

Non-equilibrium Quantum Field Theory and cosmological applications

João G. Rosa

University of Aveiro and University of Porto, Portugal
joao.rosa@ua.pt <http://gravitation.web.ua.pt/jrosa>

Abstract

This is a set of introductory lectures on non-equilibrium Quantum Field Theory and its applications in Cosmology. I will focus on describing how the interactions between a system and its environment lead to dissipative effects and associated fluctuations, which are fully encoded in a generic Langevin-like equation. First, I will discuss how this arises in the context of a simple quantum mechanical model and then move on to give a full description of these effects in Quantum Field Theory, using the real-time Schwinger-Keldysh formalism. I will give simple examples of dissipation coefficients for scalar interactions and use them to discuss how the effects of dissipation and fluctuations modify the dynamics of cosmological scalar fields, focusing on the framework of warm inflation.

Contents

1	Introduction	2
2	Dissipation in Quantum Mechanics: the Caldeira-Leggett model	3
3	The Schwinger-Keldysh formalism for non-equilibrium QFT	7
3.1	Propagators	8
3.2	Keldysh representation	10
4	Langevin-like equation for a dynamical scalar field	10
5	Examples of local dissipation coefficients	12
6	Applications to Cosmology: the case of warm inflation	15
7	Summary and further reading	21

1 Introduction

This is an introductory mini-course, aimed at graduate students and researchers, on the description of non-equilibrium systems in Quantum Field Theory (QFT), with a particular focus on aspects that are relevant for applications in Cosmology. Knowledge of basic tools in quantum mechanics and QFT is assumed and some basic knowledge of equilibrium thermal field theory is helpful, although I will try to make the course as much self-contained as possible. This is not meant to be a review, although I will give a selected bibliography for those interested in pursuing some of the topics presented. Non-equilibrium QFT is a broad subject with several different applications and methodologies, and here I will concentrate on the main tools that I use in my own research and that are particularly useful in exploring the cosmological impact of non-equilibrium dynamics. This is thus necessarily a somewhat biased discussion, but I hope to give a sufficiently broad view of the subject and methods that can be applied in different physical settings.

In many physical problems we are interested in studying the dynamical evolution of a given system that is coupled to a more or less complex environment. The focus is typically on the average properties of the system, which can often be described in terms of a few simple dynamical variables and the properties of the environment, but in many cases we are also interested in the behaviour of fluctuations about this average evolution. Non-equilibrium QFT is a very useful and powerful formalism to explore a plethora of different systems, corresponding to one or more fundamental (or composite) fields that evolve in a given environment.

Non-equilibrium dynamics is in general quite complicated to describe, but there are particular systems where a simple description is possible. This is, for example, the case of an environment that is in a state close to (thermal) equilibrium and which is only weakly perturbed by the system that we are interested in. The problem also simplifies considerably when there is a clear separation of time scales between the system and the environment, for example if the system evolves slowly compared to the typical response time of the environment to local disturbances. In these cases we may generically take the environment to be always in a local equilibrium state, described by a simple set of macroscopic variables (namely its temperature), and use this to compute its backreaction on the dynamics of the system. As we will see, this leads on average to dissipation of the system's energy into the environment, which adiabatically perturbs the latter so that it quickly returns to a (new) equilibrium state. In addition, random fluctuations in the thermal environment will also lead to random fluctuations in the system, whose amplitude is intrinsically related to the strength of the dissipative effects. This is generically known as the fluctuation-dissipation theorem, which takes a simple form in the adiabatic and quasi-equilibrium conditions given above but also extends to more general non-equilibrium scenarios.

Although these conditions may seem quite constraining, they are satisfied by a large number of physical systems, particularly in Cosmology. Firstly, there is now plenty observational evidence (e.g. the Cosmic Microwave Background) for the universe to have attained a state of local thermal equilibrium very early in its history, with the different particle species scattering off each other (and also decaying) much faster than the Universe's expansion rate. This yields an environment in local thermal equilibrium at a temperature $T(t)$, which decays in time due to Hubble expansion in the absence of entropy production processes.

The paramount example of a slowly evolving system that may interact with this environment is the inflaton field, whose vacuum energy is postulated to drive a period of accelerated expansion in the early universe in the "slow-roll" regime. As first proposed by Arjun Berera [1], interactions between this field and the environment will lead to dissipative effects, in the adiabatic regime described above, which damp the inflaton's motion, making it easier to sustain an almost constant vacuum energy for sufficiently long to explain the observed flatness and homogeneity of the present universe. Dissipation also transfers the inflaton's energy into the cosmic environment, resulting in entropy production that prevents the universe from supercooling due to the exponentially fast expansion. This generic scenario is thus known as the *warm inflation* paradigm and in many cases this entropy production is sufficient to smoothly end inflation, avoiding the need for a separate reheating period. Moreover, the fluctuations that, as explained above, are inherently related to the average dissipative effects will also modify the spectrum of inflaton perturbations and consequently the distribution of CMB anisotropies, thus modifying the observational predictions of standard inflationary models.

Many of the cosmological applications of non-equilibrium QFT have been done in the context of warm inflation, where for example dissipation has also been shown to lead to a cosmological baryon asymmetry [2] or even an asymmetry in the dark matter sector [3] for particular types of interaction. Non-equilibrium effects have, however,

a much wider range of applicability and several studies have already been conducted e.g. in the context of reheating after inflation [4–8], curvaton models [9] or cosmological phase transitions driven by a Higgs-like scalar field [10]. The fact that dissipation tends to slow down the evolution of these fields also implies in many cases that an adiabatic description can be used to describe at least part of the non-equilibrium evolution. Another example where non-equilibrium dynamics arises is in the description of the properties of the quark-gluon plasma (see e.g. [11, 12]).

This mini-course is organized as follows. As a warm up exercise, where the main features of the fluctuation-dissipation relations can be easily understood, we will explore a simple example of dissipation in (non-relativistic) quantum mechanics widely known as the Caldeira-Leggett model. I will then introduce the basic tools of the real time Closed Timed Path (CTP) or Schwinger-Keldysh formalism to study non-equilibrium effects in QFT in the path integral approach. I will focus on scalar fields, not only since this is the simplest example to study but also because the systems of interest for the cosmological applications described above naturally involve scalar fields. I will describe how dissipation and related fluctuations arise in this context, leading to a Langevin-like equation for the evolution of a generic scalar field, and give some simple (but useful!) examples of local dissipation coefficients in scalar models. I will finish by discussing cosmological applications in the context of warm inflation, which is the best-known scenario but where many of the tools employed can be applied in other cosmological problems.

At the end of these notes I also give a selected list of references that may help you go further in understanding the quantum field theory methods used to describe non-equilibrium systems and their different applications in Cosmology.

2 Dissipation in Quantum Mechanics: the Caldeira-Leggett model

We begin our discussion by considering a simple quantum mechanical system corresponding to a heavy particle of mass M , moving in one spatial dimension described by a coordinate q and its conjugate momentum p . For a generic potential $V(q)$, the Hamiltonian for our system is thus:

$$H_S = \frac{p^2}{2M} + V(q) , \quad (1)$$

which is of course written in terms of the corresponding quantum mechanical hermitian operators. Let us now assume that this particle is moving in a bath of other particles, described by a set of N harmonic oscillators that form the environment:

$$H_B = \sum_{i=1}^N \left(\frac{p_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 x_i^2 \right) . \quad (2)$$

System and environment are then coupled through bilinear interactions of the form:

$$H_{SB} = -q \sum_{i=1}^N g_i x_i + q^2 \sum_{i=1}^N \frac{g_i^2}{2m_i \omega_i^2} . \quad (3)$$

The second term is a counter-term required to ensure that $V(q)$ is the bare potential where the particle moves. For example, it is easy to convince yourself that for $V(q) = 0$ and choosing $g_i = m_i \omega_i^2$, the full Hamiltonian can be written as:

$$H = \frac{p^2}{2M} + \sum_i \left(\frac{\tilde{p}_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 \tilde{x}_i^2 \right) , \quad (4)$$

where $\tilde{x}_i = x_i - q$ and the system is invariant under translations, as must be the case if the particle truly moves in a vanishing potential.

The full Hamiltonian $H = H_S + H_B + H_{SB}$ is known as the Caldeira-Leggett Hamiltonian, since these authors were the first to describe dissipation and fluctuations in this context [13].

In the Heisenberg picture, the operators (q, p) and (x_i, p_i) evolve in time according to the Heisenberg equation of motion, generically written as:

$$\frac{dA}{dt} = \frac{i}{\hbar} [H, A] . \quad (5)$$

It is then easy to show using the canonical commutation relations that $p = Mdq/dt$, $p_i = m_i dx_i/dt$ and:

$$\frac{dp}{dt} = -V'(q) + \sum_i g_i x_i - q \sum_i \frac{g_i^2}{m_i \omega_i^2}, \quad (6)$$

$$\frac{dp_i}{dt} = -m_i \omega_i^2 x_i + g_i q, \quad (7)$$

giving the equations of motion

$$\ddot{q} + \frac{1}{M} \left(V'(q) + q \sum_i \frac{g_i^2}{m_i \omega_i^2} \right) = \frac{1}{M} \sum_i g_i x_i, \quad (8)$$

and

$$\ddot{x}_i + \omega_i^2 x_i = \frac{g_i}{m_i} q. \quad (9)$$

We are interested in determining the effects of the bath of harmonic oscillators on the evolution of the heavy particle, i.e. we would like to integrate out the coordinates x_i . To do this, we can easily check that Eq. (9) has, for a given $q(t)$, a formal solution of the form:

$$x_i = x_i(0) \cos(\omega_i t) + \frac{p_i(0)}{m_i \omega_i} \sin(\omega_i t) + \frac{g_i}{m_i \omega_i} \int_0^t dt' \sin(\omega_i(t-t')) q(t'), \quad (10)$$

where the first two terms give the homogeneous harmonic oscillator solution and it is easy to verify that the last term gives a particular solution of the full (inhomogeneous) equation. If we replace this solution into the right-hand side of Eq. (8) and integrate the inhomogeneous term by parts, we obtain the Langevin equation for the heavy particle:

$$M\ddot{q} + V'(q) + \int_0^t dt' \gamma(t-t') \dot{q}(t') = \xi(t), \quad (11)$$

where we have defined the dissipation kernel:

$$\gamma(t) = \sum_i \frac{g_i^2}{m_i \omega_i^2} \cos(\omega_i t) \quad (12)$$

and the noise term:

$$\xi(t) = \sum_i g_i \left[\left(x_i(0) - \frac{g_i}{m_i \omega_i^2} q(0) \right) \cos(\omega_i t) + \frac{p_i(0)}{m_i \omega_i} \sin(\omega_i t) \right]. \quad (13)$$

Note that the dissipation kernel depends only on the couplings and properties of the oscillators in the bath, while the noise term also depends on the initial conditions, which introduces a degree of randomness in the motion of the heavy particle.

A set of harmonic oscillators is often a good description of an environment, giving in particular the typical frequencies (i.e. time-scales) involved in its microscopic details. It is also typical for an environment to have many degrees of freedom, so that it is useful to take the limit $N \rightarrow \infty$ in the above expressions, replacing sums by integrals. It is common and convenient to introduce a spectral density of modes in the bath:

$$J(\omega) = \pi \sum_i \frac{g_i^2}{2m_i \omega_i} \delta(\omega - \omega_i), \quad (14)$$

which encodes the relative fraction of modes with a given frequency present in the bath (per logarithmic scale). The dissipation kernel can then be written, in the large N limit, as

$$\gamma(t) = \frac{2}{\pi} \int_0^\infty \frac{d\omega}{\omega} J(\omega) \cos(\omega t). \quad (15)$$

The simplest case is when $J(\omega) = \gamma\omega$, where γ is a constant, which can be seen as the leading term in the Taylor expansion of the spectral function for small frequencies (large time-scales), assuming that $J(0) = 0$ for a “well-behaved” environment. In this case, the dissipation kernel becomes:

$$\gamma(t) = \frac{2\gamma}{\pi} \int_0^\infty d\omega \cos(\omega t) = 2\gamma\delta(t) \quad (16)$$

and the equation of motion for the heavy particle takes the simple form:

$$M\ddot{q} + V'(q) + \gamma\dot{q} = \xi . \quad (17)$$

We thus see that the dissipation kernel yields the standard friction term proportional to the particle’s velocity, which will damp the motion of the particle. This is called the Ohmic dissipation kernel since the local dissipation term $\gamma\dot{q}$ appears in RLC electric circuits. For general power laws $J(\omega) = \omega^\alpha$, the spectral function is conventionally said to be super-Ohmic for $\alpha > 1$ and sub-Ohmic for $\alpha < 1$.

One must note that the linear spectral function diverges in the ultra-violet and, hence, cannot give a fully physical model. One often introduces a cut-off regularization, as with the Drude model:

$$J(\omega) = \gamma\omega \frac{\omega_D^2}{\omega^2 + \omega_D^2} , \quad (18)$$

which yields a dissipation kernel:

$$\gamma(t) = \gamma\omega_D e^{-\omega_D t} , \quad (19)$$

such that:

$$\int_0^t dt' \gamma(t-t')\dot{q}(t') = \gamma\omega_D \int_0^t dt' e^{-\omega_D(t-t')}\dot{q}(t') . \quad (20)$$

On time scales $t \gg \omega_D^{-1}$, the integral above is dominated by the contribution from $t' \sim t$ and we again obtain the local Ohmic dissipation term:

$$\int_0^t dt' \gamma(t-t')\dot{q}(t') \simeq \gamma\omega_D \dot{q}(t) \int_0^t dt' e^{-\omega_D(t-t')} \simeq \gamma\dot{q}(t) . \quad (21)$$

On the other hand, for shorter time scales this term is fully non-local, i.e. the dissipative effect has a memory of the previous interactions between the system and the environment. Thus, ω_D gives the typical time scale describing the microphysical processes in the bath, such that if the evolution of the system is slow compared to this, i.e. if we are only interested in the evolution of the system on time scales $t \gg \omega_D^{-1}$, dissipative effects become local and independent of this cut-off. This is the first example of an adiabatic regime leading to a local velocity-dependent dissipation/friction term.

One of the first systems where the Langevin equation (17) was employed to describe the dynamical evolution is the case of a Brownian particle moving in a medium (e.g. a molecular gas), where random collisions with the molecules in the medium give rise both to the friction and the noise terms in the Langevin equation. These collisions make the particle’s position and velocity fluctuate randomly and, on average, end up damping the particle’s motion. If there are many such collisions on the typical time scale set by the particle’s average velocity, the system effectively loses the memory of earlier collisions and each collision occurs, on average, independently of the previous ones.

In this example, the noise term on the right-hand side of the Langevin equation represents the randomness introduced in the evolution of the system by these random collisions and without it the description would not be complete. To see this, let us consider the case $V(q) = 0$, where we can write the Langevin equation in the adiabatic (Ohmic) form in terms of the velocity $v \equiv dq/dt$ as

$$\dot{v}(t) = -\frac{\gamma}{M}v(t) + \frac{\xi(t)}{M} . \quad (22)$$

In the absence of the random noise term, $\xi = 0$, the particle's velocity would decay exponentially,

$$v(t) = v_0 e^{-t/\tau} , \quad (23)$$

on a time scale $\tau = M/\gamma$. The average velocity in an equilibrium distribution should effectively vanish, $\langle v \rangle_{eq} = 0$, so that this is consistent with the noise term being a random function with a Gaussian distribution of zero average:

$$\langle \xi(t) \rangle_{eq} = 0 . \quad (24)$$

However, in equilibrium equipartition of energy requires non-zero variance for the velocity fluctuations,

$$\langle v^2 \rangle_{eq} = \frac{k_B T}{M} . \quad (25)$$

such that we must have for a generic system in or close to equilibrium

$$\langle \xi(t_1) \xi(t_2) \rangle_{eq} = \bar{\xi}^2 \delta(t_1 - t_2) , \quad (26)$$

where we have further assumed that random interactions between the system and the environment are Markovian, which holds for the Brownian motion example where, for a large number of collisions, each collision with the molecules in the medium happens independently of the previous ones. This will also generically be the case for systems in the adiabatic regime. To determine the variance of the noise term, we may proceed as before and note that the Langevin equation has a formal solution given by:

$$v(t) = v_0 e^{-t/\tau} + \frac{1}{M} \int_0^t dt' e^{-(t-t')/\tau} \xi(t') . \quad (27)$$

This yields for the velocity correlation function:

$$\begin{aligned} \langle [v(t_2) - \langle v(t_2) \rangle] [v(t_1) - \langle v(t_1) \rangle] \rangle &= \frac{1}{M^2} \int_0^{t_2} ds_2 \int_0^{t_1} ds_1 e^{-(t_1-s_1)/\tau} e^{-(t_2-s_2)/\tau} \langle \xi(s_1) \xi(s_2) \rangle \\ &= \frac{\bar{\xi}^2}{M^2} \int_0^{t_2} ds_2 \int_0^{t_1} ds_1 e^{-(t_1+t_2-s_1-s_2)/\tau} \delta(s_1 - s_2) \\ &= \frac{\bar{\xi}^2}{2\gamma M} \left[e^{-|t_1-t_2|/\tau} - e^{-(t_1+t_2)/\tau} \right] . \end{aligned} \quad (28)$$

From this it is easy to obtain the equal time velocity correlator:

$$\langle v^2(t) \rangle = \left(\langle v_0^2 \rangle - \frac{\bar{\xi}^2}{2M\gamma} \right) e^{-2t/\tau} + \frac{\bar{\xi}^2}{2M\gamma} . \quad (29)$$

Since in equilibrium this must be time-independent and satisfy the equipartition result (25), we thus find that:

$$\langle \xi(t_1) \xi(t_2) \rangle_{eq} = 2\gamma k_B T \delta(t_1 - t_2) . \quad (30)$$

This is known as the *fluctuation-dissipation relation* in equilibrium, which follows from a more general principle that we will find in the QFT description. It states that the noise term, i.e. the random fluctuations induced by the environment on the system's evolution, has an amplitude that is determined by the strength of the average dissipation effect.

This is a very powerful result, which states that both the average dissipation and the random fluctuations can be determined from a single dissipation coefficient γ , which can in principle be computed from the properties of the environment and its microphysical interactions with the system. The dissipation and noise terms are thus complementary effects, the former tending to damp the system's motion and bring it away from equilibrium, while the noise term keeps the system's fluctuations sufficiently large to overcome this and maintain the equilibrium properties.

3 The Schwinger-Keldysh formalism for non-equilibrium QFT

Having derived the Langevin equation in a simple quantum mechanical model, we will now describe how dissipation and associated fluctuations arise in the more general framework of (relativistic) QFT, for which it is useful to employ the so-called real-time formalism, in particular in the form first introduced by Schwinger and Keldysh [14].

Let us start by considering a quantum many-body system governed by a Hamiltonian $\hat{H}(t)$, which may in general be time-dependent. Probabilities in the corresponding many-body Hilbert space are given in terms of a density matrix, which in the distant past, $t = -\infty$, takes some form $\hat{\rho}(-\infty)$. The density matrix evolves according to the Liouville-von-Neumann equation:

$$\frac{\partial \hat{\rho}}{\partial t} = -i[\hat{H}, \hat{\rho}] , \quad (31)$$

where we consider natural units where $\hbar = c = k_B = 1$. This has the well-known formal solution:

$$\hat{\rho}(t) = \hat{U}(t, -\infty)\hat{\rho}(-\infty)\hat{U}^\dagger(t, -\infty) = \hat{U}(t, -\infty)\hat{\rho}(-\infty)\hat{U}(-\infty, t) , \quad (32)$$

where the time-evolution operator is given by the time-ordered expression:

$$\hat{U}(t, t') = \mathbb{T} \exp \left(-i \int_{t'}^t \hat{H}(\tau) d\tau \right) . \quad (33)$$

In the study of non-equilibrium quantum systems, we are generically interested in computing the expectation value of some observable, described by an operator \hat{O} , at a given time t :

$$\langle \hat{O}(t) \rangle = \frac{\text{Tr}[\hat{O}\hat{\rho}(t)]}{\text{Tr}[\hat{\rho}(t)]} = \frac{\text{Tr}[\hat{U}(-\infty, t)\hat{O}\hat{U}(t, -\infty)\hat{\rho}(-\infty)]}{\text{Tr}[\hat{\rho}(-\infty)]} , \quad (34)$$

where the trace is taken over the many-body Hilbert space and we used the cyclic property of the trace, which in particular yields the conservation of the total probability $\text{Tr}[\hat{\rho}(t)] = \text{Tr}[\hat{\rho}(-\infty)]$. We thus see that the trace in the numerator of Eq. (34) describes the evolution of the system:

1. from $t = -\infty$, where the initial density matrix is specified, to the time t where the observable is computed;
2. from t back to $t = -\infty$.

It is common and convenient to introduce the factor $\hat{U}(t, +\infty)\hat{U}(+\infty, t) = \mathbb{I}$, such that;

$$\langle \hat{O}(t) \rangle = \frac{\text{Tr}[\hat{U}(-\infty, +\infty)\hat{U}(+\infty, t)\hat{O}\hat{U}(t, -\infty)\hat{\rho}(-\infty)]}{\text{Tr}[\hat{\rho}(-\infty)]} . \quad (35)$$

Hence, the evolution takes place along a closed time path (CTP), from $t = -\infty$ to $t = +\infty$ and back, with the operator evaluated in the backwards branch in this case (the opposite being also possible). Since only the initial state of the system is specified, this is also known as the ‘‘in-in’’ formalism, as opposed to the ‘‘in-out’’ formalism conventionally used in scattering problems, where one is interested in both the initial and final state of the system.

Instead of evaluating a single quantum field (or generic operator) on the CTP contour, one can introduce two copies of the same field $\phi(x)$, where:

1. ϕ_1 is evaluated in the forward branch (from $-\infty$ to $+\infty$);
2. ϕ_2 is evaluated in the backwards branch (from $+\infty$ to $-\infty$).

In the path integral formulation of QFT, this yields the following generating functional:

$$Z[J_1, J_2] = \int \mathcal{D}[\phi_1]\mathcal{D}[\phi_2]\rho[\phi_1(-\infty), \phi_2(-\infty)] e^{iS[\phi_1] + i \int d^4x J_1 \phi_1 - iS[\phi_2] - i \int d^4x J_2 \phi_2} , \quad (36)$$

where we note that the two fields must have the same value at $t = \pm\infty$ in order to describe the same field in the CTP, so that the corresponding functional integration measures are not independent and the path integral is not factorizable. Also note that the CTP contour is well defined in the complex time plane (see e.g. [15]).

Correlation functions and operator expectation values in general can be obtained from this generating functional in the conventional way and Feynman diagrams are similar to ordinary QFT ones, except that vertices and propagators now carry a label $a = 1, 2$, with minus signs introduced for each label 2. One may actually get rid of the troublesome minus signs in the source terms by defining $J^1 = J_1$ and $J^2 = -J_2$, or equivalently by introducing a metric in field space $c_{ab} = \text{diag}(1, -1)$ ¹. Physical quantities may then be obtained by setting $\phi_1 = \phi_2$ at the end of the calculation.

3.1 Propagators

The propagator for a non-interacting field becomes a 2×2 matrix in the CTP formalism:

$$G_{ab}(x, x') = -i \frac{\delta \log Z}{\delta J^a(x) J^b(x')} = \begin{pmatrix} \langle \mathbb{T} \hat{\phi}(x) \hat{\phi}(x') \rangle_C & \langle \hat{\phi}(x') \hat{\phi}(x) \rangle_C \\ \langle \hat{\phi}(x) \hat{\phi}(x') \rangle_C & \langle \mathbb{T}^* \hat{\phi}(x) \hat{\phi}(x') \rangle_C \end{pmatrix}, \quad (37)$$

where C denotes the CTP contour and \mathbb{T}^* denotes anti-time ordering. We recognize the component G_{11} as the standard Feynman propagator, whereas for a system in thermal equilibrium, in which we are mostly interested in, G_{22} is the thermal Dyson function and G_{21} is the thermal Wightman function.

From these we may define:

1. the **spectral function**:

$$\rho(x, x') = i \langle [\hat{\phi}(x), \hat{\phi}(x')] \rangle_C \quad (38)$$

2. the **anti-commutator function**:

$$F(x, x') = \frac{1}{2} \langle \{\hat{\phi}(x), \hat{\phi}(x')\} \rangle_C \quad (39)$$

These give the imaginary and real parts of the Wightman function and we may write the propagator matrix in the form:

$$G_{ab}(x, x') = \begin{pmatrix} F(x, x') - \frac{i}{2} \sigma(x, x') & F(x, x') + \frac{i}{2} \rho(x, x') \\ F(x, x') - \frac{i}{2} \rho(x, x') & F(x, x') + \frac{i}{2} \sigma(x, x') \end{pmatrix}, \quad (40)$$

where $\sigma(x, x') = \rho(x, x') \text{sgn}(t - t')$ is the Feynman-Wheeler propagator.

In thermal equilibrium, propagators are invariant under spatial and time translations, $G_{ab}(x, x') = G_{ab}(x - x')$. A system in thermal equilibrium at temperature $T = \beta^{-1}$ may also be described in terms of an imaginary periodic time coordinate, $\tau = -it$, with $0 \leq \tau \leq \beta$. Any operator must then obey the imaginary time periodicity relation:

$$\hat{O}(t + i\beta) = e^{-\beta \hat{H}} \hat{O}(t) e^{\beta \hat{H}}. \quad (41)$$

Applying this to the Wightman function, we have:

$$\begin{aligned} \langle \hat{\phi}(t) \hat{\phi}(t') \rangle &= \frac{1}{Z} \text{Tr} \left[e^{-\beta \hat{H}} \hat{\phi}(t) \hat{\phi}(t') \right] \\ &= \frac{1}{Z} \text{Tr} \left[e^{-\beta \hat{H}} \hat{\phi}(t) e^{\beta \hat{H}} e^{-\beta \hat{H}} \hat{\phi}(t') \right] = \\ &= \frac{1}{Z} \text{Tr} \left[\hat{\phi}(t + i\beta) e^{-\beta \hat{H}} \hat{\phi}(t') \right] \\ &= \langle \hat{\phi}(t') \hat{\phi}(t + i\beta) \rangle, \end{aligned} \quad (42)$$

¹We follow the conventions and notation used in [16].

where we used the cyclic property of the trace. This implies that $G_{21}(t - t') = G_{12}(t - t' + i\beta)$, which gives the Kubo-Martin-Schwinger (KMS) relation. In Fourier space, this can be written as:

$$G_{21}(\mathbf{p}, \omega) = e^{\beta\omega} G_{12}(\mathbf{p}, \omega) , \quad (43)$$

which then yields the following relation between the anti-commutator and spectral functions:

$$F(\mathbf{p}, \omega) = -\frac{i}{2} [1 + 2n(\omega)] \rho(\mathbf{p}, \omega) , \quad (44)$$

where the Bose-Einstein distribution is given by:

$$n(\omega) = \frac{1}{e^{\beta\omega} - 1} . \quad (45)$$

Note that for fermions we obtain an analogous relation involving the Femi-Dirac distribution instead. This relation implies that the propagator matrix is completely specified by the spectral function in thermal equilibrium.

Propagators for interacting field theories satisfy the Schwinger-Dyson equation:

$$(\omega^2 - \mathbf{p}^2 - m^2)G_{ab} - \Sigma_a^c G_{cb} = ic_{ab} , \quad (46)$$

where Σ_{ab} is the self-energy matrix, which accounts for loop corrections to the free 2-point correlation function.

As for the propagator, the self-energy can also be specified in terms of two functions:

$$i\Sigma_\rho = i(\Sigma_{21} - \Sigma_{12}) , \quad i\Sigma_F = \frac{1}{2}(\Sigma_{21} + \Sigma_{12}) . \quad (47)$$

These satisfy the equilibrium KMS relation:

$$\Sigma_F(\mathbf{p}, \omega) = -\frac{i}{2} [1 + 2n(\omega)] \Sigma_\rho(\mathbf{p}, \omega) . \quad (48)$$

As we will see below, this KMS relation will lead to the fluctuation-dissipation relation that we have obtained earlier in the context of quantum mechanics. Additionally, the imaginary part of the self-energy yields, in particular, the decay width of the field, as a consequence of the (generalized) optical theorem:

$$\Gamma(\mathbf{p}, \omega) = \frac{i\Sigma_\rho(\mathbf{p}, \omega)}{4\omega_p} , \quad (49)$$

where $\omega_p = \sqrt{\mathbf{p}^2 + m^2}$ is the energy of an on-shell particle. Note that Γ includes information about both decays ('d') and inverse decays ('i'), with $\Gamma = \Gamma_d - \Gamma_i$ (for bosons) setting the time scale $\tau_d = \Gamma^{-1}$ of approach to equilibrium. A general non-equilibrium distribution $f(\omega, t)$ satisfies (see e.g. [17]):

$$\frac{df}{dt} = -f\Gamma_d + (1 + f)\Gamma_i , \quad (50)$$

such that

$$f(\omega, t) = \frac{\Gamma_i}{\Gamma_d - \Gamma_i} + A(\omega)e^{-(\Gamma_d - \Gamma_i)t} . \quad (51)$$

Since in equilibrium $\Gamma_d = e^{\beta\omega}\Gamma_i$, we see that $f(\omega, t) \rightarrow n(\omega)$ independently of the initial distribution for $t \gg \Gamma^{-1}$.

3.2 Keldysh representation

For analyzing non-equilibrium effects using the real-time formalism, it is useful to introduce an alternative field representation known as the Keldysh representation:

$$\phi_c = \frac{1}{2}(\phi_1 + \phi_2) , \quad \phi_\Delta = \phi_1 - \phi_2 . \quad (52)$$

It is easy to see that in this basis the propagator matrix can be written as:

$$G_{a'b'}(x, x') = \begin{pmatrix} F(x, x') & G_R(x, x') \\ G_A(x, x') & 0 \end{pmatrix} , \quad (53)$$

where the advanced and retarded propagators are given in terms of the spectral function:

$$G_A(x, x') = i\rho(x, x')\theta(t' - t) , \quad G_R(x, x') = -i\rho(x, x')\theta(t - t') . \quad (54)$$

Similarly, for the self-energy matrix (which contributes to the inverse propagator matrix), we have:

$$\Sigma^{a'b'}(x, x') = \begin{pmatrix} 0 & \Sigma_A(x, x') \\ \Sigma_R(x, x') & -i\Sigma_F(x, x') \end{pmatrix} , \quad (55)$$

where

$$\Sigma_A(x, x') = -\Sigma_\rho(x, x')\theta(t' - t) , \quad \Sigma_R(x, x') = \Sigma_\rho(x, x')\theta(t - t') . \quad (56)$$

As an example of application of this formalism and the convenience of using the Keldysh representation, let us consider a massive scalar field with a quartic self-coupling:

$$S[\phi_1, \phi_2] = \int d^4x \left[\frac{1}{2}\phi_1(-\partial^2 - m^2)\phi_1 - \frac{\lambda}{4!}\phi_1^4 \right] - (\phi_1 \leftrightarrow \phi_2) . \quad (57)$$

In the Keldysh representation, this action then takes the form:

$$S[\phi_c, \phi_\Delta] = \int d^4x \left[\phi_\Delta(-\partial^2 - m^2)\phi_c - \frac{\lambda}{4!} (4\phi_\Delta\phi_c^3 + \phi_\Delta^3\phi_c) \right] . \quad (58)$$

Varying the action with respect to ϕ_Δ and setting $\phi_\Delta = 0$, we obtain the standard classical equation of motion:

$$(\partial^2 + m^2)\phi_c + \frac{\lambda}{3!}\phi_c^3 = 0 . \quad (59)$$

We thus see that, even though we have doubled the number of fields, this is simply an auxiliary procedure, with physical results corresponding to setting $\phi_1 = \phi_2$ at the end of the calculation.

4 Langevin-like equation for a dynamical scalar field

Having introduced the real-time formalism and its different representations, we will now describe how a Langevin-like equation appears in non-equilibrium QFT. Quantum corrections to the classical action of a quantum field theory are determined by the effective action. As in conventional QFT, only the 1-particle-irreducible (1PI) diagrams contribute to the effective action in the Schwinger-Keldysh formalism, which is then given by²:

$$e^{i\Gamma[\phi_1, \phi_2]} = \int_{1\text{PI}} \mathcal{D}[\phi'_1] \mathcal{D}[\phi'_2] \rho[\phi'_1(-\infty), \phi'_2(-\infty)] e^{i(S[\phi_1 + \phi'_1] - S[\phi_2 + \phi'_2])} . \quad (60)$$

²For other approaches using n -PI effective actions see e.g. [11].

The quantum field equation can then be obtained by varying the effective action with respect to the fields and setting $\phi_1 = \phi_2$ in the end:

$$\left. \frac{\delta \Gamma}{\delta \phi_a} \right|_{\phi_1 = \phi_2} = 0 . \quad (61)$$

In the Keldysh representation, this corresponds to:

$$\left. \frac{\delta \Gamma}{\delta \phi_\Delta} \right|_{\phi_\Delta = 0} = 0 , \quad (62)$$

as we had seen for the quartic example above. Besides the classical terms that appeared in this example, the effective action receives contributions from loop-diagrams, the leading corrections being given by the elements of the self-energy matrix. We then have the generic expansion:

$$\begin{aligned} \Gamma[\phi_c, \phi_\Delta] &= - \int d^4 x \phi_\Delta \left[(\partial^2 + m^2) \phi_c + \frac{\partial V(\phi_c)}{\partial \phi_c} \right] - \int d^4 x \int d^4 x' \phi_\Delta \Sigma_R(x, x') \phi_c(x') \\ &+ \frac{1}{2} \int d^4 x \int d^4 x' \phi_\Delta(x) i \Sigma_F(x, x') \phi_\Delta(x') + \mathcal{O}(\phi_\Delta^3) , \end{aligned} \quad (63)$$

where we have included terms up to quadratic order in the auxiliary field and $V(\phi)$ is a generic tree-level potential. At a first glance, only the linear terms in ϕ_Δ will contribute to the effective equation of motion, which is given by:

$$(\partial^2 + m^2) \phi_c + V'(\phi_c) + \int d^4 x' \Sigma_R(x, x') \phi_c(x') = 0 , \quad (64)$$

which explicitly exhibits causality from the use of the retarded self-energy $\Sigma_R(x, x')$. However, the quadratic term can be rewritten in the effective action by introducing an auxiliary field $\xi(x)$ and using the properties of gaussian functional integrals:

$$e^{-\frac{1}{2} \int d^4 x \int d^4 x' \phi_\Delta(x) \Sigma_F(x, x') \phi_\Delta(x')} = |\det(\Sigma_F)|^{1/2} \int \mathcal{D}[\xi] e^{-\frac{1}{2} \int d^4 x \int d^4 x' \xi(x) \Sigma_F^{-1}(x, x') \xi(x')} e^{i \int d^4 x \xi(x) \phi_\Delta(x)} . \quad (65)$$

This thus introduces an additional linear term in $\phi_\Delta(x)$, such that the effective equation of motion becomes:

$$(\partial^2 + m^2) \phi_c + V'(\phi_c) + \int d^4 x' \Sigma_R(x, x') \phi_c(x') = \xi(x) . \quad (66)$$

Comparing with the results obtained in the context of non-relativistic quantum mechanics, we see that this corresponds to a Langevin-like equation for the dynamics of a quantum field, with the auxiliary field $\xi(x)$ being interpreted as a Gaussian stochastic noise:

$$\langle \xi(x) \rangle = 0 , \quad \langle \xi(x) \xi(x') \rangle = \Sigma_F(x, x') . \quad (67)$$

The term in $\Sigma_R(x, x')$ gives the dissipation term that we had found earlier, which can be seen more explicitly by defining a dissipation kernel:

$$\Sigma_R(x, x') = - \frac{\partial \mathcal{D}}{\partial t'}(x, x') \quad (68)$$

in such a way that, upon integration by parts and assuming the fields vanish at infinity, the Langevin-like equation becomes:

$$(\partial^2 + m^2) \phi_c + V'(\phi_c) + \int d^4 x' \mathcal{D}(x, x') \dot{\phi}_c(x') = \xi(x) . \quad (69)$$

We can now see that the KMS relation between the self-energy functions yields the fluctuation-dissipation relation for a system close to thermal equilibrium:

$$\Sigma_F(\mathbf{p}, \omega) = 2\omega \left(n(\omega) + \frac{1}{2} \right) \mathcal{D}(\mathbf{p}, \omega) . \quad (70)$$

For $\omega \gg T$, $n(\omega) \simeq (\beta\omega)^{-1} \gg 1$ and this gives:

$$\langle \xi(\mathbf{p}, t) \xi(\mathbf{p}', t') \rangle = 2T \int \frac{d\omega}{2\pi} \mathcal{D}(\mathbf{p}, \omega) e^{i\omega(t-t')} . \quad (71)$$

This is a powerful result, which tells us that both the effects of dissipation and fluctuation on the dynamics of the field from the interaction with the surrounding environment are, close to equilibrium, given in terms of the self-energy, which can be easily computed from its interactions with the fields that make up the environment.

Particularly useful for cosmological applications are scenarios where the field moves slowly compared to the typical relaxation time of the environment or thermal bath. In this case, we can expand the field for small velocities to yield, for a homogeneous field (see [18] for the effect of inhomogeneities):

$$\int d^4x' \Sigma_R(x, x') \phi_c(x') = \int d^4x' \Sigma_R(x - x') \left[\phi_c(t) + \dot{\phi}_c(t)(t' - t) + \dots \right] \quad (72)$$

By absorbing the term in ϕ_c in the quantum effective potential, the equation of motion then takes the local form:

$$(\partial^2 + m^2)\phi_c + V'_{eff}(\phi_c) + \Upsilon \dot{\phi}_c = \xi , \quad (73)$$

with the dissipation or friction coefficient given by:

$$\Upsilon = - \int d^4x' \Sigma_R(x') t' = - \frac{i}{2} \left. \frac{\partial \Sigma_\rho}{\partial \omega}(\mathbf{0}, \omega) \right|_{\omega=0} . \quad (74)$$

Hence, we again find that the adiabatic regime yields a local Ohmic dissipation regime with the fluctuation-dissipation relation yielding:

$$\langle \xi(\mathbf{p}, t) \xi(\mathbf{p}', t') \rangle = 2\Upsilon T (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') \delta(t - t') , \quad (75)$$

which is the field theory analogue of Eq. (30).

5 Examples of local dissipation coefficients

Although in deriving the Langevin-like equation it was convenient to introduce the Keldysh representation, practical computations of dissipation coefficients are more easily performed in the original field basis. Using Eq. (47) and that $\Sigma_{12} = -\Sigma_{21}^*$, we then have:

$$\Upsilon = - \int_{-\infty}^t dt' \Sigma_\rho(\mathbf{0}, t') t' = 2 \int_0^{+\infty} \text{Re} [\Sigma_{21}((0), t')] t' . \quad (76)$$

Let us consider, as an example for the calculation of dissipation coefficients that will be useful for cosmological applications, the interaction between two real scalar fields ϕ and χ , the former being the system that we are interested in and the latter describing the degrees of freedom in an ambient thermal bath near equilibrium:

$$\mathcal{L}_{int} = -\frac{1}{2} g^2 \phi_a^2 \chi_a^2 , \quad (77)$$

with $a = 1, 2$ for the two copies of each field in the CTP contour.

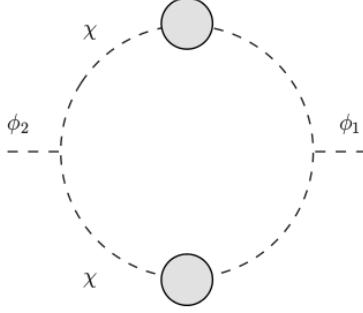


Figure 1: Leading 1-loop contribution for the Σ_{21} self-energy of the scalar field ϕ from a biquadratic interaction with another scalar field χ . The shaded circles represent the self-energy of the χ field, so that the propagators are effectively ‘dressed’.

The leading part of the self-energy of the ϕ field induced by the interaction with the χ field is illustrated in Figure 1 and is given by:

$$i\Sigma_{21} = \frac{1}{2}(2g^2\phi)^2 \int \frac{d^3p}{(2\pi)^3} [G_{21}^\chi(t')]^2 . \quad (78)$$

The corresponding dissipation coefficient can then be written in the form:

$$\Upsilon = 4g^4\phi^2 \text{Im} \int \frac{d^3p}{(2\pi)^3} \int dt' [G_{21}^\chi(t')]^2 t' . \quad (79)$$

We can obtain the Wightman function for the χ field in terms of its spectral function using (44), such that:

$$G_{21}^\chi(t') = -i \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} [n(\omega) + 1] \rho_\chi(\omega) e^{-i\omega t'} . \quad (80)$$

If we use the identity:

$$\text{Im} \left(\int_0^{+\infty} dt' t' e^{-i\omega t'} \right) = -\pi \delta'(\omega) , \quad (81)$$

and integrate by parts, taking into account that $\rho(-\omega) = -\rho(\omega)$ and that $n'(-\omega) = -n'(\omega) = \beta n(\omega)(1 + n(\omega))$, we may write the dissipation coefficient after some algebra as:

$$\Upsilon = \frac{2g^4\phi^2}{T} \int \frac{d^4p}{(2\pi)^4} \rho_\chi^2(\omega) n(\omega)(n(\omega) + 1) . \quad (82)$$

Now, the spectral function corresponds to the imaginary part of the propagator, such that:

$$\begin{aligned} \rho_\chi &= \frac{i}{p^2 + m_{\chi,R}^2 + 2i\omega_p\Gamma_\chi} - \frac{i}{p^2 + m_{\chi,R}^2 - 2i\omega_p\Gamma_\chi} \\ &= \frac{4\omega_p\Gamma_\chi}{(\omega^2 - \omega_p^2)^2 + 4\omega_p^2\Gamma_\chi^2} , \end{aligned} \quad (83)$$

where we have used the relation between the imaginary part of the self-energy and the decay width in Eq. (49) and absorbed the real part of the self-energy into the renormalized mass of the field, $m_{\chi,R}^2$, such that $\omega_p^2 = |\mathbf{p}|^2 + m_{\chi,R}^2$.

Hence, to compute the dissipation coefficient resulting from the interaction between the fields ϕ and χ , we just need to compute the latter's mass and decay width and perform a loop-integral at finite temperature. Neglecting the quantum corrections to the field's mass, we have in our example:

$$m_\chi = g\phi . \quad (84)$$

The decay width of the field receives contributions from all possible processes at finite temperature, including direct and inverse decays as well as thermal scatterings (Landau damping). If the χ field has cubic self-interactions, for example, the process $\chi(p_1)\chi(p_2) \rightarrow \chi(p_3)$ is possible at finite temperature, even though it would be kinematically forbidden at zero temperature in the final χ rest frame. This is due to the fact that the heat bath induces a preferred frame that explicitly breaks Lorentz invariance. We will nevertheless focus on a case where the χ field is truly unstable and can decay into scalar particles σ , which we take to have a small mass $m_\sigma \ll T$, independent of the ϕ field. The decay corresponds to a trilinear interaction:

$$\mathcal{L}_{\chi\sigma} = \frac{1}{2}hM\chi\sigma^2 , \quad (85)$$

where M is a mass scale. The decay width of the χ field is then given by:

$$\Gamma_\chi = \frac{h^2}{32\pi} \frac{M^2}{m_\chi} + \text{finite } T \text{ corrections} . \quad (86)$$

We will be particularly interested in the case $M = m_\chi = g\phi$, which arises in supersymmetric theories with superpotential:

$$W = \frac{g}{2}\Phi X^2 + \frac{h}{2}XY^2 , \quad (87)$$

where the scalar fields ϕ , χ and σ correspond to the (complex) scalar components of the chiral superfields Φ , X and Y , respectively. Note that the term of the form (85) arises through the F -term of the χ field and that the σ mass is indeed independent of ϕ as assumed above.

The 4-momentum integral has to be performed numerically in the general case, but there are two particular limits where the integration can be done analytically and yields simple results for the dissipation coefficient:

1. High temperature or narrow width regime

For $T \gg m_\chi$ or when the field has a small decay width $\Gamma_\chi \ll m_\chi$, the main contribution to the integral in Eq. (82) comes from the pole in the spectral function at $\omega = \omega_p$, i.e. to the resonant production of on-shell χ particles in the thermal bath, and we can make the replacement:

$$\rho_\chi^2 \rightarrow \frac{\pi}{2\omega_p^2\Gamma_\chi} \delta(\omega - \omega_p) . \quad (88)$$

This yields, using the zero-temperature result for the decay width, which is a good approximation in this regime:

$$\Upsilon \simeq \frac{8}{\pi} \frac{g^4}{h^2} \frac{\phi^2 m_\chi}{M^2} \int_0^{+\infty} \frac{x^2 dx}{x^2 + \alpha^2} \frac{e^{\sqrt{x^2 + \alpha^2}}}{(e^{\sqrt{x^2 + \alpha^2}} - 1)^2} , \quad (89)$$

where $x = |\mathbf{p}|/T$ and $\alpha = m_\chi/T$. For $\alpha \ll 1$, the integral above is well approximated by α^{-1} (up to an $\mathcal{O}(1)$ numerical factor), yielding:

$$\Upsilon \simeq \frac{8}{\pi} \frac{g^4}{h^2} \frac{\phi^2}{M^2} T \simeq \frac{8}{\pi} \frac{g^2}{h^2} T , \quad (90)$$

where in the last line we have used $M = m_\chi$.

2. Low temperature regime

For $m_\chi \gtrsim T$, the contribution from the poles of the spectral function to the 4-momentum integral become Boltzmann-suppressed, since $n(\omega_p) \lesssim e^{-m_\chi/T} \ll 1$. If the decay width is sufficiently small, this suppression may be compensated, but in this case one has to worry about whether the evolution of the field is truly adiabatic to yield a local dissipation coefficient, since $\tau_\chi = \Gamma_\chi^{-1}$ sets the relaxation time of the thermal bath (namely the relevant time scale for the self-energy of the field). Except for this case, the main contribution to dissipation will come from off-shell or virtual modes of low momenta, $|\mathbf{p}|, \omega \sim T \lesssim m_\chi$. We may then approximate the spectral function by:

$$\rho_\chi \simeq \frac{4\Gamma_\chi}{m_\chi^3}. \quad (91)$$

It is then easy to see that $\Upsilon \propto h^4 T^3 / \phi^2$ for $M = m_\chi$ with a coefficient that must be determined by numerical integration, which typically requires including the finite temperature corrections to the field's decay width. A detailed analysis including thermal corrections to the χ mass (proportional to h^2) yields [19]:

$$\Upsilon = C_\phi \frac{T^3}{\phi^2}, \quad C_\phi = 0.02 h^2 N_Y N_X, \quad (92)$$

where we have generalized for N_X χ fields coupled to ϕ , each of them decaying into N_Y light species, the notation referring to the SUSY realization in Eq. (87). We note that the dependence on the coupling g between the ϕ and χ fields has cancelled out in this regime, which is particularly relevant for cosmological applications as we describe below. In addition, in a SUSY model there are also fermionic contributions to dissipation, but these are sub-dominant compared to the scalar contribution above at low temperatures, as shown in [4, 20].

Both on-shell and off-shell modes contribute to the dissipation coefficient in general, and numerically one observes that adding both contributions gives a very good approximation to the overall result [19]. In practice, well within the regimes described above, the dissipation coefficient can be taken to have the simple forms given above, which can thus be both field and temperature-dependent.

6 Applications to Cosmology: the case of warm inflation

Having now obtained the Langevin-like equation that describes the motion of a quantum field interacting with other fields in a thermal bath and computed the associated dissipation coefficient in some particular cases, let us now discuss some of the effects that this may have in Cosmology.

Scalar fields are ubiquitous in particle physics models, the best example being the Standard Model Higgs boson that has been recently discovered at the LHC. Several extensions of the Standard Model predict the existence of additional scalar fields, such as the superpartners of quarks and leptons in SUSY theories, extra-dimensional moduli in string theory or Higgs-like fields in Grand Unification Theories.

In many cases such scalar fields have a non-trivial cosmological dynamics, since their potential may depend on the temperature through their interactions with the degrees of freedom in the cosmic heat bath, as for example during cosmological phase transitions. In this case the effects of dissipation and associated fluctuations may play an important role in the dynamics and we will illustrate this in the best-known case of warm inflation.

The standard paradigm for inflation considers a homogeneous scalar field whose potential energy dominates the energy balance in the early universe and leads to a period of accelerated expansion. This is achieved if there is a sufficiently flat section of the scalar potential such that the field evolves slowly and behaves effectively as a cosmological constant. In the standard case the inflaton is taken as a weakly coupled field, effectively isolated from other degrees of freedom in the cosmic plasma, such that accelerated expansion drives the temperature of the universe to negligible values and the inflationary dynamics is entirely determined by the scalar field alone.

In warm inflation one considers the possibility of having significant interactions between the inflaton and other degrees of freedom in the cosmic bath, such that it experiences the effects of dissipation and fluctuations that we

have described before. Since it is evolving slowly, the Ohmic form of the dissipation kernel is generically a good approximation, and its effective equation of motion is Langevin-like:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) + \Upsilon\dot{\phi} = \xi , \quad (93)$$

where the fluctuation-dissipation relation yields:

$$\langle \xi(\mathbf{k}, t) \xi(\mathbf{k}', t') \rangle = \frac{2\Upsilon T}{a^3} (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \delta(t - t') , \quad (94)$$

where \mathbf{k} is the comoving momentum, which justifies the inclusion of the scale factor $a(t)$ in the above expression [16]. It is easy to see that the Langevin-like equation can be rewritten in the form:

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -\Upsilon\dot{\phi}^2 , \quad (95)$$

where $\rho_\phi = \dot{\phi}^2/2 + V(\phi)$ and $p_\phi = \dot{\phi}^2/2 - V(\phi)$ are the energy density and pressure of a homogeneous scalar field. Thus, we see that dissipation decreases the field's energy density, which must be transferred into the heat bath. Although we are not going to show this explicitly in these lectures, it is worth mentioning that this can be rigorously shown [21] by computing the self-energy of the fields in the heat bath. If the particles that result from dissipation are light compared to the ambient temperature, as in the example considered in the previous section, the inflaton's energy will be dissipated into radiation, which must thus follow:

$$\dot{\rho}_R + 4H\rho_R = \Upsilon\dot{\phi}^2 , \quad (96)$$

where in equilibrium $\rho_R = 3p_R = (\pi^2/30)g_*T^4$ for g_* relativistic degrees of freedom. In the absence of dissipation, $\rho_R \sim e^{-4Ht} \rightarrow 0$ during inflation and the universe *supercools* as mentioned above, requiring a separate period of reheating after inflation. On the other hand, if the inflaton is varying slowly, the source term on the right-hand side of Eq. (96) is approximately constant, yielding a slow-roll solution:

$$\rho_R \simeq \frac{\Upsilon\dot{\phi}^2}{4H} . \quad (97)$$

If this yields a temperature $T \gtrsim H$, we say that the universe remains warm during inflation. In the slow-roll regime, the inflaton's equation of motion can be written as:

$$3H(1 + Q)\dot{\phi} \simeq -V'(\phi) , \quad (98)$$

where the dissipative ratio is defined as:

$$Q \equiv \frac{\Upsilon}{3H} . \quad (99)$$

Additionally, if the inflaton dominates the energy density, as required in order to yield accelerated expansion, the Friedmann equation reads:

$$H^2 \simeq \frac{V(\phi)}{3M_P^2} , \quad (100)$$

where $M_P = (8\pi G_N)^{-1/2}$ is the reduced Planck mass. One can then use Eqs. (98) and (100) to show that:

$$\frac{\dot{H}}{H^2} \simeq -\frac{\epsilon_\phi}{1 + Q} , \quad (101)$$

where $\epsilon_\phi = (M_P^2/2)[V'(\phi)/V(\phi)]^2$ is the standard slow-roll parameter. Hence, a quasi-exponential period of inflation can be obtained provided that:

$$\epsilon_\phi \ll 1 + Q . \quad (102)$$

Consistency of the slow-roll solution also requires that the acceleration term is negligible in the field equation of motion and that the time derivative of the radiation energy density can also be neglected. These yield the condition:

$$|\eta_\phi| \ll 1 + Q, \quad (103)$$

where $\eta_\phi = M_P^2 V''(\phi)/V(\phi)$ is the second slow-roll parameter measuring the curvature of the potential. We thus see that the standard conditions on the inflationary potential $\epsilon_\phi, |\eta_\phi| \ll 1$ are relaxed in the presence of dissipation, since friction slows down the motion of the field and makes it easier to obtain 50-60 e-folds of inflation. In particular, if $Q \gg 1$ at some stage in the evolution, this may offer a solution to the so-called ‘‘eta-problem’’ typically found in supergravity models of inflation [22, 23]. We note that Q is a dynamical variable, since the dissipation coefficient generically depends on dynamical quantities such as the temperature or the field value and the Hubble parameter is also not exactly constant. We will see below that Q satisfies a slow-roll equation as well.

Using the slow-roll equations for the radiation and the field, one can show that:

$$\frac{\rho_R}{\rho_\phi} \simeq \frac{1}{2} \frac{Q}{1+Q} \frac{\epsilon_\phi}{1+Q}, \quad (104)$$

so that, for $\epsilon_\phi \ll 1 + Q$, radiation is indeed a sub-dominant component and expansion is accelerated. However, at the end of the slow-roll regime, $\epsilon_\phi \sim 1 + Q$ and we can have $\rho_R \sim \rho_\phi/2$ if $Q \gtrsim 1$. Thus, in models where Q grows during inflation and attains large values towards the end, radiation can become a significant component at the end of inflation and, in fact, smoothly take over the inflaton as the dominant component as it exits the slow-roll regime. Thus, in warm inflation one may have a smooth exit from inflation into the standard ‘Big Bang’ Cosmology, with no need for a separate reheating period.

The best-studied examples of successful warm inflation models are based on the type of scalar interactions found in supersymmetric models that we have described in the previous section, with the inflaton coupled to a scalar χ field (and its fermion superpartner), which decay into light degrees of freedom in a SUSY multiplet Y as given in Eq. (87). Supersymmetric models allow for successful realizations of warm inflation since the cancellation between bosonic and fermionic radiative corrections to the inflaton’s effective potential protects the flatness of the potential while allowing for significant dissipative effects. However, SUSY cancellations are not fully effective at finite temperature, since bosons and fermions have distinct equilibrium distributions, and in particular both types of species will generate inflaton thermal masses if the fields in the X sector are light. We may, however, consider the low temperature regime, $T \lesssim m_\chi$, where inflaton thermal masses are Boltzmann-suppressed and the effective potential takes the standard 1-loop Coleman-Weinberg form:

$$\Delta V_{eff}(\phi) = \frac{1}{32\pi^2} \text{Str} \left[m_X^4 \left(\log \left(\frac{m_X^2}{\mu^2} \right) - \frac{3}{2} \right) \right] = \frac{g^2 N_X}{32\pi^2} \log \left(\frac{m_{\chi,T}^2}{\mu^2} \right) V(\phi), \quad (105)$$

where the supertrace is taken over the bosonic and fermionic components of the X chiral multiplet, $V(\phi)$ is the tree-level potential and μ is the renormalization scale. Note that the effective potential does depend on the temperature through thermal mass corrections in $m_{\chi,T}^2 = g^2 \phi^2 + h^2 N_Y T^2/8$ [19], which are due to the coupling of the χ fields to the light Y species in the thermal bath. Since these only contribute logarithmically and are typically negligible for $m_\chi \gtrsim T$, we see that the flatness of the tree-level potential (chosen through a suitable analytic function in the superpotential) is preserved.

As we had seen in the previous section, the dissipation coefficient in the low-temperature regime is essentially due to the excitation of virtual low-momentum χ modes in the heat bath and independent of the coupling g . This implies that we may obtain arbitrarily strong dissipation for a large number of X fields if the coupling g is kept small such that $g^2 N_X \lesssim 1$ and radiative/thermal corrections to the inflaton potential are suppressed. This also ensures that the perturbative calculation of the effective potential and dissipation coefficient is under control, provided also that the effective coupling $h^2 N_Y \lesssim 1$ [19] (see also [24] for a discussion of the effective potential).

For this dissipation coefficient, $\Upsilon = C_\phi T^3/\phi^2$, we can write the slow-roll equation for radiation (97) in the form:

$$\frac{T}{H} \simeq \frac{15}{2\pi^2} \frac{C_\phi}{g_*} \left(\frac{\phi'}{\phi} \right)^2, \quad (106)$$

where $\phi' = d\phi/dN_e = \dot{\phi}/H$, with N_e denoting the number of e-folds of inflationary expansion until the end of the slow-roll regime. The flat space calculation of the dissipation coefficient that we have illustrated in the previous section is only consistent for $T \gtrsim H$, a regime where we can neglect the effects of space-time curvature. Since $\phi'/\phi \sim N_e^{-1}$, the above expression shows that this implies $C_\phi/g_* \gtrsim N_e^2$. Since only the fields in the Y sector are light and hence $g_* \propto N_Y$, this yields $h^2 N_X \gtrsim N_e^2$ up to numerical factors. Consistent realizations of warm inflation can thus be obtained by taking a sufficiently large number of heavy fields coupled directly to the inflaton, pointing towards beyond the Standard Model constructions such as e.g. string theory or extra-dimensions, where explicit models with large field multiplicities have been constructed (see e.g. [25, 26]).

Let us now consider the dynamics of warm inflation, corresponding to the coupled evolution of the inflaton field and the ambient radiation. It is convenient to express the slow-roll equations in terms of the number of e-folds of inflation defined above. It is easy to show that the slow-roll equation for the inflaton (98) can be written in the form:

$$\frac{\phi'}{\phi} \simeq -\frac{\sigma_\phi}{1+Q}, \quad (107)$$

where we defined the slow-roll parameter $\sigma_\phi = M_P^2 V'(\phi)/[\phi V(\phi)]$. From Eq. (106), this yields:

$$\frac{T}{H} \simeq \frac{C_\phi}{4C_R} \left(\frac{\sigma_\phi}{1+Q} \right)^2, \quad (108)$$

where we have defined $C_R = (\pi^2/30)g_*$ for simplicity. We can then use this to relate the dissipative ratio and the inflaton field value, yielding after some algebra:

$$Q(1+Q)^6 \simeq \frac{C_\phi^4}{24C_R} \epsilon_\phi^3 \left(\frac{M_P}{\phi} \right)^8 \left(\frac{H^2}{M_P^2} \right). \quad (109)$$

If we now differentiate both sides with respect to the number of e-folds and use Eqs. (101), (107) and also that:

$$\epsilon'_\phi = \frac{2\epsilon_\phi}{1+Q} (2\epsilon_\phi - \eta_\phi), \quad (110)$$

we obtain:

$$\frac{Q'}{Q} = \frac{10\epsilon_\phi - 6\eta_\phi + 8\sigma_\phi}{1+7Q}, \quad (111)$$

which explicitly shows that the dissipative ratio also evolves slowly during inflation but is nevertheless a dynamical variable. We may thus express the system of equations in terms of the dynamical variables (ϕ, Q) instead of (ϕ, T) , which turns out to be typically more convenient. To see this, let us consider the example of a quartic potential:

$$V(\phi) = \lambda^2 \phi^4 \quad (112)$$

corresponding to including a cubic term in the superpotential $\lambda\Phi^3/3$. For this potential, the slow-roll parameters take a simple form:

$$\epsilon_\phi = \frac{2}{3}\eta_\phi = 2\sigma_\phi = 8 \left(\frac{M_P}{\phi} \right)^2. \quad (113)$$

In the absence of dissipation, this requires superplanckian values of the inflaton field in order to yield a slow-roll regime. This is not the case if dissipation is strong throughout inflation, $Q \gg 1$, but as we will see the most interesting models correspond to cases where dissipation is weak when the relevant CMB scales leave the horizon 50-60 e-folds before the end of inflation. In this case, one also requires superplanckian values at horizon-crossing, although somewhat smaller than in the supercooled inflation case.

For this potential, we can express Eq. (109) in the form:

$$Q(1+Q)^6 \simeq \frac{64}{9} \lambda^2 \frac{C_\phi^4}{C_R} \left(\frac{M_P}{\phi} \right)^{10} \quad (114)$$

and use this to obtain:

$$\frac{Q'}{Q} = \frac{40}{1+7Q} \left(\frac{M_P}{\phi_*} \right)^2 \left(\frac{Q}{Q_*} \right)^{1/5} \left(\frac{1+Q}{1+Q_*} \right)^{6/5}, \quad (115)$$

where the subscript ‘*’ denotes quantities evaluated at horizon-crossing. Firstly, this shows that $Q' > 0$, such that the effects of dissipation will become progressively stronger as inflation proceeds, with the quartic potential being one of the examples where radiation smoothly takes over at the end of the slow-roll regime. In addition, we see that the coupled dynamics can be expressed as a single first order differential equation for the dissipative ratio Q . This equation can actually be solved exactly in terms of hypergeometric functions, but it is simpler to understand the effects of dissipation by considering the limit where $Q_* \ll 1$ and dissipation becomes significant towards the end of inflation. This yields for the number of e-folds of inflation after horizon-crossing:

$$N_e \simeq \frac{1}{8} \left(\frac{\phi_*}{M_P} \right)^2 \left(1 + 2.8Q_*^{1/5} \right), \quad (116)$$

which reduces to the standard supercooled result in the limit $Q_* \rightarrow 0$. It is thus explicit that dissipation makes inflation last longer, with significant effects for $Q_* \gtrsim 10^{-6}$.

Having determined the effects of dissipation on the average evolution of the homogeneous inflaton field, we now turn to the related effects of fluctuations, which we will only briefly discuss due to our limited time. While for the background field we neglect the effects of the noise term on the right-hand side of the Langevin-like equation (93) since $\langle \xi \rangle = 0$, these need to be considered when computing the spectrum of inflaton fluctuations, which in Fourier space satisfy³ [1, 29, 30]:

$$\delta \ddot{\phi}_k + 3H(1+Q)\delta \dot{\phi}_k + \left(\frac{k^2}{a^2} + V''(\phi) \right) \delta \phi_k = \xi_k. \quad (117)$$

We thus see that the noise term, whose amplitude is determined by the dissipation coefficient, will source the fluctuations of the inflaton field. As in the supercooled case, the amplitude of the field perturbations will freeze out. However, due to dissipation, this may happen even before their physical size becomes larger than the horizon.

Solving the Langevin equation in the general case can be rather involved and here we will focus on the simpler scenario where dissipation is weak when the observable CMB scales become super-horizon, $Q_* \ll 1$, such that fluctuations freeze-out when crossing the Hubble horizon as in the standard inflationary models. The main difference between this case and the standard supercooled inflationary model is that inflaton fluctuations need not be in the de Sitter vacuum and may in fact have a near-equilibrium distribution (see [31, 32] for a more detailed discussion). In this case, we can simply employ the canonical quantization procedure at finite temperature, where the creation and annihilation operators yield $\langle \hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}'}^\dagger \rangle = [n(\omega_k) + 1](2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}')$ and $\langle \hat{a}_{\mathbf{k}'}^\dagger \hat{a}_{-\mathbf{k}} \rangle = n(\omega_k)(2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}')$ where $\omega_k \simeq k/a$. The calculation is otherwise analogous to the computation of the inflaton spectrum in supercooled scenarios, yielding:

$$\begin{aligned} \langle \delta \phi^2 \rangle &\simeq \int_{k < k_F} \frac{d^3 k}{(2\pi)^3} \frac{1}{2k} |\delta \phi_k|^2 [1 + 2n(\omega_k)] \\ &\simeq 2 \left(\frac{T_*}{H_*} \right) \int_{k < k_F} \frac{dk}{k} \left(\frac{H_*}{2\pi} \right)^2, \end{aligned} \quad (118)$$

³This equation actually neglects the dependence of the dissipation coefficient on both ϕ and T . While discarding the former is typically a good approximation in the slow-roll regime, the temperature dependence generates a coupling between the inflaton and radiation perturbations that makes the analysis more complex and is beyond the scope of these lectures. However, in our working example with $Q_* \ll 1$ at horizon-crossing, this effect may safely be neglected [27, 28].

where $k_F = aH$ is the comoving momentum of the scale crossing the Hubble horizon at a given time, $|\delta\phi_k| \sim H/k$ for super-horizon scales and in the last step we have used that $T \gtrsim H$. We thus have that inflaton fluctuations have a larger amplitude than in the standard supercooled scenarios:

$$\langle \delta\phi^2 \rangle \simeq 2 \left(\frac{T_*}{H_*} \right) \langle \delta\phi^2 \rangle_{cold} . \quad (119)$$

This consequently enhances the spectrum of primordial curvature perturbations, which is given by:

$$\Delta_{\mathcal{R}}^2 \simeq \frac{1}{12\pi^2} \frac{V(\phi_*)}{M_P^4} \epsilon_{\phi_*}^{-1} \left(\frac{T_*}{H_*} \right) . \quad (120)$$

This allows one to relate the temperature and dissipative ratio at horizon-crossing by substituting this result into Eq. (104), yielding:

$$Q_* \simeq \frac{8\pi^2}{3} C_R \Delta_{\mathcal{R}}^2 \left(\frac{T_*}{H_*} \right)^3 \simeq 2 \times 10^{-8} g_* \left(\frac{T_*}{H_*} \right)^3 , \quad (121)$$

where we have used the observational value for the amplitude of the primordial curvature spectrum $\Delta_{\mathcal{R}}^2 \simeq 2.2 \times 10^{-9}$ [33]. This thus sets a lower limit on the strength of dissipative effects at horizon-crossing from the requirement that $T_* \gtrsim H_*$ for consistency of our calculation.

From the amplitude of the spectrum we can also deduce two other observables. Firstly, the scalar spectral index is given by:

$$\begin{aligned} n_s - 1 &\equiv \frac{d \log \Delta_{\mathcal{R}}^2}{d \log k} \simeq \frac{d \log \Delta_{\mathcal{R}}^2}{d N_e} \\ &\simeq 2\eta_{\phi_*} - 6\epsilon_{\phi_*} + \left. \frac{d \log(T/H)}{d N_e} \right|_* , \end{aligned} \quad (122)$$

where we have used the result from supercooled inflation. From Eq. (108) and using that $\sigma'_\phi/\sigma_\phi \simeq \sigma_\phi + 2\epsilon_\phi - \eta_\phi$ for weak dissipation, we obtain:

$$n_s - 1 \simeq 2\sigma_{\phi_*} - 2\epsilon_{\phi_*} . \quad (123)$$

This is independent of the second derivative of the inflaton potential, as opposed to supercooled scenarios. Secondly, since neither dissipation nor the presence of a heat bath will change the evolution of tensor perturbations, as gravity waves interact extremely weakly with matter, the tensor-to-scalar ratio is reduced compared to non-dissipative models:

$$r \equiv \frac{\Delta_t^2}{\Delta_{\mathcal{R}}^2} = 8\epsilon_{\phi_*} \left(\frac{T_*}{H_*} \right)^{-1} . \quad (124)$$

Since the tensor spectral index is unaffected by dissipation, this yields a modified consistency relation for warm inflation models in general:

$$n_t \simeq -2\epsilon_\phi \simeq -\frac{r}{8} \left(\frac{T_*}{H_*} \right) , \quad (125)$$

which may thus be used to observationally probe the significance of dissipative effects during inflation.

If we apply the above results to our example of a quartic scalar potential, we obtain:

$$n_s - 1 \simeq -\frac{1}{N_e} \left(1 + 2.8Q_*^{1/5} \right) \quad r \simeq \frac{8}{N_e} \left(1 + 2.8Q_*^{1/5} \right) \left(\frac{T_*}{H_*} \right)^{-1} . \quad (126)$$

These two observables are then determined by the duration of inflation and a single additional parameter, which we may take to be either Q_* or the ratio T_*/H_* using Eq. (121). For example, if $Q_* \simeq 0.005$, corresponding to $T_*/H_* \simeq 10$ if all the MSSM degrees of freedom are present the heat bath ($g_* = 228.75$), we obtain:

$$n_s \simeq 0.961 - 0.967 \quad (0.94 - 0.95) \quad r \simeq 0.026 - 0.031 \quad (0.27 - 0.32) \quad (127)$$

for 50-60 e-folds of inflation, with the values given in brackets corresponding to the supercooled inflation case. We thus see that the quartic potential is an example of an inflationary model that is only in agreement with the observations of the Planck satellite in the presence of dissipation (see [32] for a more complete discussion), with the results of the Planck satellite giving $n_s = 0.968 \pm 0.006$ [33] and $r < 0.12$ (in combination with BICEP2/Keck [34]).

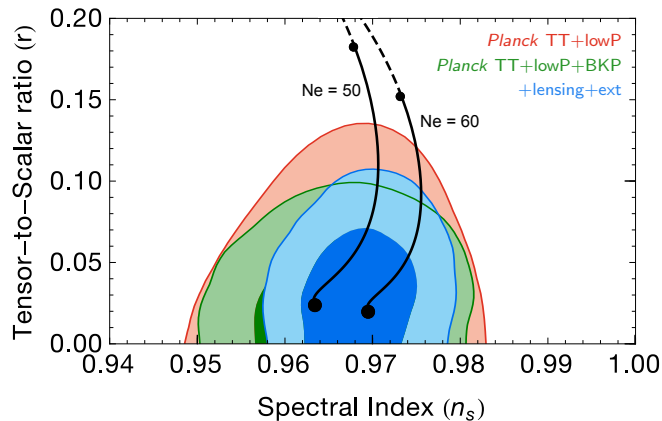


Figure 2: Observational predictions for a quartic potential in warm inflation with weak dissipation at horizon crossing and $g_* = 228.75$ (black curves) compared to the recent limits obtained by the Planck satellite [33]. The dashed part of the curves corresponds to $T_* < H_*$, where the underlying assumptions fail. The black circles indicate the predictions for $T_* = H_*$ (small black circles) and $Q_* = 0.01$ (large black circles), the latter setting the limit of validity of the weak dissipation analysis. [Courtesy of Sam Bartrum]

7 Summary and further reading

To summarize the main points of these lectures, we have seen that two effects arise when one considers the interactions between a dynamical system and the surrounding environment - dissipation and fluctuations. This applies to many different systems in nature and in particular to quantum fields. These are generically non-local effects but, when the evolution of the field is slow compared to the typical time scale of microphysical processes in the environment (adiabatic regime), dissipation results in a local friction coefficient in the field's equation of motion and a Gaussian white noise term. The variance of the noise term is entirely determined by the dissipation coefficient in what is known as the fluctuation-dissipation relation.

Dissipation coefficients can be computed using the Schwinger-Keldysh formalism from the self-energy of the dynamical field and depend on the properties of the degrees of freedom present in the heat bath, namely their masses and decay width. Dissipation coefficients generically depend on the field value and the temperature of the heat bath (for a near-equilibrium configuration). For further reading on the more formal aspects of non-equilibrium QFT and computation of dissipation coefficients, I suggest [35–46] in addition to the references mentioned in the text.

These effects may have a large impact on the dynamics of cosmological scalar fields. As we have seen for the case of warm inflation, friction slows down the average field motion and leads to entropy production as the field's energy is transferred into the heat bath. Additionally, the noise term sources random fluctuations of the field, with an amplitude that depends on the size of the dissipation coefficient and on the field's statistical state. In warm

inflation, these effects prolong the duration of the slow-roll period, lead to a graceful exit into a radiation-dominated era and modify the form of the primordial spectrum of curvature perturbations. There is an extensive literature on warm inflation models, their dynamics, particle physics realizations and observational predictions and besides the references mentioned above I would suggest [47–58] for those interested in knowing more about this topic and also in applying similar techniques to the dynamics of other cosmological scalar fields.

The field of applications of non-equilibrium QFT in Cosmology is still in its infancy and there are several research avenues to pursue. I thus hope that these lectures will motivate you to learn more about these and other related topics.

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References

- [1] A. Berera, Phys. Rev. Lett. **75**, 3218 (1995) [astro-ph/9509049]; A. Berera and L. Z. Fang, Phys. Rev. Lett. **74**, 1912 (1995) [astro-ph/9501024].
- [2] M. Bastero-Gil, A. Berera, R. O. Ramos and J. G. Rosa, Phys. Lett. B **712**, 425 (2012) [arXiv:1110.3971 [hep-ph]].
- [3] M. Bastero-Gil, A. Berera, R. O. Ramos and J. G. Rosa, JCAP **1410**, no. 10, 053 (2014) [arXiv:1404.4976 [astro-ph.CO]].
- [4] M. Bastero-Gil, A. Berera and R. O. Ramos, JCAP **1109**, 033 (2011) [arXiv:1008.1929 [hep-ph]].
- [5] K. Mukaida and K. Nakayama, JCAP **1301**, 017 (2013) [JCAP **1301**, 017 (2013)] [arXiv:1208.3399 [hep-ph]].
- [6] K. Mukaida and K. Nakayama, JCAP **1303**, 002 (2013) [arXiv:1212.4985 [hep-ph]].
- [7] M. Drewes and J. U. Kang, Nucl. Phys. B **875**, 315 (2013) [Erratum-ibid. B **888**, 284 (2014)] [arXiv:1305.0267 [hep-ph]].
- [8] K. Harigaya and K. Mukaida, JHEP **1405**, 006 (2014) [arXiv:1312.3097 [hep-ph]].
- [9] K. Mukaida, K. Nakayama and M. Takimoto, JCAP **1406**, 013 (2014) [arXiv:1401.5821 [hep-ph]].
- [10] S. Bartrum, A. Berera and J. G. Rosa, arXiv:1412.5489 [hep-ph].
- [11] J. Berges, AIP Conf. Proc. **739**, 3 (2005) [hep-ph/0409233].
- [12] E. S. Fraga, Eur. Phys. J. A **29**, 123 (2006) [hep-ph/0510344].
- [13] A. O. Caldeira and A. J. Leggett, Phys. Rev. Lett. **46**, 211 (1981); A. O. Caldeira and A. J. Leggett, Annals Phys. **149**, 374 (1983).
- [14] J. Schwinger, Jour. Math. Phys. (N.Y.) **2**, 407 (1961); P. M. Bakshi and K. T. Mahanthappa, Jour. Math. Phys. (N.Y.) **4**, 1 (1963); **4**, 12 (1963); L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964); A. Niemi and G. Semenoff, Ann. Phys. (NY) **152**, 105 (1984); Nucl. Phys. B **230**, 181 (1984).

- [15] T. S. Evans and A. C. Pearson, Phys. Rev. D **52**, 4652 (1995).
- [16] A. Berera, I. G. Moss and R. O. Ramos, Rept. Prog. Phys. **72**, 026901 (2009) [arXiv:0808.1855 [hep-ph]].
- [17] H. A. Weldon, Phys. Rev. D **28**, 2007 (1983).
- [18] M. Bastero-Gil, A. Berera, I. G. Moss and R. O. Ramos, JCAP **1412**, no. 12, 008 (2014) [arXiv:1408.4391 [astro-ph.CO]].
- [19] M. Bastero-Gil, A. Berera, R. O. Ramos and J. G. Rosa, JCAP **1301**, 016 (2013) [arXiv:1207.0445 [hep-ph]].
- [20] I. G. Moss and C. Xiong, hep-ph/0603266.
- [21] I. G. Moss and C. M. Graham, Phys. Rev. D **78**, 123526 (2008) [arXiv:0810.2039 [hep-ph]].
- [22] M. Bastero-Gil and A. Berera, Int. J. Mod. Phys. A **24**, 2207 (2009) [arXiv:0902.0521 [hep-ph]].
- [23] M. Dine, L. Randall and S. D. Thomas, Phys. Rev. Lett. **75**, 398 (1995) [hep-ph/9503303].
- [24] M. Bastero-Gil, A. Berera, T. P. Metcalf and J. G. Rosa, JCAP **1403**, 023 (2014) [arXiv:1312.2961 [hep-ph]].
- [25] M. Bastero-Gil, A. Berera and J. G. Rosa, Phys. Rev. D **84**, 103503 (2011) [arXiv:1103.5623 [hep-th]].
- [26] T. Matsuda, Phys. Rev. D **87**, no. 2, 026001 (2013) [arXiv:1212.3030 [hep-th]].
- [27] M. Bastero-Gil, A. Berera and R. O. Ramos, JCAP **1107**, 030 (2011) [arXiv:1106.0701 [astro-ph.CO]].
- [28] I. G. Moss and C. Xiong, JCAP **0704**, 007 (2007) [astro-ph/0701302].
- [29] A. Berera, Nucl. Phys. B **585**, 666 (2000) [hep-ph/9904409].
- [30] L. M. Hall, I. G. Moss and A. Berera, Phys. Rev. D **69**, 083525 (2004) [astro-ph/0305015].
- [31] R. O. Ramos and L. A. da Silva, JCAP **1303**, 032 (2013) [arXiv:1302.3544 [astro-ph.CO]].
- [32] S. Bartrum, M. Bastero-Gil, A. Berera, R. Cerezo, R. O. Ramos and J. G. Rosa, Phys. Lett. B **732**, 116 (2014) [arXiv:1307.5868 [hep-ph]].
- [33] P. A. R. Ade *et al.* [Planck Collaboration], arXiv:1502.01589 [astro-ph.CO].
- [34] P. A. R. Ade *et al.* [BICEP2 and Planck Collaborations], [arXiv:1502.00612 [astro-ph.CO]].
- [35] R. Kubo, Rep. Prog. Phys. **29**, 255 (1966).
- [36] A. Hosoya and M. Sakagami, Phys. Rev. D **29**, 2228 (1984).
- [37] M. Morikawa and M. Sasaki, Phys. Lett. B **165**, 59 (1985).
- [38] M. Morikawa, Phys. Rev. D **33**, 3607 (1986).
- [39] R. Rivers, “Path Integral Methods in Quantum Field Theory” (Cambridge University Press, Cambridge, 1987).
- [40] B. L. Hu, J. P. Paz and Y. Zhang, in “The Origin of Structure in the Universe”, Ed. E. Gunzig and P. Nardone (Kluwer Acad. Publ. 1993); E. Calzetta and B. L. Hu, Phys. Rev. D **49**, 6636 (1994); *ibid* **52**, 6770 (1995); B. L. Hu and A. Matacz, Phys. Rev. D **51**, 1577 (1995); A. Matacz, Phys. Rev. D **55**, 1860 (1997); S. A. Ramsey and B. L. Hu, Phys. Rev. D **56**, 678 (1997), Erratum-*ibid.* **57**, 3798 (1998).
- [41] D. Lee and D. Boyanovsky, Nucl. Phys. B **406**, 631 (1993).
- [42] D. Boyanovsky, H. J. de Vega, R. Holman, D. -S. Lee and A. Singh, Phys. Rev. D **51**, 4419 (1995).

- [43] M. Gleiser and R. O. Ramos, Phys. Rev. D **50**, 2441 (1994).
- [44] M. Le Bellac, “Thermal Field Theory” (Cambridge University Press, Cambridge, 1996).
- [45] A. Anisimov, W. Buchmuller, M. Drewes and S. Mendizabal, Annals Phys. **324**, 1234 (2009) [arXiv:0812.1934 [hep-th]].
- [46] E. A. Calzetta and B. L. Hu, “Non-equilibrium Quantum Field Theory” (Cambridge University Press, Cambridge, 2008).
- [47] A. Berera, M. Gleiser and R. O. Ramos, Phys. Rev. D **58**, 123508 (1998) [hep-ph/9803394].
- [48] J. Yokoyama and A. D. Linde, Phys. Rev. D **60**, 083509 (1999) [hep-ph/9809409].
- [49] A. Berera and R. O. Ramos, Phys. Lett. B **567**, 294 (2003) [hep-ph/0210301].
- [50] M. Bastero-Gil and A. Berera, Phys. Rev. D **76**, 043515 (2007) [hep-ph/0610343].
- [51] J. C. Bueno Sanchez, M. Bastero-Gil, A. Berera and K. Dimopoulos, Phys. Rev. D **77**, 123527 (2008) [arXiv:0802.4354 [hep-ph]].
- [52] C. Graham and I. G. Moss, JCAP **0907**, 013 (2009) [arXiv:0905.3500 [astro-ph.CO]].
- [53] Y. -F. Cai, J. B. Dent and D. A. Easson, Phys. Rev. D **83**, 101301 (2011) [arXiv:1011.4074 [hep-th]].
- [54] K. Kamada and J. Yokoyama, Int. J. Mod. Phys. Conf. Ser. **01**, 114 (2011).
- [55] S. Bartrum, A. Berera and J. G. Rosa, Phys. Rev. D **86**, 123525 (2012) [arXiv:1208.4276 [hep-ph]].
- [56] R. Cerezo and J. G. Rosa, JHEP **1301**, 024 (2013) [arXiv:1210.7975 [hep-ph]].
- [57] S. Bartrum, A. Berera and J. G. Rosa, JCAP **1306**, 025 (2013) [arXiv:1303.3508 [astro-ph.CO]].
- [58] X. M. Zhang and J. Y. Zhu, arXiv:1412.4366 [gr-qc].