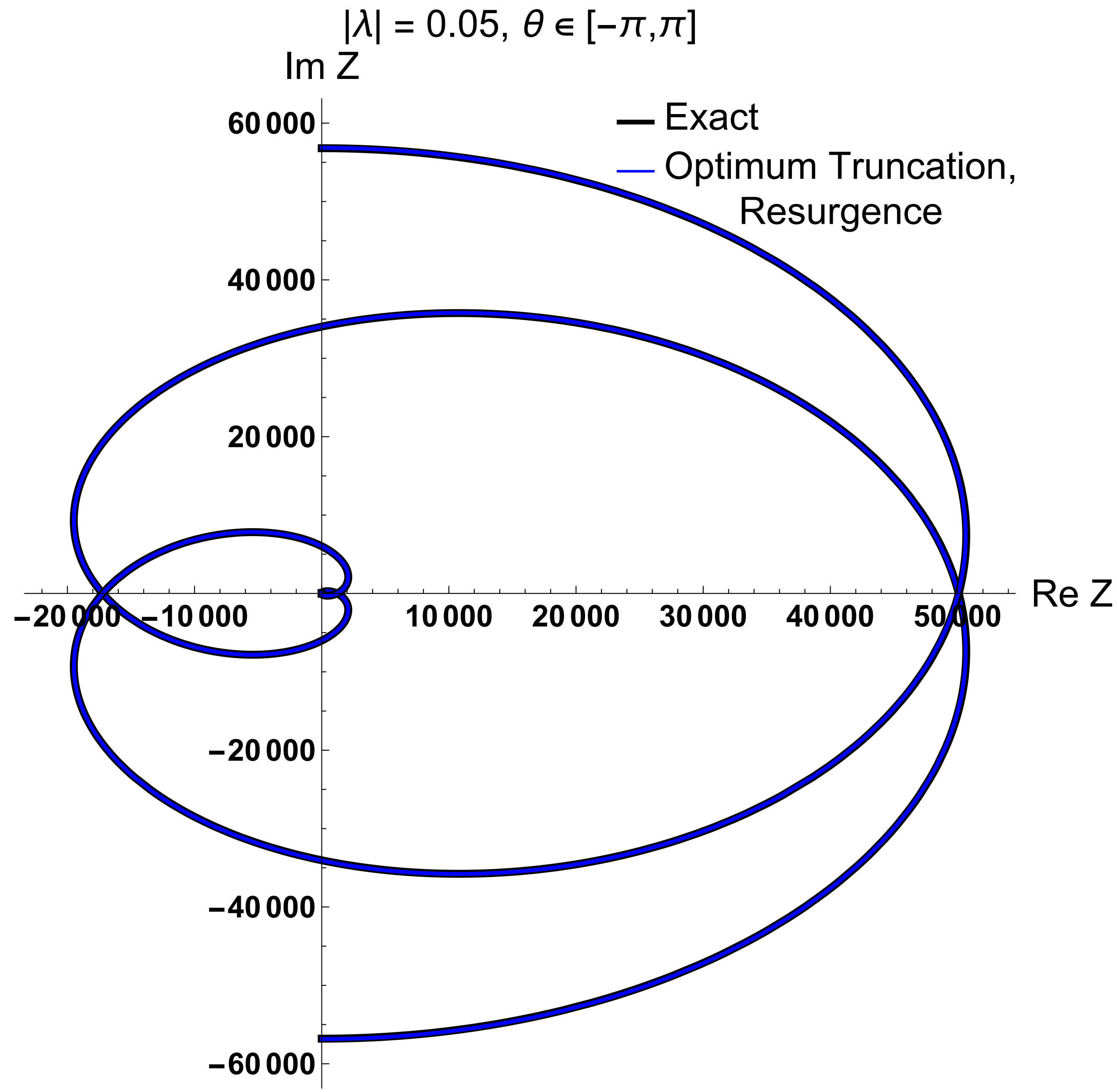
Note that result is not just a naive sum of saddle point contributions, due to '+/-' factor above.

Now we can explore whether this helps deal with the two failures of perturbation theory.

The payoff from resurgence theory, 1

AC, Koroteev, Unsal, 2014

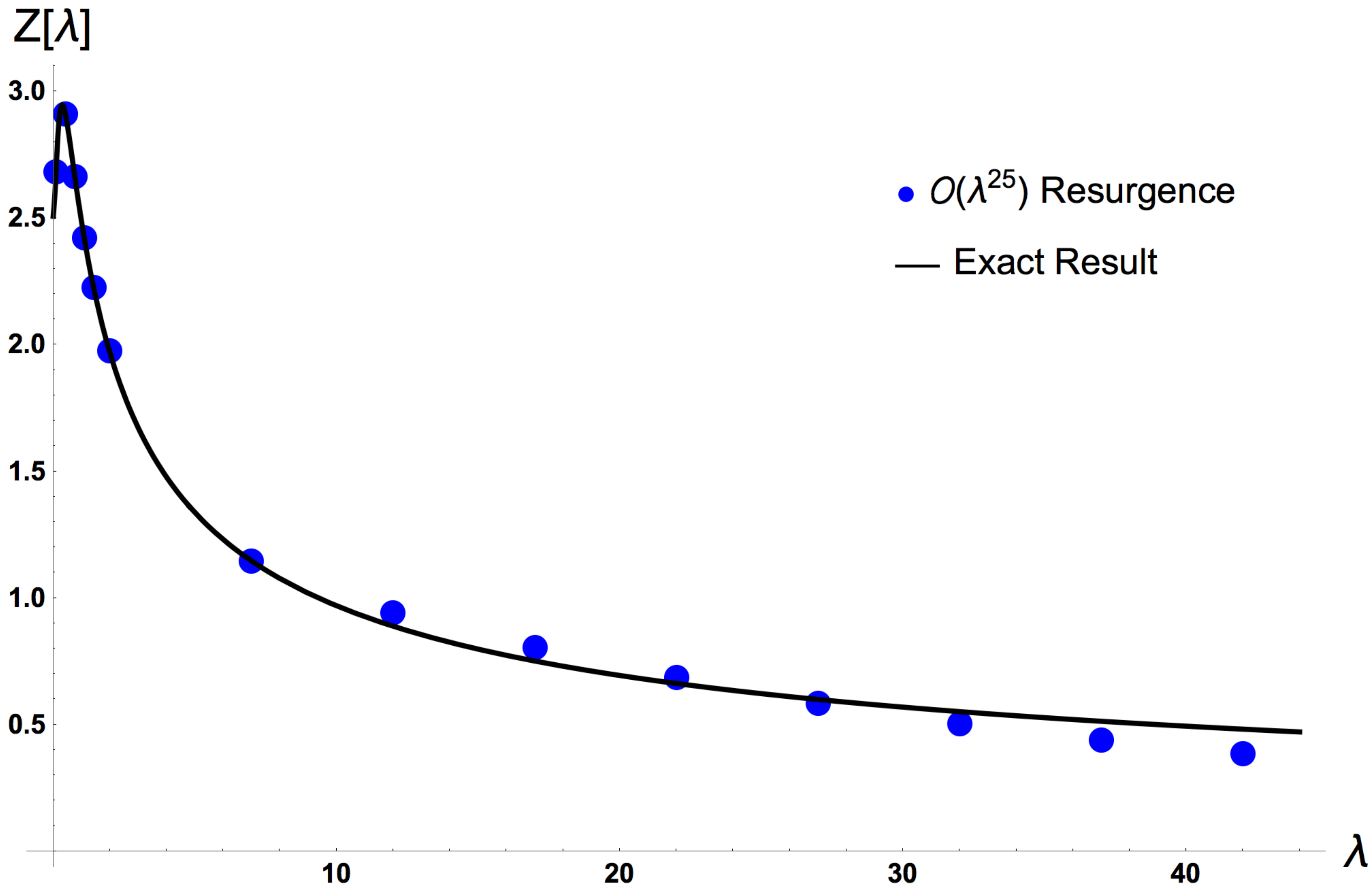




Resurgence theory yields arbitrarily accurate results at arbitrary arg λ and e.g. small |λ|

The payoff from resurgence theory, 2

AC, Koroteev, Unsal, 2014



Resurgence theory allows arbitrarily accurate strong-coupling predictions from small-g perturbative data

Summary of Part 1

\`Resurgence' is the statement that perturbation expansions around any one saddle-point of QFT-type integrals contains quantitative data about expansions around the other saddle points

In short: P = NP, and resurgence theory decodes the '=' !

As a result, naively ambiguous asymptotic series can be assembled into unambiguous transseries representations of observables.

Many interesting subtleties and lessons appear when these ideas are applied to quantum mechanics and quantum field theory.

To hear more about them, you are welcome to stay for Part 2!

End of Part 1

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Part 2

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Perturbation theory as a semiclassical expansion

$$\langle \mathcal{O}[\lambda] \rangle = Z[\lambda]^{-1} \int d[U]\, e^{-S(U;\lambda)} \mathcal{O}$$

regularized path integral

For small λ tempting to use saddle-point approximation

$$Z(\lambda) \stackrel{?}{=} \sum_{n=0}^{\infty} p_n \lambda^n + \sum_c e^{-S_c/\lambda} \sum_{k=0}^{\infty} p_{c,k} \lambda^k$$

Usually *all* of these series are sick, suffer from divergences!

Traditional view is that semiclassical expansions have an inherent and irreducible 'vagueness' of order $e^{-1/\lambda}$

Modern approach, based on resurgence theory:

'transseries' expansions are faithful and unambiguous (but subtle) representations of observables.

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regularized path integral

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Usually *all* of these series are sick, suffer from divergences!

If above `transseries' is to encode well-defined smooth function of λ, need intricate relations connecting $p_{c,n}$ for different saddles

$\Longrightarrow$ relations between perturbative and non-perturbative physics

Vainshtein, 1964;
Bender+Wu 1969;
Lipatov 1977

Resurgence theory is the detailed implementation of this idea

Dingle, Berry 1960+...

Ecalle: 1980s

Argyres, Dunne, Unsal… QFT

Aniceto, Marino, Schiappa… strings

What happens in quantum field theory?

The approach in part 1 generalizes to multi-dimensional integrals.

Hence might expect resurgence theory to apply to QFTs, at least when they are regularized.

Then perturbative and non-perturbative effects should be correlated in QFT!

But once we consider quantum field theory more carefully, there are some important challenges…

$$Z(\lambda) \stackrel{?}{=} \sum_{n=0}^{\infty} p_n \lambda^n + \sum_c e^{-S_c/\lambda} \sum_{k=0}^{\infty} p_{c,k} \lambda^k$$

(1) What do we even *mean* by λ in QFT? Couplings run!

(2) Is it actually plausible that ambiguities are controlled by finite-action saddles?

Resurgence theory in asymptotically-free QFT

To deal with (1), might try to introduce large momentum Q into problem, and define $\lambda = \lambda(Q/\Lambda) \ll 1$

Indeed, in asymptotically-free QFTs like QCD, $p_n \sim n!$ for large n

This results in ambiguities in resummations of perturbation theory

Could they be due to NP saddle point contributions?

If so, reasonable to expect resurgence theory technology to apply.

Many (but not all) asymptotically-free QFTs have instantons, which would be associated to ambiguities of order

$$\text{ambiguity: } \sim e^{-N/\lambda} \sim e^{-1/g^2} \qquad \lambda = g^2 N$$

But physically, leading ambiguities must be ~ strong scale Λ

$$\text{ambiguity: } \sim e^{-1/\lambda}$$

Mismatch?

Borel plane singularities in QCD

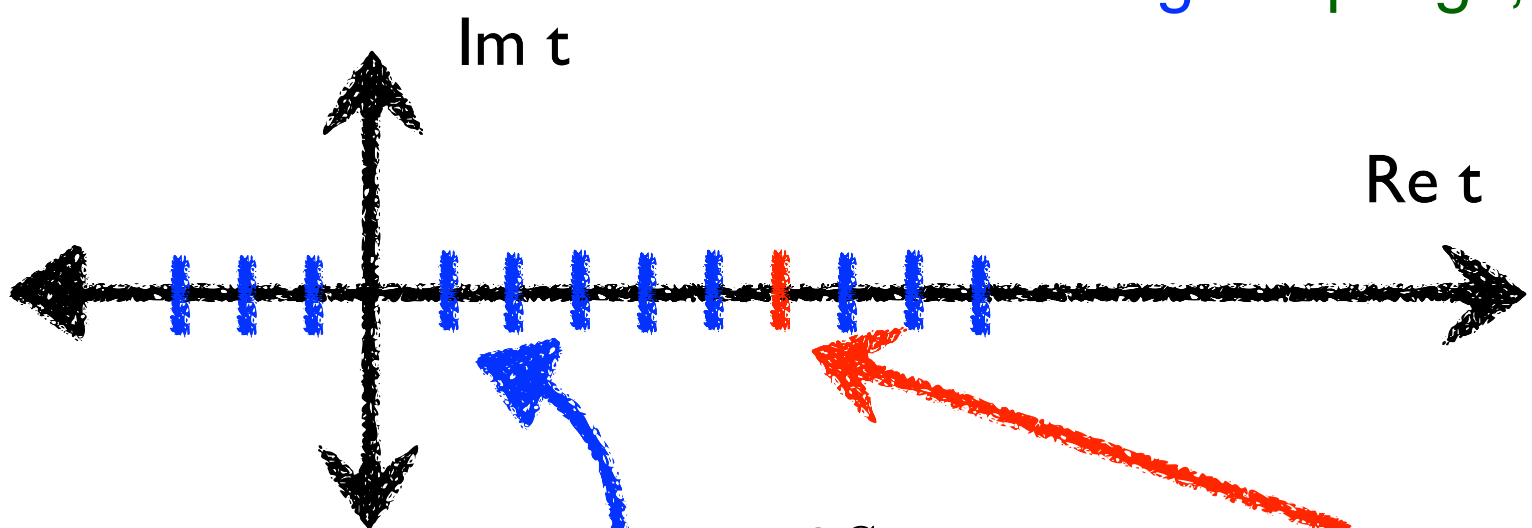
't Hooft, 1979

(1) Combinatorial singularities

related to the number of diagrams at order n growing as n!

(2) Renormalon singularities

related to `single' planar diagrams with n running couplings, scale as n!



$$t = 2S_I/|\beta_0| \Rightarrow e^{-\frac{2S_I}{|\beta_0|g^2}} \qquad t = 2S_I \Rightarrow e^{-2S_I/g^2}$$

$\beta_0 = 11N/3$ so renormalon ambiguity >> 'instanton' ambiguity

Not just a formal problem!

Renormalons arise in pQCD calculations relevant for e.g. collider physics

effect parametrized by introducing phenomenological 'power corrections'

Borel singularities for QCD and its relatives

In quantum mechanics, perturbation theory is also asymptotic.
But instanton-anti-instanton contributions are ambiguous, too!

Bogomolny; Zinn-Justin early 1980s

Instanton ambiguities precisely cancels against perturbative ambiguities, at least in QM.

’t Hooft’s dream: QFT renormalons associated to some kind of fractional instantons, related to confinement

No such configurations known on in QCD on R^4, or in other asymptotically-free theories

Moreover, many asymptotically-free theories don’t have instantons at all, let alone `fractional instantons’!

Argyres, Dunne, Unsal 2012-13

Key idea: find smooth compactification which preserves confinement, while driving theory to weak coupling.

Desired fractional instantons emerge, allow application of resurgence theory, yield systematic ambiguity cancellations.

SU(N) Principal Chiral Model

Focus for the rest of the talk:

$$S = \frac{1}{2g^2} \int_M d^2x \, \mathrm{Tr} \; \partial_\mu U \partial^\mu U^\dagger, \qquad U \in SU(N)$$

Why is it interesting?

Asymptotically free, like QCD
Dynamically generated mass gap, like QCD
Matrix-like large N limit, like QCD
Large N confinement-deconfinement transition, like QCD
Perturbation theory suffers from combinatorial and renormalon ambiguities, just like QCD
Integrable, $M = R^2$ S-matrix known, so easier than QCD

Fateev, Kazakov, Wiegmann

But $\pi_2[SU(N)] = 0$, so no instantons, unlike QCD!

Lack of known NP saddles seems like big difference from QCD.

Almost a nice toy model for QCD

Resurgence theory for the PCM

Will see that structure of perturbative series is inconsistent with lack of NP saddles.

Resurgence demands finite-action field configurations exist - whether or not topology seems to allow it!

To justify semiclassics in asymptotically-free theory, first task is to find adiabatic weakly-coupled limit.

Guided by resurgence, we find ‘fracton’ NP saddles in PCM

Cancellation of renormalon ambiguities driven by fractons

Renormalons are indeed tied to the mass gap, as guessed by ’t Hooft!

Results reinforce lesson that naive topological classification of saddle points in path integrals is insufficient.

Dealing with strong coupling

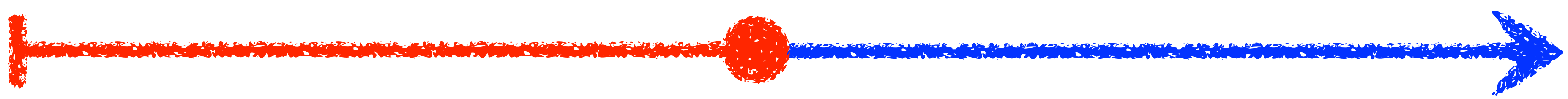
'Coupling is small' assumption for saddle-point expansion doesn't make sense in PCM: $\beta<0$

Need a weakly coupled limit, while keeping mass gap etc, with physics adiabatically connected to original theory

Our approach is to put the theory on $M = R^{time} \times S^1(L)$

For small enough L, weak coupling guaranteed by asymptotic freedom

But with periodic boundary conditions, looks like a thermal circle!



small L — $F/N^2 \sim 1$

large L — $F/N^2 \sim 0$

Resembles confinement/deconfinement transition in 4D YM!

In PCM, large N phase transition, finite N cross-over

Twisted boundary conditions

PCM has an $SU(N)_L x SU(N)_R$ symmetry

$$U \to \Omega_L U \Omega_R^\dagger$$

Wide variety of sensible spatial boundary conditions:

$$U(x_1, x_2 + L) = e^{iL^{-1} H_L} U(x_1, x_2) e^{-iL^{-1} H_R}$$

Working with a gapped theory - when $L \gg \Lambda^{-1}$,
choice of BCs doesn't matter

But at small L, dialing H_L, H_R parametrizes
a wide family of distinct theories

Claim: unique choice of H_L, H_R such that physics
appears to be adiabatically connected to large L limit

small L large L

Twisted boundary conditions

Convenient to trade fields with twisted BCs for background gauge fields + fields with periodic BCs

$$\partial_\mu U \to \partial_\mu \tilde{U} - i\delta_{\mu,x_2}\left([H_V, \tilde{U}] + \{H_A, \tilde{U}\}\right)$$

$\tilde{U}$ is periodic

$$2H_{V,A} = H_L \pm H_R$$

Essentially 'chemical potentials' for spatial $SU(N)_{L,R}$ currents

$$J^L_\mu = i\, U^\dagger \partial_\mu U, \qquad J^R_\mu = i\, \partial_\mu U\, U^\dagger$$

Partition function now depends on $H_{V,A}$

$$Z \to Z(L; H_V, H_A)$$

What are the desirable 'adiabaticity conditions' in terms of Z?

(A) A free energy scaling as $F/N^2 \sim 0$ at large N

(B) Insensitivity of theory to changes in BCs

Adiabaticity conditions

At small L, completely insensitivity to BCs is not possible. Closest we can come is to demand

1

$$\frac{\partial \left[\mathcal{V}^{-1} \log Z(L)\right]}{\partial H_V} = \langle J_x^V \rangle_{H_V, H_A} = 0$$

$$\frac{\partial \left[\mathcal{V}^{-1} \log Z(L)\right]}{\partial H_A} = \langle J_x^A \rangle_{H_V, H_A} = 0$$

Picks out BCs which extremize the free energy F

2

$$\frac{L^2}{N^2} \mathcal{F}\big|_{H_V, H_A} \sim 0$$

Make sure we stay in ‘confining’ phase

Our task: compute F(L; H_A, H_V) at small L, where theory is weakly coupled, and look at large N scaling of extrema

Small L Free Energy

$$\frac{T}{\mathcal{V}} \log Z = V = V_{\text{classical}} + V_{1-\text{loop}} + \ldots$$

$V_{\text{classical}} = V(H_A) > 0$, so 1-loop correction to $V(H_A)$ is negligible.

$V_{\text{classical}}(H_A)$ only has $H_A=0$ extremum, so $H_A=0$

First contribution to $V_{1\text{-loop}} = V(H_V)$ comes from one-loop level

$$\Omega = e^{iH_V/L} = \mathcal{P}e^{i \oint dx_2 \, H_V}$$

$$V_{1-\text{loop}}(\Omega) = \frac{-1}{\pi L^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (|\text{Tr } \Omega^n|^2 - 1)$$

Same form as Coleman-Weinberg potential for dynamical gauge field Wilson loop in YM theory on $R^3 x S^1$, but different interpretation!

Small L Free Energy

$$V_{1-\mathrm{loop}}(\Omega = \Omega_V, \Omega_A = 1) = \frac{-1}{\pi L} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(|\mathrm{Tr}\, \Omega^n|^2 - 1 \right)$$

One extremum corresponds to $H_V=0$

Thermal BCs! $\Omega = \Omega_T \equiv 1_N$ broken Z_N symmetry

$$F = -\frac{\pi}{6L^2}(N^2 - 1) = \mathcal{O}(N^2)$$

This is a deconfined small L limit.

Clearly not what we want…

Small L Free Energy

$$V_{1-\mathrm{loop}}(\Omega) = \frac{-1}{\pi L^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (|\mathrm{Tr}\ \Omega^n|^2 - 1)$$

The only other (non-degenerate, Z_N preserving) extremum:

$$\Omega = \Omega_S \equiv e^{i\frac{\pi}{N}\nu} \begin{pmatrix} 1 & & & \\ & e^{i\frac{2\pi}{N}} & & \\ & & \ddots & \\ & & & e^{i\frac{2\pi(N-1)}{N}} \end{pmatrix}$$

v = 0,1 for N odd, even

$$\log Z = \frac{-1}{\pi L^2} \times \frac{\pi^2}{6} = \mathcal{O}(N^0)$$

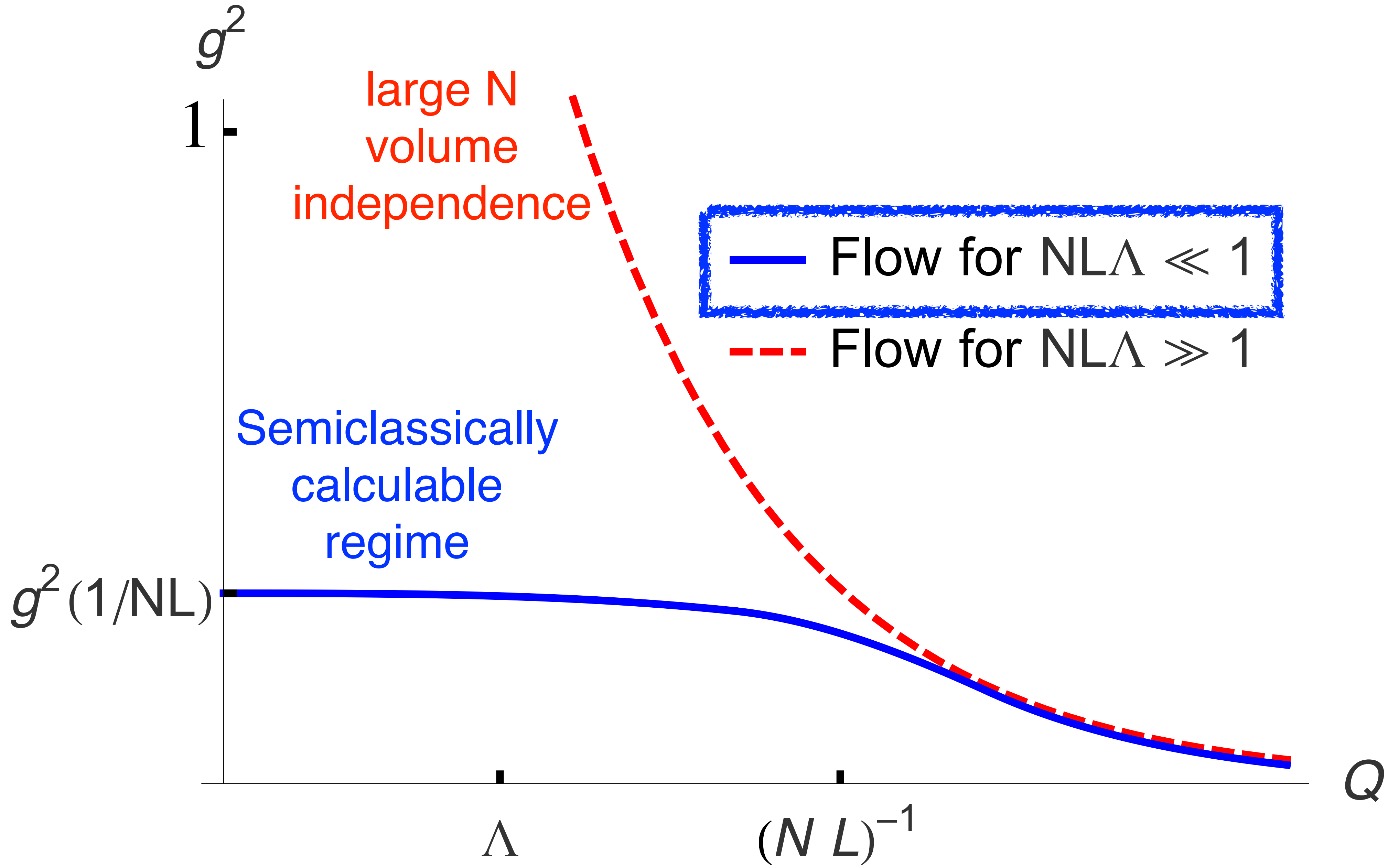
'Confinement' even at small L

Z_N-symmetric BCs give desired adiabatic small-volume limit.

Related construction of an adiabatic small L limit known for 4D YM theories

Unsal, Yaffe; Shifman, Unsal; ...

Flow of coupling constant in Z_N-twisted PCM



Scale NL appears due to Z_N-symmetric form of H_V

We focus on $NL\Lambda << 1$ to get a weakly-coupled theory

Physics remains very rich - mass gap, renormalons still remain!

Perturbation theory at small L

For small L, 2D PCM describable via to 1D EFT: quantum mechanics with a Z_N-symmetric background gauge field

Are renormalons still present?

In PCM, $|\beta| = N$. Renormalon means an ambiguity in perturbation theory of order

$$\sim \pm i e^{-\frac{\#}{g^2 N}}$$

On R^2, integrability calculations of Kazakov, Fateev, Wiegmann give:

$$\sim \pm i e^{-\frac{8\pi}{g^2 N}}$$

If small-L limit is adiabatic, expect size of renormalon ambiguity to move by order-1 amount as L goes from large to small.

But result should still involve $\#/g^2N$

Perturbation theory at small L

SU(2) Example

$$U = \begin{pmatrix} \cos\theta e^{i\phi_1} & i\sin\theta e^{i\phi_2} \\ i\sin\theta e^{-i\phi_2} & \cos\theta e^{-i\phi_1} \end{pmatrix}$$

Hopf parametrization

$$S = \frac{1}{g^2}\int_{\mathbb{R}\times S^1} dt\,dx\,\left[(\partial_\mu\theta)^2 + \cos^2\theta(\partial_\mu\phi_1)^2 + \sin^2\theta(\partial_\mu\phi_2 + \xi\delta_{\mu,x})^2\right]$$

KK reduction

Imprint of Z_N-sym. twist $\xi = 2\pi/(NL) = \pi/L$

$$S = \frac{L}{g^2}\int dt\left[\dot\theta^2 + \cos^2\theta\,\dot\phi_1^2 + \sin^2\theta\,\dot\phi_2^2 + \xi^2\sin^2\theta\right]$$

Compute e.g. series for ground state energy:

$$\mathcal{E}(g^2) = E\xi^{-1} = \sum_{n=0}^{\infty} p_n (g^2)^n$$

Perturbation theory at small L

SU(2) Example

To get high-order small L behavior, easiest to work in Hamiltonian formalism

$$H = \frac{g^2}{4L} P_\theta^2 + \frac{L\xi^2}{g^2}\sin^2\theta + \frac{g^2}{4L\sin^2\theta} P_{\phi_1}^2 + \frac{g^2}{4L\cos^2\theta} P_{\phi_2}^2$$

Will see that mass gap is $L^{-1}e^{-1/\lambda}$, while f_1, f_2 quanta cost L^{-1}

Means we can treat H in Born-Oppenheimer approximation, freezing f_1, f_2.

Now ground state energy found from solution of Schrodinger equation

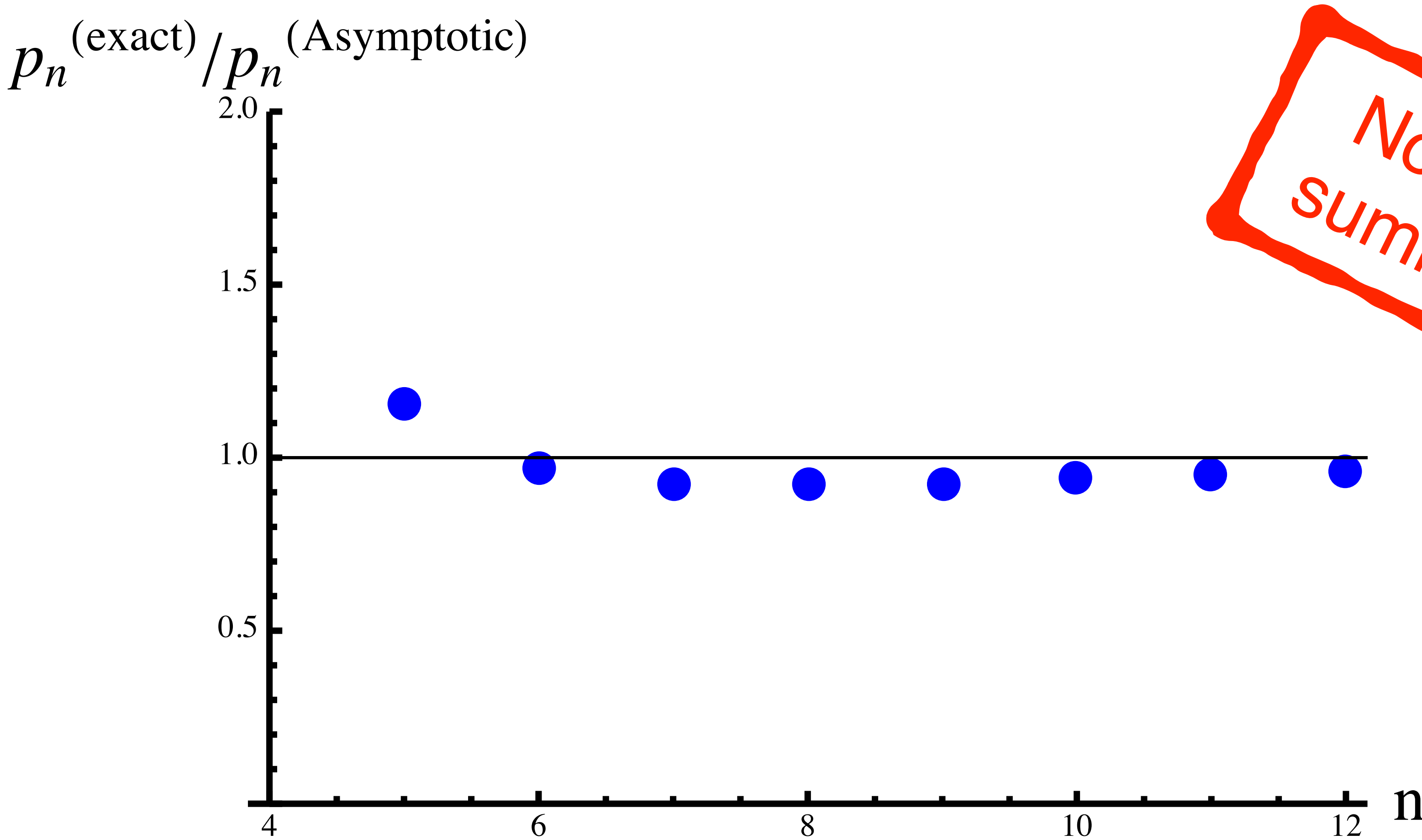
$$\left[-\frac{1}{2}\frac{d}{d\theta^2} + \frac{\xi^2}{g^2}\sin^2(\theta)\right]\psi = E\psi \qquad \theta \in [0,\pi]$$

Large order structure of perturbation theory

Stone, Reeve 1978

Large-order behavior can be shown to be

$$p_n \sim -\frac{2}{\pi}\left(\frac{1}{\frac{16\pi}{N}}\right)^n n! \left[1 - \frac{5}{2n} + \mathcal{O}(n^{-2})\right]$$



Not Borel summable!

Factorially growing and non-alternating series!

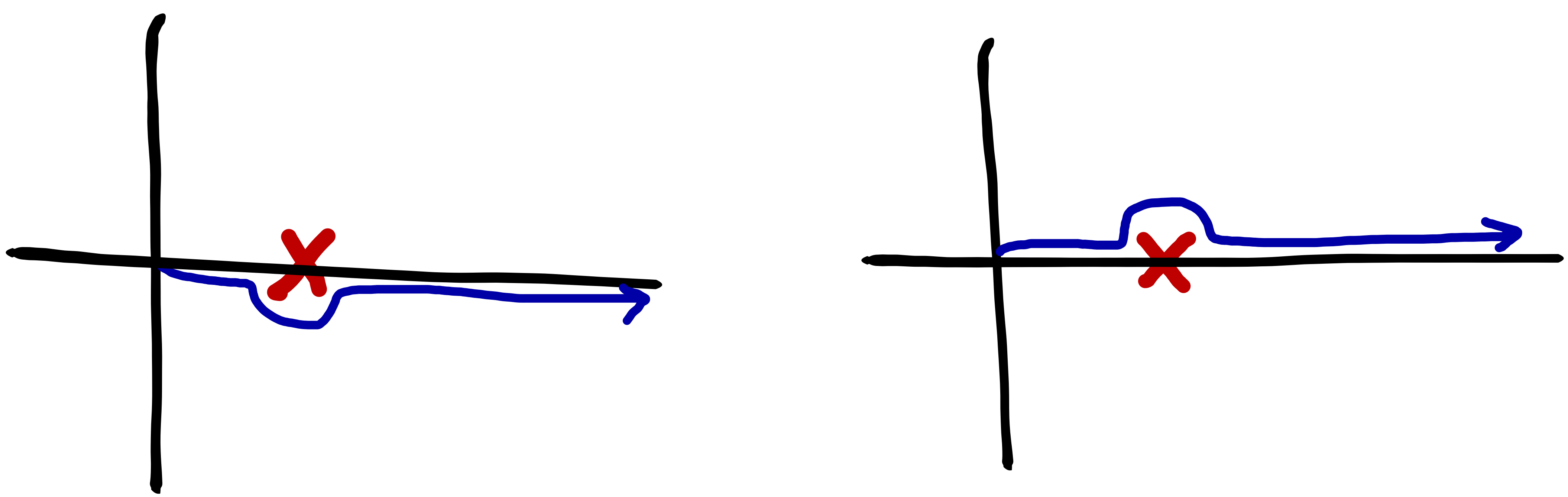
Non-perturbative ambiguity

Borel transform of leading n! piece is

$$B\mathcal{E}(t) \sim \text{polynomial} + \frac{2}{\pi}\sum_{n=0}^{\infty}\left(\frac{t}{\left[\frac{16\pi}{N}\right]}\right)^n = \text{polynomial} - \frac{2}{\pi}\frac{1}{1-\frac{t}{\left[\frac{16\pi}{N}\right]}}$$

$$\mathcal{S}\mathcal{E}(g^2) = \int_0^\infty dt e^{-t/g^2} B\mathcal{E}(t)$$

Singularity on C=R⁺ at t = 16π/N, Borel sum does not exist!



Non-perturbative ambiguity

$$\mathcal{S}_{\pm}\mathcal{E}(\lambda) = \int_{C_\pm} dt e^{-t/g^2} B\mathcal{E}(t)$$

$$= \mathrm{Re}\mathcal{S}\mathcal{E}(\lambda) \mp i\frac{32\pi}{\lambda}e^{-16\pi/\lambda}$$

$\lambda = g^2N$
renormalon!

What to make of red term?

(1) System is stable, ground state energy must be real!

(2) E must be well-defined - no sign-ambiguous bits allowed!

If E is a `resurgent function', perturbation ambiguities must cancel against ambiguities of some non-perturbative saddle F

$$\mathrm{Im}\left[\mathcal{S}_{\pm}\mathcal{E}(g^2) + [\mathcal{F}\bar{\mathcal{F}}]_{\pm}\right] = 0, \text{ up to } \mathcal{O}\left(e^{-4S_F}\right)$$

plus more intricate relations between P and NP physics at higher orders

But what are the relevant saddle points in the PCM?

Recall $\pi_2[SU(N)] = 0$…

Uhlenbeck 1985...

Non-topological saddle points

Finite-action `uniton' solutions of PCM EoMs are known

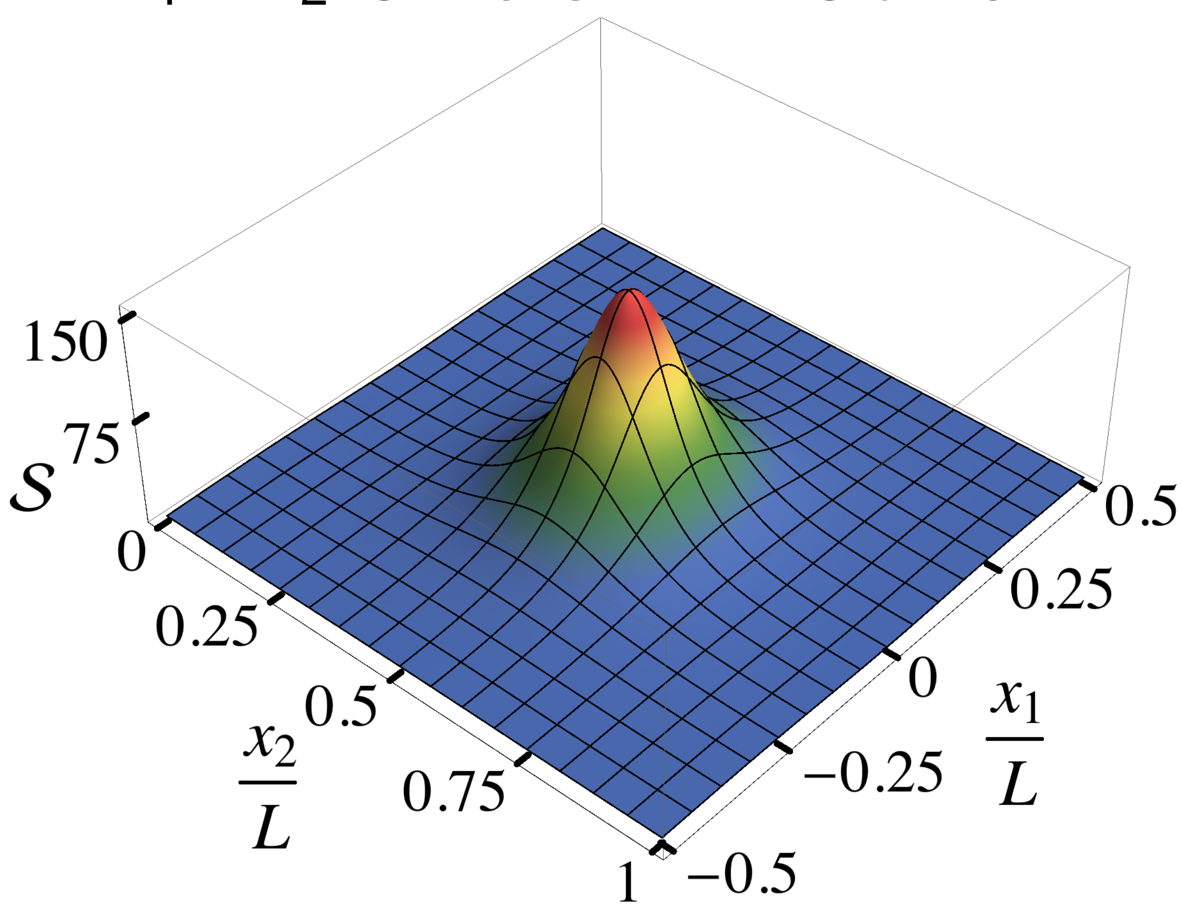
Based on observation that CP^{N-1} is a geodesic submanifold of SU(N)

CP^{N-1} instantons lift to uniton solutions in SU(N) PCM

Stable solutions within CP^{N-1} submanifold, but not in the full SU(N) manifold!

$$U(z,\bar{z}) = e^{i\pi/N}(1-2\mathbb{P}) \qquad \mathbb{P} = \frac{v \cdot v^\dagger}{v^\dagger \cdot v}$$

v(z), $z = x_1 + i\, x_2$ is the CP^{N-1} instanton in homogeneous coordinates



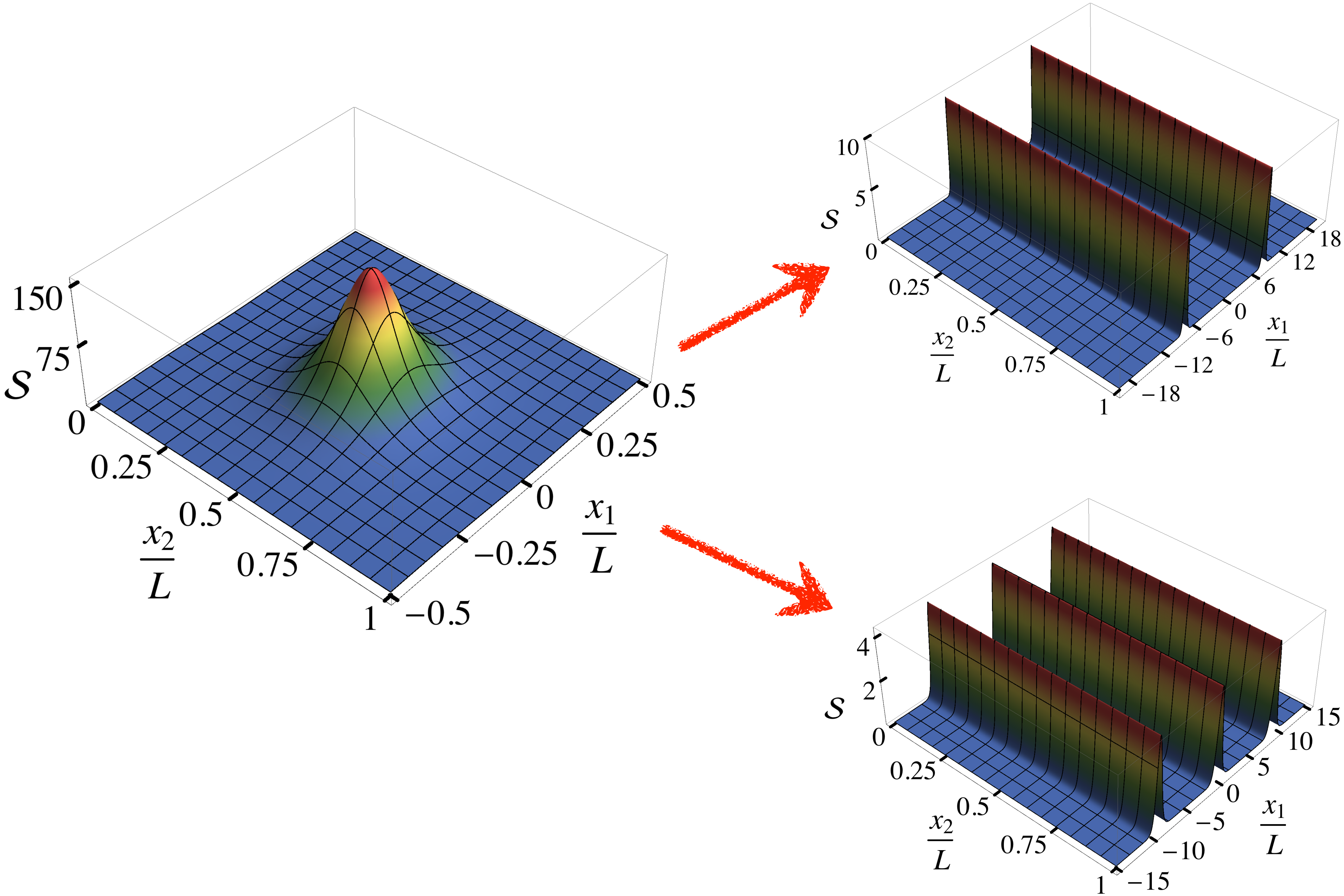
$$S_{\text{uniton}} = \frac{8\pi}{g^2}$$

see also Smilga, Shifman in Schwinger model, 1994

Fractons

AC, Dorigoni, Dunne, Unsal

Uniton appearance with Z_N-twisted BCs depends on size modulus



SU(2)

SU(3)

Unitons fractionalize into N `fracton' constituents on small S^1

Fractons

SU(2) Example, small L

$$S = \frac{L}{g^2} \int dt \left[\dot{\theta}^2 + \cos^2\theta \dot{\phi}_1^2 + \sin^2\theta \dot{\phi}_2^2 + \xi^2 \sin^2\theta \right]$$

Explicit solutions:

$$\theta(t;t_0) = 2\,\mathrm{arcCot}\left[e^{-\xi(t-t_0)} \right]$$

$$\bar{\theta}(t;t_0) = \pi - 2\,\mathrm{arcCot}\left[e^{-\xi(t-t_0)} \right]$$

ϕ_1=const

ϕ_2=const

$$S_{\text{fracton}} = \frac{8\pi}{g^2 N} = \frac{S_{\text{uniton}}}{N}$$

N types of minimal-action fractons in SU(N)

N-1 fractons associated to N-1 simple roots of su(N)
The other - called KK fracton - associated to \`affine root'

KK Fractons

Lee+Yi;
Kraan+van Baal

KK fractons in PCM appear same way as KK monopoles in compactified YM theories with non-trivial Wilson lines

Do KK reduction with n units of winding in compact scalar φ_2

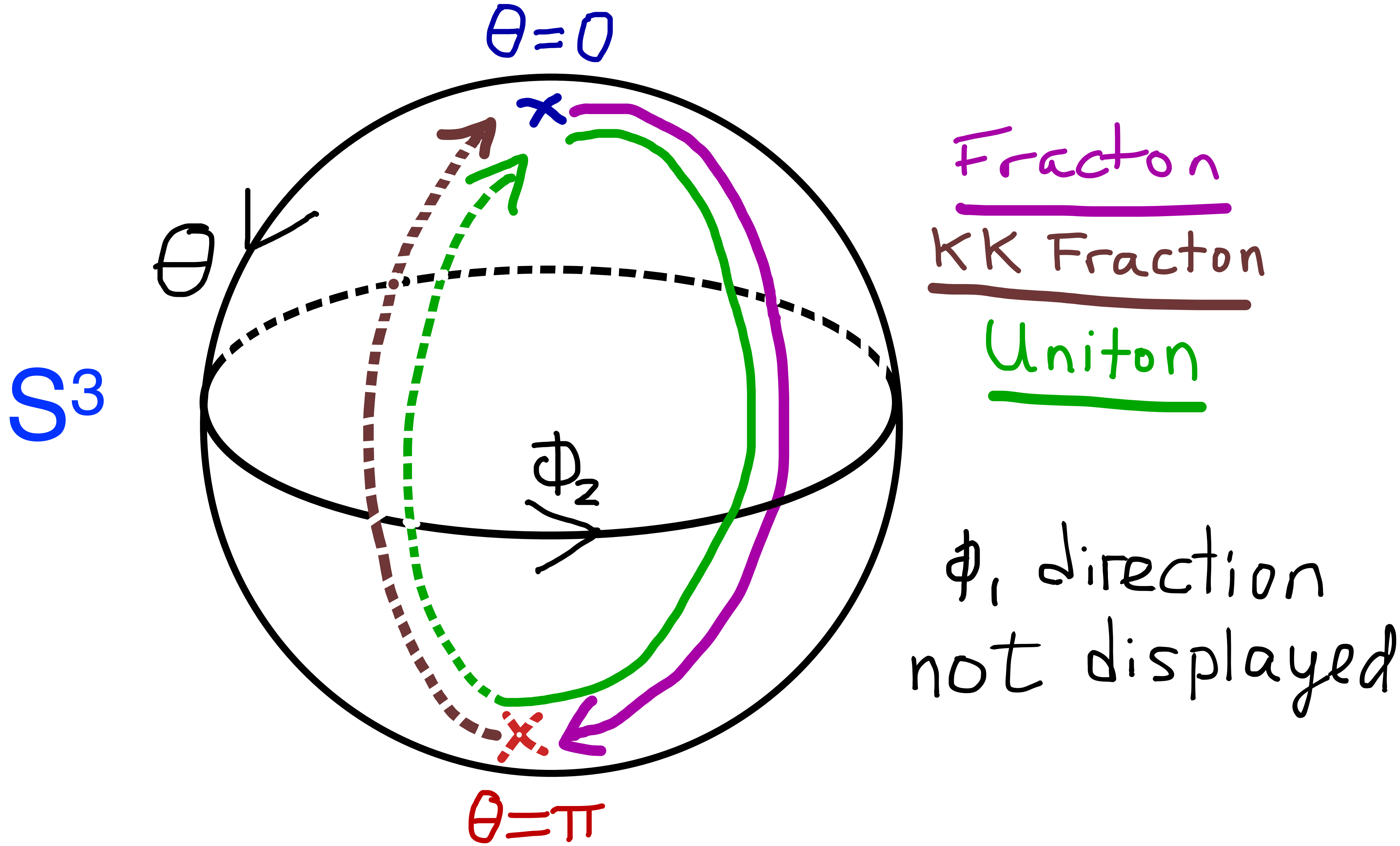
$$S = \frac{L}{g^2} \int dt \left[\dot{\theta}^2 + \cos^2\theta \dot{\phi}_1^2 + \sin^2\theta \dot{\phi}_2^2 + \left(\frac{2\pi n}{L} + \xi \right)^2 \sin^2\theta \right]$$

= ξ^2 when ξ is at center-symmetric value and n=-1

Example: take N=2, then $\xi=\pi/L$. Then $2\pi(-1)/L + \pi/L = -\pi/L$

$$S_{\text{KKfracton}} = \frac{8\pi}{g^2 N}$$

Unitons, Fractons, and KK fractons in SU(2)



SU(2) Uniton = fracton + KK fracton

The sum over finite-action configurations

$$\langle \mathcal{O}(\lambda) \rangle = \sum_{n=0}^{\infty} p_{0,n} \lambda^n + \sum_{c} e^{-S_c/\lambda} \sum_{k=0}^{\infty} p_{c,n} \lambda^n$$

How can NP saddles give ambiguous contributions to path integral?

Small-L theory weakly coupled, dilute fracton gas approximation is valid

Contributions entering NP sum:

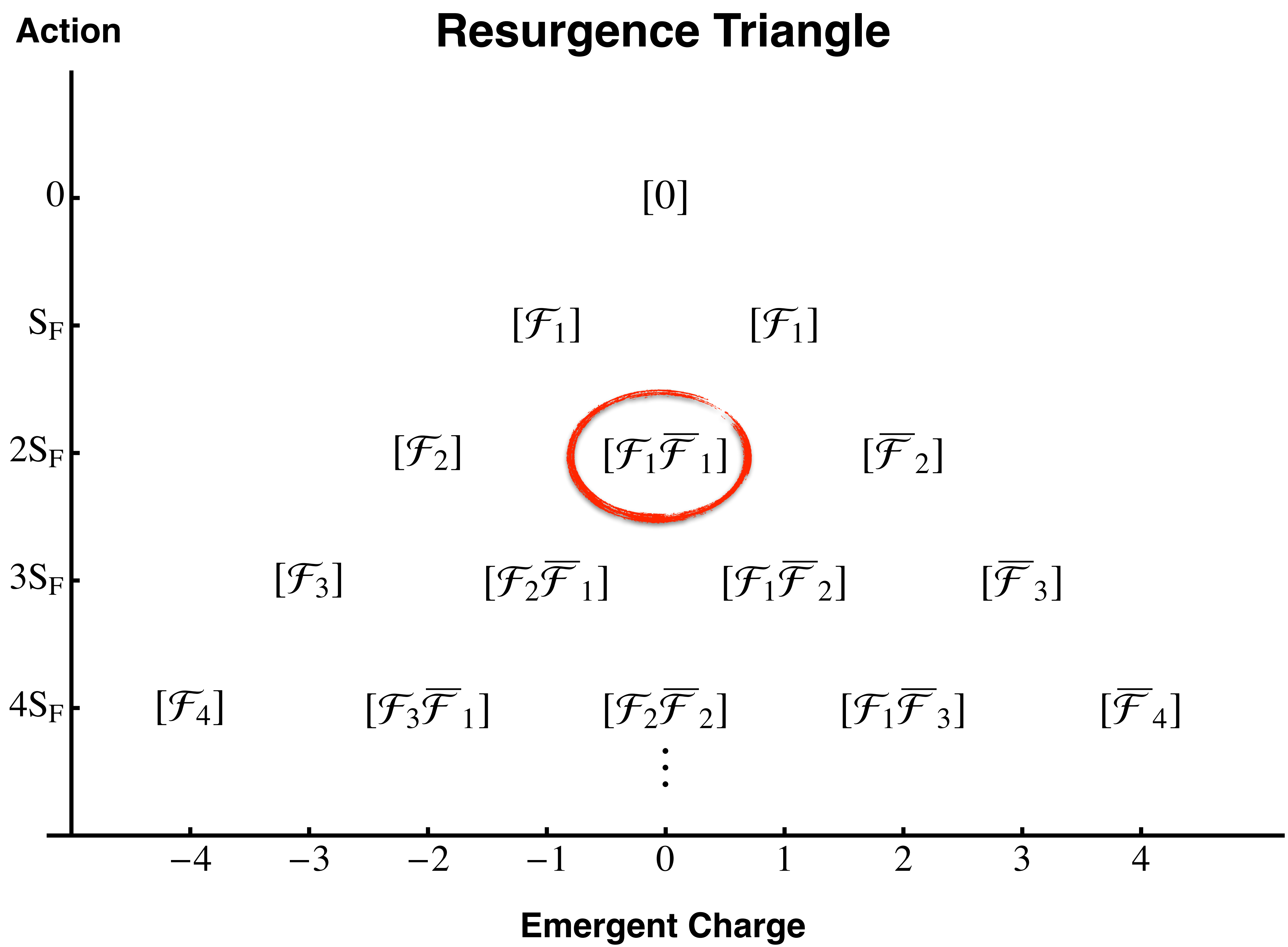
(1) Arbitrarily widely separated 'fundamental' fracton events

Within small-L EFT, individual fractons are just instantons, and are stable - 1-fracton events have unambiguous amplitudes

(2) Correlated multi-fracton events

Fluctuation sum includes zero modes, perturbative modes, and quasi-zero modes such as constituent separation

Gives rise to `correlated' events



Resurgence Triangle
Action
0
S_F
$2S_F$
$3S_F$
$4S_F$
$[0]$
$[\mathcal{F}_1]$ $[\mathcal{F}_1]$
$[\mathcal{F}_2]$ $[\mathcal{F}_1\overline{\mathcal{F}}_1]$ $[\overline{\mathcal{F}}_2]$
$[\mathcal{F}_3]$ $[\mathcal{F}_2\overline{\mathcal{F}}_1]$ $[\mathcal{F}_1\overline{\mathcal{F}}_2]$ $[\overline{\mathcal{F}}_3]$
$[\mathcal{F}_4]$ $[\mathcal{F}_3\overline{\mathcal{F}}_1]$ $[\mathcal{F}_2\overline{\mathcal{F}}_2]$ $[\mathcal{F}_1\overline{\mathcal{F}}_3]$ $[\overline{\mathcal{F}}_4]$
-4 -3 -2 -1 0 1 2 3 4
Emergent Charge

Contribution from fracton-anti-fracton events

Turns out: the interesting events are correlated ones.

Three types of fluctuations for multi-fracton configurations

(1) Zero modes
(2) Quasi-zero modes like fracton separation
(3) Gaussian modes + perturbative corrections

(2) is the subtle part - gives rise to notion of correlated events

Fracton size ~ $LN = \xi^{-1}$

Typical fracton separation ~ $LNe^{+8\pi/\lambda}$

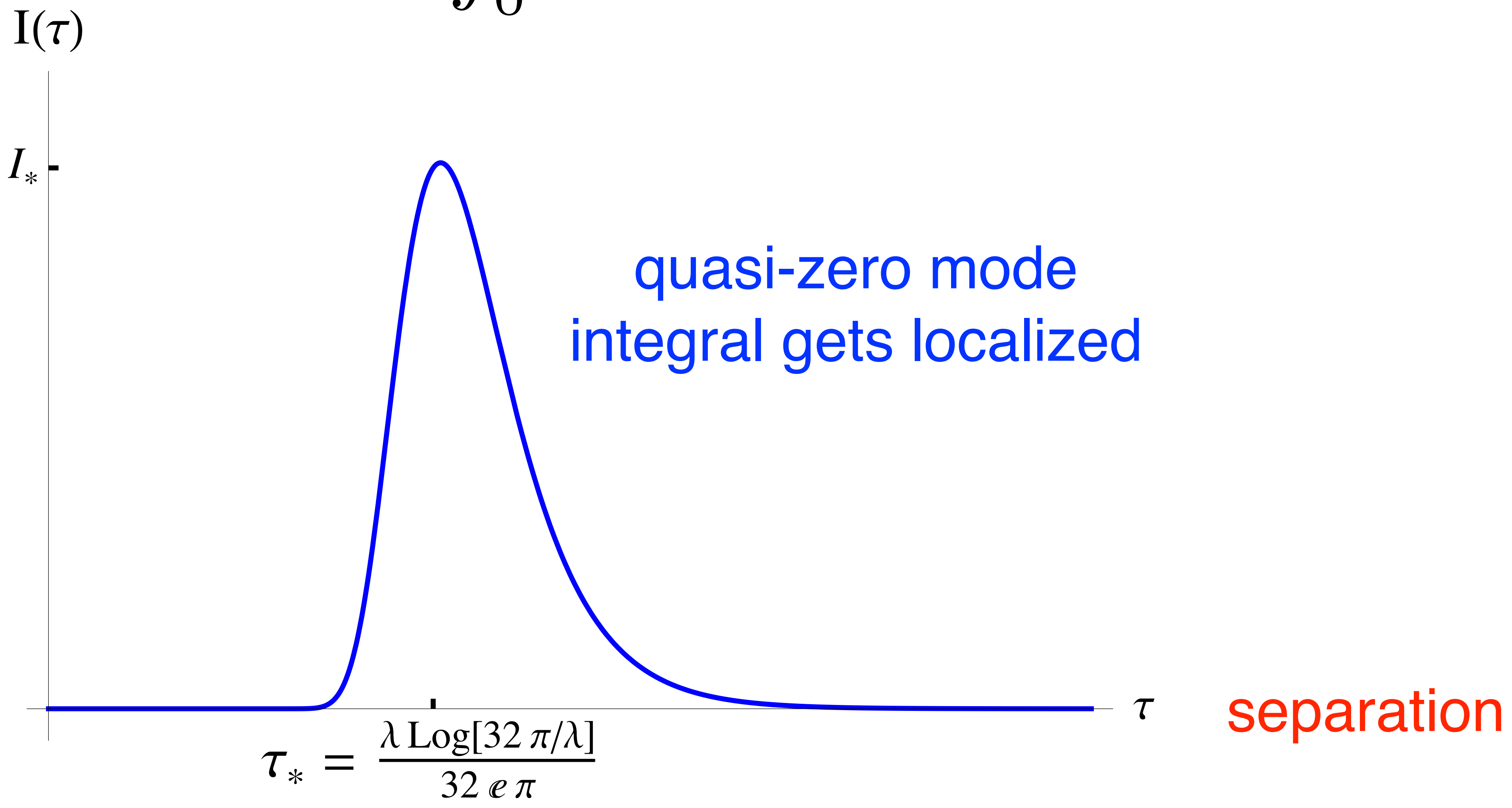
Quasi-zero mode integration reveals another scale!

Correlated fluctuation size ~ $LN \log\left(\frac{32\pi}{\lambda}\right)$

Correlated multi-fracton events

Correlated fracton-fracton events are unambiguous

$$I_{\mathcal{F}\mathcal{F}} \sim e^{-2S_F} \int_0^\infty d\tau \tau e^{-\left(\frac{32\pi}{\lambda} e^{-\tau} + \tau\right)}$$



amplitude: $$[\mathcal{F}\mathcal{F}] = \left(-\log\left[\frac{32\pi}{\lambda}\right] - \gamma\right)\frac{16}{\lambda} e^{-2S_F}$$

Correlated multi-fracton events

Correlated fracton-anti-fracton events are ambiguous

$$I_{\mathcal{F}\bar{\mathcal{F}}} \sim e^{-2S_F} \int_0^\infty d\tau \tau e^{-\left(-1\times\frac{32\pi}{\lambda}e^{-\tau}+\tau\right)}$$

The anti-fracton-fracton interaction is `attractive'!

Fracton-anti-fractons `want' to get close to annihilate

Since dilute gas approximation means all fractons must be widely separated, we should expect subtleties….

Making sense of fracton-anti-fracton events

Quasi-zero-mode integrals dominated by
τ=0 region, do not make sense as written

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This is a feature, not a bug.

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This is a feature, not a bug.

Bogomolny,
Zinn-Justin

Analytically continue $g^2 \to g^2(1 \pm i\epsilon)$

Remember, we had to this for perturbation theory too!

Away from $\mathrm{Im}[g^2]=0$, integral dominated by well-separated fractons

$$\begin{array}{ccccccc}
r_{\mathcal{F}} & \ll & r_{[\mathcal{F}\overline{\mathcal{F}}]_\pm} & \ll & d_{[\mathcal{F}][\overline{\mathcal{F}}]} & \ll & d_{[\mathcal{F}\overline{\mathcal{F}}][\mathcal{F}\overline{\mathcal{F}}]} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\frac{1}{\xi} & \ll & \frac{1}{\xi}\log(4S_F) & \ll & \frac{1}{\xi}e^{+S_F} & \ll & \frac{1}{\xi}e^{2S_F}
\end{array}$$

Analytic continuation back to positive g^2 is ambiguous!

$$[\mathcal{F}_j \bar{\mathcal{F}}_j]_\pm = \left(-\log\left[\frac{32\pi}{g^2 N}\right] - \gamma\right)\frac{16}{g^2 N}e^{-\frac{16\pi}{g^2 N}} \pm i\frac{16\pi}{g^2 N}e^{-\frac{16\pi}{g^2 N}}$$

Cancellation of ambiguities

Contribution from P saddle is ambiguous. So are some from NP saddles.

Neither is directly physical, only sum is. Resurgence predicts:

$$\mathrm{Im}\left[\mathcal{S}_{\pm}\mathcal{E}(g^2) + [\mathcal{F}\bar{\mathcal{F}}]_{\pm}\right] = 0, \text{ up to } \mathcal{O}\left(e^{-4S_F}\right)$$

Preceding result implies that this works in PCM

Systematic demonstration that leading renormalon ambiguities of perturbation theory cancel against ambiguities in saddle-point sum

Illustrates that exact information about NP physics is present in perturbation theory, albeit in coded form!

At higher order resurgence implies more intricate relations:

$$\begin{aligned} F(\lambda) &= \mathrm{Re}\mathcal{S}P_0 + \mathrm{Re}[\mathcal{F}\bar{\mathcal{F}}]\mathrm{Re}\mathcal{S}P_{\mathcal{F}\bar{\mathcal{F}}} + \mathrm{Im}[\mathcal{F}\bar{\mathcal{F}}]_{\pm}\mathrm{Im}\mathcal{S}_{\pm}P_{\mathcal{F}\bar{\mathcal{F}}} \\ &+ \mathrm{Re}[\mathcal{F}_2\bar{\mathcal{F}}_2]\mathrm{Re}\mathcal{S}P_{\mathcal{F}_2\bar{\mathcal{F}}_2} + \mathcal{O}(e^{-6S_F}) \end{aligned}$$

Mass gap at small L

The mass gap ~ one-fracton amplitude

$$\text{renormalon} \sim e^{-\frac{2\times 8\pi}{g^2 N}} = e^{-\frac{2\times 8\pi}{\lambda}}$$

Gap between ground state and first excited state in

$$\Delta_{\text{SU(N) PCM}} \sim \frac{1}{NL}\frac{8\pi}{\sqrt{\lambda}}e^{-\frac{8\pi}{\lambda}}$$

Same relation in all small-L cases checked so far: PCM, CP^N, YM

$$\Delta \sim \text{renormalon}^{1/2}$$

Relation also holds when massless fermions are added

Resurgent trans-series to all orders

Changed perspective: semiclassical expansions may be exact representations of QFT observables

$$\mathcal{O}(g^2) \approx \sum_{n=0,k=0}^{\infty} \sum_{q=1}^{k-1} c_{n,k,q}\, g^{2n} \left[e^{-S/g^2}\right]^k \left[\log\left(\frac{1}{g^2}\right)\right]^q$$

All series divergent, ambiguous sum if $c_{n,k,q}$ were random

Demanding $O(g^2)$ be *well-defined* implies relations between $c_{n,k,q}$

Resurgence means relations, not just cancellations - `ambiguities' at one order give unambiguous contribution at the next order.

Resurgence theory gives technology to find the relations

Perturbative and non-perturbative contributions intimately related

We've only just begun exploring the implications...

What we learned so far…

Even when there's no topology, resurgence predicts existence of NP saddle points with specific properties, which can then be found.

In semiclassical domain, renormalon ambiguities systematically cancel against contributions of non-BPS NP saddles

Renormalons closely related to mass gap, as 't Hooft dreamt

All results so far fit conjecture of resurgent nature of QFTs

Lots left to do!

Now exploring relations to analytic continuation of path integrals

Lefshetz thimble decomposition of integration cycles appears to geometrize resurgence

Witten 2010

AC, Dorigoni, Unsal 2014

There are likely to be many practical implications!

Better understanding of QFTs with complex actions?

Resurgence theory and Lefshetz thimble technology play vital role in seeing how instantons appear in real-time Feynman path integrals.

AC, Unsal 2014

Improved understanding of connections between strong and weak coupling regimes?

AC, Koroteev, Unsal 2014

Resurgence in SUSY QFTs?

Aniceto, Russo, Schiappa, 2014

End of Part 2