Shintani descent and character sheaves on algebraic groups

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Introduction

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- Conjecturally these sets can be described geometrically using "character sheaves", a geometric analogue of irreducible characters.
- Let $\mathscr{D}_G(G)$ be the braided monoidal category of conjugation equivariant $\overline{\mathbb{Q}}_\ell$-complexes on G (under convolution with compact support). Character sheaves on G are supposed to be certain special objects in $\mathscr{D}_G(G)$.

Lang's Theorem and pure inner forms

Consider the action of G on itself by F-twisted conjugation:

$$g : h \longmapsto ghF(g)^{-1}.$$

Denote by $H^1(F, G)$ the set of F-twisted conjugacy classes.

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Let G be connected. Then the Lang map $L : G \to G$

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is surjective, i.e. $H^1(F, G)$ is singleton.
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It is more natural to consider all pure inner forms $\mathrm{ad}(g) \circ F$ of the Frobenius F parameterized by $\langle g \rangle_F \in H^1(F,G)$. For $g \in G$ its stabilizer for the F-twisted conjugation action is the finite group $G^{\mathrm{ad}(g)\circ F}$.

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- In particular if $C_G(g)$ is connected then being geometrically conjugate to g is equivalent to being rationally conjugate.

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$$\begin{array}{cccc} \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} & \begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix} & \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} & \begin{pmatrix} x & y \\ \epsilon y & x \end{pmatrix} \\ t \in \mathbb{F}_q^\times & t \in \mathbb{F}_q^\times & t_1 \neq t_2 & y \neq 0 \end{array}$$

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On the other hand, for $SL_2(\mathbb{F}_q)$ there is a difference:

$$\underset{\pm I}{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \quad \underset{\pm u}{\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} \quad \underset{\pm u'}{\pm\begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}} \quad \underset{\substack{t\in\mathbb{F}_q^\times\setminus\{\pm 1\} \\ t}}{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}} \quad \underset{\substack{y\neq 0 \\ s}}{\begin{pmatrix} x & y \\ \epsilon y & x \end{pmatrix}}$$

Here the two geometric conjugacy classes of u and $-u$ each split up into two rational conjugacy classes.

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- In general the twist of an irreducible character can be complicated.

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tr	1	1	1	1	1	1	1	1
St	q	q	0	0	0	0	1	-1
$R_{T,\chi}$	$q+1$	$(q+1)\chi(-1)$	1	1	$\chi(-1)$	$\chi(-1)$	$\chi(t)+\overline{\chi(t)}$	0
$R_{S,\eta}$	$q-1$	$(q-1)\eta(-1)$	-1	-1	$-\eta(-1)$	$-\eta(-1)$	0	$-\eta(s)-\overline{\eta(s)}$
R'_{T,χ_0}	$\frac{q+1}{2}$	$\frac{\varepsilon(q+1)}{2}$	$\frac{1+\sqrt{\varepsilon q}}{2}$	$\frac{1-\sqrt{\varepsilon q}}{2}$	$\frac{\varepsilon(1+\sqrt{\varepsilon q})}{2}$	$\frac{\varepsilon(1-\sqrt{\varepsilon q})}{2}$	$\chi_0(t)$	0
R''_{T,χ_0}	$\frac{q+1}{2}$	$\frac{\varepsilon(q+1)}{2}$	$\frac{1-\sqrt{\varepsilon q}}{2}$	$\frac{1+\sqrt{\varepsilon q}}{2}$	$\frac{\varepsilon(1-\sqrt{\varepsilon q})}{2}$	$\frac{\varepsilon(1+\sqrt{\varepsilon q})}{2}$	$\chi_0(t)$	0
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Θ

We see that $SL_2(\mathbb{F}_q)$ has 4 exceptional characters that are not preserved by the twist Θ.

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$\frac{R'_T+R''_T+R'_S+R''_S}{2}$	q	ε	0	0	ε	ε	$\chi_0(t)$	$-\eta_0(s)$
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We see that the first 3 "almost characters" above are fixed by Θ, whereas the last one is an eigenvector with eigenvalue -1. The unitary matrix

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

relating the 4 special characters and almost characters is the S-matrix of a certain modular category, namely the Drinfeld double of $\mathbb{Z}/2\mathbb{Z}$.

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- The Frobenius induces an automorphism of G^{F^m} of order m, which we also denote by F.
- We first define the norm map

$$N_m : (G^{F^m}/\sim_F) \xrightarrow{\cong} (G^F/\sim)$$

defined by

$$\langle g\rangle_F \mapsto \langle x^{-1}gF(g)\cdots F^{m-1}(g)x\rangle = \langle F^m(x)^{-1}x\rangle,$$

where $G^{F^m} \ni g = xF(x)^{-1}$.

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- We will now define Shintani descent, which is a map

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- Finally we set $\mathrm{Sh}_m(\chi) = N_m^{-1*}(\widetilde{\chi})$.

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Conjecture (partly proved by T.Shoji)

There exists an integer m_0 such that if m is divisible by m_0, then the image of Sh_m in $\mathrm{Fun}(G^F/\sim)$ is independent of m (up to scaling by roots of unity). Moreover for such an m, $\mathrm{Sh}_m(\chi)$ is an eigenvector for the twisting operator Θ^*.

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We call this conjectural common image of Sh_m as the set of almost characters of G^F. They form an orthonormal basis of $\mathrm{Fun}(G^F/\sim)$.

Remark on disconnected groups

- Everything that we have said can also be defined for disconnected groups G. In this case we should consider all pure inner forms $gF := \mathrm{ad}(g) \circ F$ of the Frobenius. (They are parameterized by $H^1(F, G)$.)

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- Instead of $\mathrm{Irrep}(G^F), \mathrm{Irrep}(G^{F^m})$ we should rather work with

$$\coprod_{\langle g \rangle_F \in H^1(F,G)} \mathrm{Irrep}(G^{gF}), \qquad \coprod_{\langle g \rangle_{F^m} \in H^1(F^m,G)} \mathrm{Irrep}(G^{gF^m}).$$

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- Once we make such adjustments, everything goes through for disconnected groups as well.

The triangulated braided category $\mathscr{D}_G(G)$

- We now describe a geometric approach to studying the almost characters described previously in terms of triangulated braided monoidal category $\mathscr{D}_G(G)$ of conjugation equivariant $\overline{\mathbb{Q}}_\ell$-complexes on G. (We fix some prime number $\ell \neq p$.)

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$$C_1 * C_2 = \mu_!(C_1 \boxtimes C_2) = \mu_!(p_1^* C_1 \otimes p_2^* C_2).$$

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$$\theta_{C_1 * C_2} = \beta_{C_2,C_1} \circ \beta_{C_1,C_2} \circ (\theta_{C_1} * \theta_{C_2}).$$

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- $\mathscr{D}_G^F(G)$ is a $\mathscr{D}_G(G)$-module category, namely if $C \in \mathscr{D}_G(G), M \in \mathscr{D}_G^F(G)$, then $C * M \in \mathscr{D}_G^F(G)$.

The sheaf-function correspondence

- Let X be a scheme with a Frobenius $F : X \to X$. Let $M \in \mathscr{D}(X)$ be such that we have $\psi : F^*M \xrightarrow{\cong} M$. Then we have the "trace of Frobenius" function

$$t_{M,\psi} : X^F \longrightarrow \overline{\mathbb{Q}}_\ell$$

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- If $C \in \mathscr{D}_G(G)$ is such that we have $\psi : F^*C \xrightarrow{\cong} C$, then we can define its associated trace of Frobenius function on each pure inner form and obtain a function $T_{C,\psi} \in \operatorname{Fun}([G], F)$.

The trace of Frobenius functions and twists

The trace functions obtained from F-stable objects in $\mathscr{D}_G(G)$ behave nicely with respect to the twist Θ

Lemma

Let $C \in \mathscr{D}_G(G)$ be such that its associated twist $\theta_C : C \to C$ is a scalar and such that we have $\psi : F^*C \xrightarrow{\cong} C$. Then we have

$$\Theta^*(T_{C,\psi}) = \theta_C \cdot T_{C,\psi}.$$

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We say that an object $C \in \mathscr{D}_G(G)$ is simple if $\mathrm{End}(C) = \overline{\mathbb{Q}}_\ell$. Hence if $C \in \mathscr{D}_G(G)$ is an F-stable simple object then its associated trace function is an eigenvector for the twisting operator with eigenvalue θ_C.

Main conjecture

Our main goal is to define a set of some special simple objects in $\mathscr{D}_G(G)$ called character sheaves and they should satisfy the following conjectural properties:

Conjecture 1

There exists a (possibly infinite) set $CS(G)$ of isomorphism classes of some special simple objects of $\mathscr{D}_G(G)$ such that

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- Each $CS_f(G)$ is further partitioned into families called $\mathbb{L}$-packets. Each $\mathbb{L}$-packet is a finite set.
- Associated with each $\mathbb{L}$-packet, there is a modular category whose simple objects are the character sheaves in that $\mathbb{L}$-packet.

Main conjecture

Conjecture 1 (contd.)

- Let $\varphi : \mathscr{D}_G(G) \to \mathscr{D}_G(G)$ be any braided triangulated autoequivalence, hence it preserves the set of minimal idempotents. Then φ also preserves the set $CS(G)$ and respects its $\mathbb{L}$-packet decomposition. In particular F^* acts on the set $CS(G)$ and the set of $\mathbb{L}$-packets.

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- For each $C \in CS(G)^F$, we fix an isomorphism $F^*C \xrightarrow{\psi_C} C$ such that $|T_{C,\psi_C}| = 1$. Then the set $\{T_{C,\psi_C}\}_{C \in CS(G)^F} \subset \mathrm{Fun}([G], F)$ is an orthonormal basis.

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- The conjectural orthonormal basis of $\mathrm{Fun}([G], F)$ of almost characters defined using Shintani descent agrees with the basis $\{T_{C,\psi_C}\}_{C \in CS(G)^F}$ (up to scalings).

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- The unitary matrix relating the two bases $\mathrm{Irrep}(G, F)$ and $\{T_{C,\psi_C}\}_{C \in CS(G)^F}$ of $\mathrm{Fun}([G], F)$ is block diagonal with small blocks. The blocks correspond to F-stable $\mathbb{L}$-packets.

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- Associated with each F-stable $\mathbb{L}$-packet is an invertible module category over the corresponding modular category. The corresponding block in the change of basis matrix is the crossed S-matrix associated with this module category.

Our approach towards a theory of character sheaves

- We aim to break up the category $\mathscr{D}_G(G)$ into more manageable pieces using minimal idempotents/admissible pairs.

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- Let $G' \subset G$ be the normalizer of an admissible pair. Then $e_{H,\mathcal{L}} := \mathcal{L} \otimes \mathbb{K}_H$ is a closed idempotent in $\mathscr{D}_{G'}(G')$ and $f_{H,\mathcal{L}} := \operatorname{ind}_{G'}^{G} e_{H,\mathcal{L}}$ is an idempotent in $\mathscr{D}_G(G)$.

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- If $G = G'$, we say that the admissible pair $(H, \mathcal{L})$ and closed idempotent $e_{H,\mathcal{L}} = f_{H,\mathcal{L}}$ are *Heisenberg*.
- If we have an admissible pair with normalizer G', we have the reductive group $G'_{red} := G'/\mathscr{R}_u(G')$ associated with the admissible pair.

An auxiliary conjecture

Conjecture 2

- The idempotent $f_{H,\mathcal{L}} \in \mathscr{D}_G(G)$ associated with an admissible pair is locally closed and we have a braided equivalence $\mathrm{ind}_{G'}^{G} : e_{H,\mathcal{L}}\mathscr{D}_{G'}(G') \xrightarrow{\cong} f_{H,\mathcal{L}}\mathscr{D}_G(G)$.

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In [De1], we reduce this conjecture to the following apriori weaker conjecture:

Conjecture 2'

Any Heisenberg idempotent $e_{H,\mathcal{L}} = f_{H,\mathcal{L}}$ is a minimal idempotent.

Examples

- If G is reductive, then $(1, \overline{\mathbb{Q}}_\ell)$ is the only admissible pair for G. It is Heisenberg.

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  The isomorphism class of $f_{\mathbb{G}_a,\mathcal{L}}$ is independent of $\mathcal{L}$.
- (Geometric Weil representation) Let V be a symplectic vector space over $\mathtt{k}$ and let U $(= V \times \mathbb{G}_a$ as a variety) be the corresponding Heisenberg group. Then $(\mathbb{G}_a, \mathcal{L})$ is a Heisenberg admissible pair for the unipotent group U and also for the group $G := Sp(V) \ltimes U$. We have $G_{red} = Sp(V)$. In this case we can prove that $e_{\mathcal{L}}\mathscr{D}_G(G) \cong \mathscr{D}_{G_{red}}(G_{red})$.

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- We remark that $e\mathscr{D}_G(G)$ is a "twisted version" of $\mathscr{D}_{G_{red}}(G_{red})$.
- The conjectures 1,2,3 for neutrally unipotent groups follow from [B], [BD], [De2].
- For reductive groups, many aspects of Conjecture 1 are proved in the works of G. Lusztig, T. Shoji and others. However many of these conjectures are still open in this case.

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