Extremal chiral ring states in AdS/CFT are described by free fermions

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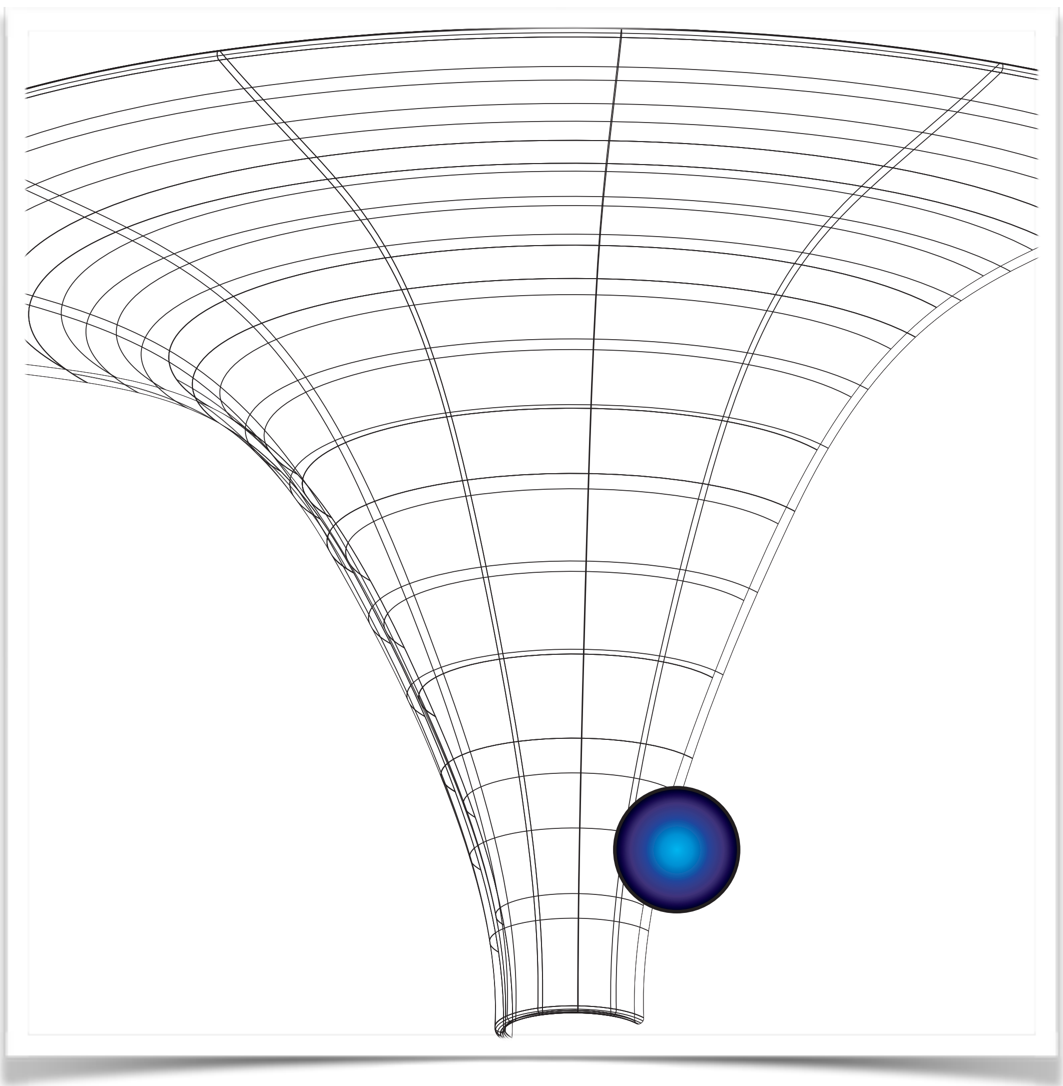
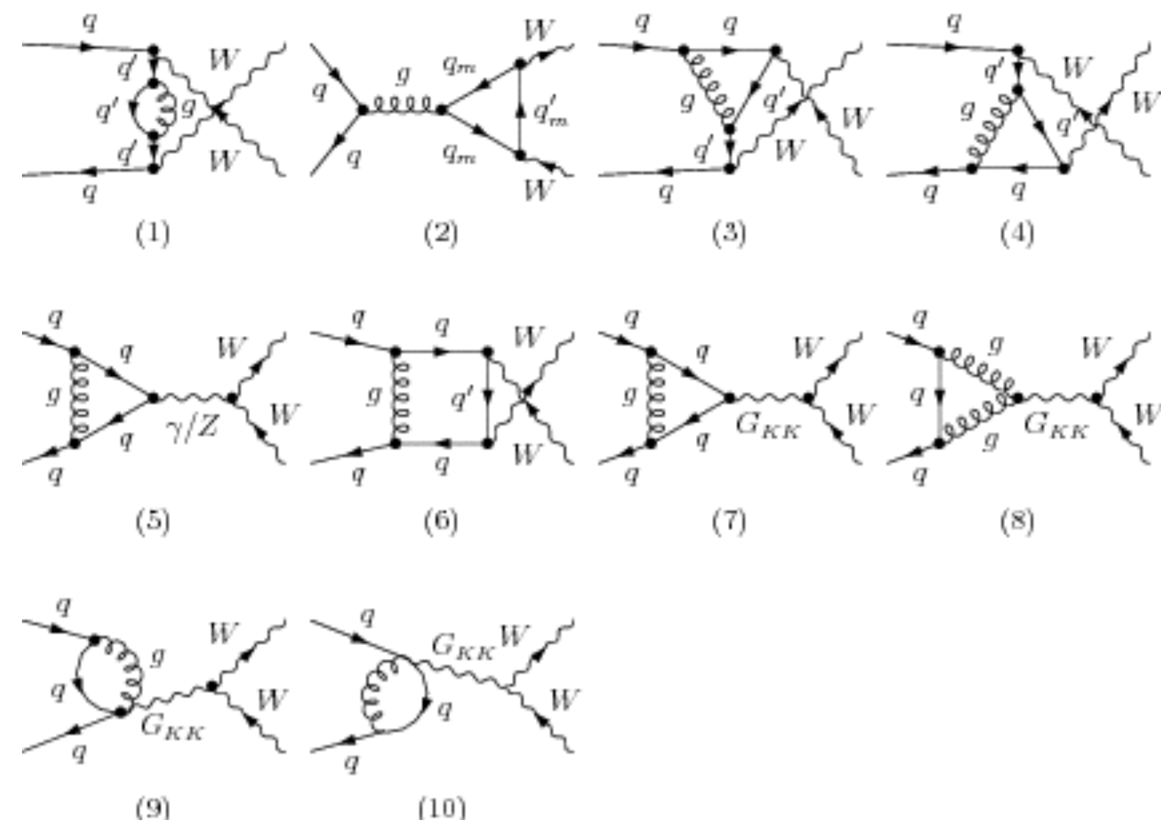
Mostly based on arXiv:1504.05389

IPMU, July 2015

INTRODUCTION

Gauge/gravity duality in a nutshell





Mathematical equivalence of certain quantum gravity theories and certain quantum field theories.

Standard dictionary

Correlators in field theory are dual to "gravity solutions" with modified boundary conditions.

This can be expressed in terms of a generating series for correlation functions of local operators.

$$\langle \exp(i \int \alpha(x)\mathcal{O}(x)) \rangle = Z_{grav}[g, \alpha]$$

$$\simeq \exp(IS[g])|_{\lim_{x\to\infty} g[x]\to\alpha}$$

Best understood cases

(Super) Conformal Field Theory are dual to (super) gravity theory on (Asymptotically) AdS spaces

Known as AdS/CFT correspondence

Conformal field theory two point functions are simple and take the following form

$$\langle \mathcal{O}_I(0)\mathcal{O}_J(x)\rangle = \frac{Z_{IJ}}{|x|^{2\Delta_I}}\delta_{\Delta_I,\Delta_J}$$

This is the Zamolodchikov metric.
Sometimes it is orthnormalized, but not necessarily.

Three point functions are really simple as well

$$\langle \mathcal{O}_I(x)\mathcal{O}_J(y)\mathcal{O}_K(z)\rangle \simeq \frac{C_{IJK}}{|x-y|^{\Delta_I+\Delta_J-\Delta_Z}|y-z|^{\Delta_J+\Delta_K-\Delta_I}|z-x|^{\Delta_K+\Delta_I-\Delta_J}}$$

The C are structure constants.

Also written as

$$\mathcal{O}_I(x)\mathcal{O}_J(y) \simeq \sum C_{IJK} Z^{-1} \mathcal{O}_K(x) \frac{1}{|x-y|^{\Delta_I+\Delta_J-\Delta_k}}$$

And this is called the OPE expansion.

The C are then also called the OPE coefficients.

Consider

$$AdS_5 \times X$$

Freund-Rubin compactification of IIB string theory.

What do we know about the dual field theory?

How do we go beyond SUGRA?

Some dual CFT's have been classified (Toric Sasaki-Einstein, Orbifolds of N=4 SYM) (Hanany et al.)

Matter content + superpotential

a-maximization (Intrilligator-Wecht)

Permits calculating R-charges of fields

We can compute the chiral ring

Cohomology of D

Gauge invariant operators, modulo relations generated by

$$\partial_\phi W = 0$$

List of some states with their energy (R-charge)

however…

Don’t even know the (Zamolodchikov) norm of states.
This is equivalent to knowing the Kahler potential

What else

- We can compute in free field theories
- We can do perturbation theory
- Most field theories of interest do not have a free field limit (they have non-trivial anomalous dimensions).

Goal

Tell you a conjecture about the norms of
a special set of those states for a special subset of
superconformal field theories.

Outline

- Rep. Theory of SC group.
- Half BPS states in N=4 SYM
- Generalization to orbifolds
- Going further: Extremal chiral ring states
- Establishing the main conjecture
- Free fermions

REP. THEORY OF SCFT

$$\{Q_{I\dot{\alpha}}, S_{J\dot{\beta}}\} \propto \delta^I_J M_{\dot{\alpha}\dot{\beta}} + \delta^I_J \epsilon_{\dot{\alpha}\dot{\beta}} \Delta + \epsilon_{\dot{\alpha}\dot{\beta}} R^I_J$$

Lowest state representations are annihilated by S, descendants are produced by acting with Q.

When working on cylinder geometry

$$S^3 \times \mathbb{R}$$

The operator Δ is the energy (Hermitian)

And unitarity of Hilbert space symmetries requires that

$$S = Q^\dagger$$

Unitarity condition leads to

$$\Delta \geq R + Spin$$

Saturation implies that some Q act by 0 on l.w.s.

States where some Q act as zero on l.w.s are called BPS or short representations.

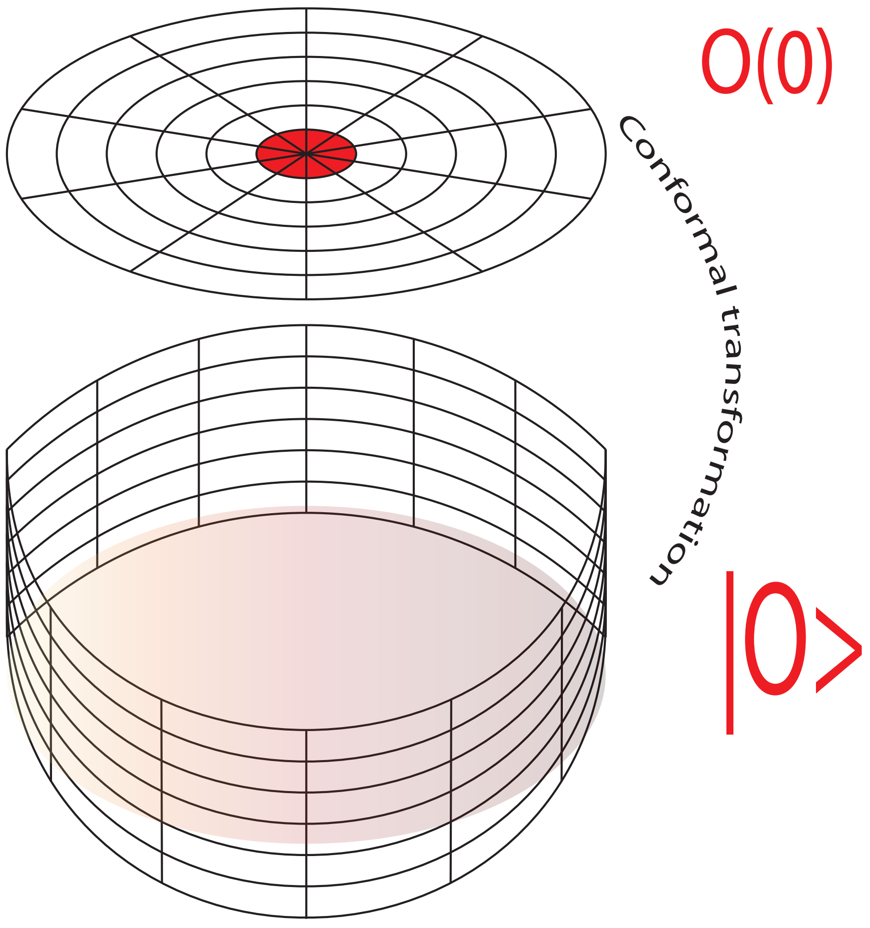
A particular subset is the chiral ring (cohomology of Q).

Operator-state correspondence

Weyl equivalence of cylinder and plane establishes a 1-1 correspondence between local operator insertions and states for the conformal field theory on cylinder.

We can talk about states and operators interchangeably

$$ds^2 = r^2\left(\frac{dr^2}{r^2} + d\Omega_3^2\right)$$



$$\mathcal{O}(0) \sim |\mathcal{O}\rangle$$

Correspondence requires that Hamiltonian is scaling dimension

Half BPS states in N=4 SYM

Preserve SO(4) rotation group (scalar)

They also preserve an SO(4) R-symmetry group (remember that there R-symmetry group is SO(6)~SU(4)).

They satisfy

$$\Delta = R$$

States that do this can be constructed in free limit by studying Fock space of states of N=4 SYM.

$$Z(\theta) \simeq \sum Y_{\ell m}(\theta) Z^\dagger_{\ell m} + Y_{\ell m}(\theta) \bar{Z}_{\ell m}$$

Only states built from Z can be half-BPS

$$Z^\dagger_{00} \simeq Z$$

They also need to be Gauge invariant.
This is accomplished by taking traces.

$$A^{j_1 \cdots j_m}_{i_1, \cdots i_m} Z^{i_1}_{j_1} \dots Z^{i_m}_{j_m}$$

The A need to be built out of U(N) invariant tensors.
Upper indices need to be contracted with lower indices.

$$A^{j_1 \cdots j_m}_{i_1, \cdots i_m} \simeq \delta^{[i]}_{[j]}$$

States are of the form

$$\sum \prod_s \mathrm{Tr}(Z^s)^{N_s}$$

Built out of only one matrix.

They belong to the chiral ring.

Technical note:

Elements of chiral ring have non-singular OPE

$$\mathcal{O}_1(0)\mathcal{O}_2(x) \simeq \mathcal{O}_1\mathcal{O}_2(0) + \sum x^{\Delta - R_1 - R_2}\mathcal{O}_\Delta(0)$$

The OPE coefficients are ‘trivial’ due to factorization.

The only thing that is non-trivial is the Zamolodchikov metric.

Another characterization

$$V \simeq N \text{ rep. of U(N)}$$

$$g \in SU(N)$$

This induces an action on tensor products

$$g : V^{\otimes s} \to V^{\otimes s}$$

$$g(v_1 \otimes v_2 \cdots \otimes v_s) = gv_1 \otimes gv_2 \ldots gv_s$$

Which preserves the induced norm

Decomposing this tensor product into irreps of U(N) gives an induced action on each such irrep.

Decomposition is done by summing over permutations of the factors.

$$V^{\otimes s} = \oplus R_{U(N)} \otimes R_{S_s}$$

These are characterized by Young diagrams.

The character of g in each such irrep.is Gauge invariant.

this can be extended to

$$g \in GL(N, \mathbb{C})$$

$$g \in Mat_N(\mathbb{C})$$

The result is invariant under conjugation (this is, Gauge invariant)

It is also polynomial in entries, of degree s.

These are called Schur functions

$$\chi_R(Z)$$

they are in one to one correspondence with sum of multi-trace states

the R are represented by Young tableaux

These are orthogonal in the Zamolodchikov metric (Corley, Jevicki, Ramgoolam- hep-th/0111222)

N	$N+1$	$N+2$
$N-1$	N	$N+1$
$N-2$		

$$|Y|^2 = \prod(\text{ Labels of boxes})$$

Product of Young tableaux is governed by Littlewood-Richardson coefficients.

$$|\ \mathrm{Tr}(Z^s)|^2 \simeq sN^s(1+O(1/N^2))$$

And

$$\langle Tr(Z^{s_1})Tr(Z^{s_2})|Tr(Z^{s_3})\rangle = \frac{\sqrt{s_1 s_2 s_3}}{N}\delta_{s_3,s_1+s_2}|Tr(Z^{s_1})||Tr(Z^{s_2})||Tr(Z^{s_3})|$$

Result for 3pt functions in N=4 SYM is identical in free field theory and gravity dual.

Lee, Minwalla, Rangamani, Seiberg, hep-th/9806074

All half BPS states are described by free fermion droplets in 2d flat phase space

D.B. hep-th/0403110

All half BPS non-singular geometries are parametrized by pictures of an incompressible fluid in a flat 2D plane

Lin, Lunin, Maldacena hep-th/0409174

Goal

Generalize this story to other setups.

May lead to a new way to think about calculations, even in theories that do not have a perturbative free field limit.

Problem

This only seems to work if we have an analog of Z.

Lets start with an example that does: orbifolds.

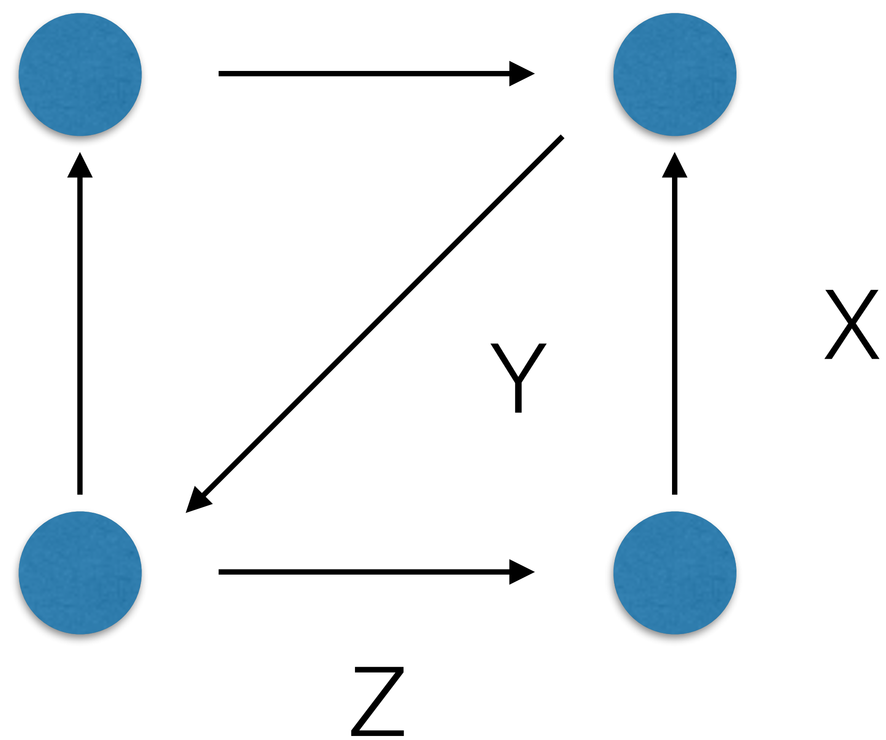
Orbifold group should map states made out of Z to themselves.

$$\Gamma \subset SU(2) \times U(1) \subset SU(3)$$

this means it has at least one additional U(1) symmetry.

Start with abelian

get a quiver diagram



$$W = Tr(XYZ - ZYX)$$

$$Tr(Z^s) \simeq \sum_i Tr(Z_{i,i+1} Z_{i+1,i+2} \dots)$$

The operators made only out of Z do not mix with other operators in the F-term relations.

They maximize the U(1) charge that counts Z relative to the R-charge of operator (extremal)

Closed loop implies that s is a multiple of the length of closed path in torus.

Easy to show that gauge invariance implies that everything can be written as multitraces of

$$\tilde{Z}_\ell = Z_{\ell,\ell+1} \dots Z_{\ell-1,\ell}$$

And that origin in a loop does not matter.

IF we want orbifold non-singular, then quiver has only one row.

Tautologically, states can be defined in terms of Young tableaux

$$\chi_R(\tilde{Z}_\ell)$$

Multiplication is trivial in same sense as before.

Orthogonality

Young tableaux is formed from a projector in symmetric group.

Projector Sums with weights over upper indices in

$$(\tilde{Z}_\ell)^{[i]}_{[j]}$$

but this is same as sum over upper indices of

$$(Z_{\ell,\ell+1})^{[i]}_{[j']}$$

Projects in Fock space to a unique irrep. of U(N) under which this field is in fundamental.

Different irreps are orthogonal.

Bose symmetry of labels forces Young tableaux of upper indices to be same Young tableaus as lower indices.

$$|Y|^2 = \prod (\text{Labels of boxes})^k$$

Generalization of k=2 case by

Dey, arXiv:1105.0218

Extremal chiral ring states

Any case that is “almost like orbifolds”

Want extra U(1) charge, maximize U(1) charge relative to R-charge, want this to lead to unique Z, no relations, states are multi-traces.

Toric field theories.

In dual SUGRA, want the extra U(1) to lead to a unique circle in Sasaki-Einstein geometry

In type IIB compactifications, we have a string coupling constant that we can vary and take to zero.

Formally, this makes the gauge coupling constants in quiver go to zero, keeping anomalous dimensions of fields fixed.

Can argue for orthogonality of Young tableaux.

Consider for example the flow from N=2 SYM SU(N) x SU(N) to the Klebanov Witten theory.

$$g^1_{YM} \simeq 0$$

Then theory has a superpotential of the form

$$W \simeq m\phi_2^2 + \phi_2(Q_1\tilde{Q}_1 - \tilde{Q}_2 Q_2)$$

Has global U(2N) flavor symmetry.

$$Q_2 Q_1 \simeq (\bar{N}, N) \text{ of } SU(N) \times SU(N)$$

Even though it has some terms that don't belong to KW super potential, can deform to only terms that do belong keeping SU(N)x SU(N) symmetry (also, h-deformation = 0).

Some point in conformal manifold with enhanced global symmetry, which is weakly gauged to diagonal.

Main conjecture

Young tableaux states are orthogonal.

Let

$$t_s = \mathrm{Tr}(\tilde{Z}_\ell^s)$$

At large N we are supposed to get an approximate Fock space with the t as generators.

$$|\prod_i t_i^{n_i}|^2 = \prod n_i! |t_i|^{2n_i} (1 + O(1/N^2))$$

And 1/N corrections to overlaps

$$\langle t_s | t_a t_b \rangle \simeq N^{-1} |t_s||t_a||t_b| \delta_{s,a+b} A_{s;a,b}$$

$$\langle t_s t_u | t_a t_b \rangle \simeq |t_s||t_u||t_a||t_b| (\delta_{sa}\delta_{ub} + \delta_{sb}\delta_{ua} + N^{-2}\delta_{s+u,a+b} A_{s,u;a,b})$$

$$t_1 = \square$$

Define

$$|t_1|^2 = T$$

Orthogonality of Young tableaux implies relations

$$t_1^2 = \square\square + \begin{smallmatrix}\square\\\square\end{smallmatrix}$$

$$t_2 = \square\square - \begin{smallmatrix}\square\\\square\end{smallmatrix}$$

$$\Rightarrow \quad |\square\square|^2 + \left|\begin{smallmatrix}\square\\\square\end{smallmatrix}\right|^2$$

$$|t_1^2|^2 = |\,\square\square\,|^2 + |\,\begin{smallmatrix}\square\\\square\end{smallmatrix}\,|^2 \simeq 2T^2(1 + O(1/N^2))$$

$$|t_2|^2 = |\,\square\square\,|^2 + |\,\begin{smallmatrix}\square\\\square\end{smallmatrix}\,|^2 \simeq 2T^2$$

And we expect that

$$\langle t_2|t_1^2\rangle = |\,\square\square\,|^2 - |\,\begin{smallmatrix}\square\\\square\end{smallmatrix}\,|^2 \simeq N^{-1}\sqrt{2}|T|^2 A_{2;1,1}$$

From here it follows that

$$|\,\square\square\,|^2 = T^2(1 + \eta/N + O(1/N^2))$$

$$\left|\,\begin{array}{c}\square\\[-4pt]\square\end{array}\,\right|^2 = T^2(1 - \eta/N + O(1/N^2))$$

$$t_1^3 = (t_1^2)t_1 = \left(\yng(2) + \yng(1,1)\right)\yng(1) = \yng(3) + 2\,\yng(2,1) + \yng(1,1,1)$$

$$t_2 t_1 = \left(\yng(2) - \yng(1,1)\right)\yng(1) = \yng(3) - \yng(1,1,1)$$

$$t_3 = \yng(3) - \yng(2,1) + \yng(1,1,1)$$

It follows that

$$\left|\yng(2,1)\right|^2 = T^3(1 + O(1/N^2))$$

$$\left|\yng(3)\right|^2 = T^3(1 + 3\eta/N + O(1/N^2))$$

$$\left|\yng(1,1,1)\right|^2 = T^3(1 - 3\eta/N + O(1/N^2))$$

tes that

$$|t_3|^3 = |\,\ydiagram{3}\,|^2 + |\,\ydiagram{2,1}\,|^2 + |\,\ydiagram{1,1,1}\,|^2 = 3(T^3)(1 + O(1/N^2))$$

$$\langle t_3 | t_2 t_1 \rangle = \langle t_1^3 | t_2 t_1 \rangle \simeq N^{-1} |t_3| |t_2| |t_1| A_{3;2,1} = \frac{3 \times 2}{N} T^3 \eta$$

Best way to organize it?

$$\chi_R(Z) \simeq \sum_{\sigma} \chi_R(\sigma) \mathrm{tr}(\tilde{Z}_\ell)^{[i]}_{[\sigma(i)]}$$

Explicit way to write it in terms of multi-traces, using character
of the symmetric group.

There is only one new trace at each order.

There are finitely many new 3 point functions

$$A_{m;m-n,n}/N$$

Then use the exact orthogonality between different objects on the left hand side: one gets linear relations on the right hand side order by order in powers of 1/N.

One expects this procedure to have a new finite number of unknowns order by order in 1/N

After a bit of pattern recognition

$$|Y|^2 = T^{\#\text{ boxes of }Y}\left(1+\frac{\eta}{N}\sum_{\text{boxes}}(\text{label of box})^2\right)$$

The integer k for orbifolds can in principle be exchanged by a positive real number

$$k \to \eta$$

From there it follows that

$$|t_n|^2 = nT^n(1 + O(1/N^2))$$

$$\langle t_n | t_i t_j \rangle = \delta_{n,i+j} \frac{\eta}{N} |t_n||t_i||t_j| \sqrt{(n)(i)(j)}$$

Extremal correlators are the same as N=4 SYM, except for a constant!

k was R-charge of word

We get a conjecture: always R-charge of word.

Full conjecture:

$$|Y|^2 = \prod_{\text{boxes}} (\text{labels of boxes})^{R_{\tilde{Z}_\ell}}$$

This represents free fermions for generalized oscillator.

Consider algebra

$$[\hat{N}, a^\dagger] = a^\dagger$$

$$[\hat{N}, a] = -a$$

Where N is hermitian with a unique irrep. and bounded from below spectrum.

$$\hat{N}a|\alpha\rangle = a\hat{N}|\alpha\rangle - a|\alpha\rangle = (\alpha - 1)a|\alpha\rangle \propto |\alpha - 1\rangle$$

Can choose ground state to have eigenvalue 0 and orthonormal

$$a^{\dagger}|n\rangle = f_{n+1}|n+1\rangle$$

$$G_k = |(a^{\dagger})^k|0\rangle| = \prod_{i=1}^{k} |f_k|^2$$

Go to a tensor product of these and impose Fermi statistics.

wave functions are Slater determinants

$$Y = N_0 \frac{1}{\sqrt{N!}} \det \begin{pmatrix} (a_1^\dagger)^{N_1} & (a_1^\dagger)^{N_2} & \dots & (a_1^\dagger)^{N_N} \\ (a_2^\dagger)^{N_1} & (a_2^\dagger)^{N_2} & \dots & (a_2^\dagger)^{N_N} \\ \vdots & \vdots & \ddots & \vdots \\ (a_N^\dagger)^{N_1} & (a_N^\dagger)^{N_2} & \dots & (a_N^\dagger)^{N_N} \end{pmatrix} (|0\rangle)^{\otimes N}$$

$$N_1 > N_2 \dots$$

Where the Young tableaux has rows of sizes

$$N_i - (N - i)$$

Norm becomes

$$\prod_{\text{boxes}} |f_{\text{label of box}}|^2$$

Nice large N limit requires

$$\frac{f_{N+1}}{f_N} = 1 + \frac{\eta}{N} + \ldots$$

This can be used to show that asymptotically

$$f_{N+k} \simeq (N+k)^{\eta}$$

Can define coherent states

$$a|\lambda\rangle = \lambda|\lambda\rangle$$

$$|\lambda\rangle = N_\lambda \sum \frac{\lambda^k}{\sqrt{G_k}}|k\rangle$$

And this allows one to define interesting wave functions on Fermion system.

$$|\lambda_1, \lambda_2, \ldots\rangle \propto N_0 \frac{1}{\sqrt{N!}} \det \begin{pmatrix} |\lambda_1\rangle_1 & |\lambda_2\rangle_1 & \ldots & 1 \\ |\lambda_1\rangle_2 & |\lambda_2\rangle_2 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ |\lambda_1\rangle_N & |\lambda_2\rangle_2 & \ldots & 1 \end{pmatrix} (|0\rangle)^{\otimes N}$$

$$\langle Tr(Z_\ell^s)\rangle = \sum \bar{\lambda}_i^s$$

These are states in the Coulomb branch of the theory.

For single brane Energy is approximated by

$$R[\tilde{Z}_\ell](|\lambda|^2)^{\eta^{-1}} - R[\tilde{Z}_\ell](N-1)$$

And only makes sense when the parameters are sufficiently large

State is in the Coulomb branch=
0 energy in flat space.

Energy on 3-sphere comes from curvature coupling to scalars and kinetics

Large vev for operators of dimension R determines a mass gap by dimensional analysis.

in equations

$$[\phi] = R_\phi$$
$$[R_{\mu\nu}] = 2$$

$$E/Vol \simeq R_{\mu\nu}|\phi|^{2/R_\phi}$$

But this must also be the Hamiltonian

$$R[\tilde{Z}_\ell](|\lambda|^2)^{\eta^{-1}} - R[\tilde{Z}_\ell](N-1)$$

Comparing

$$\eta = R[\tilde{Z}_\ell]$$

More precisely, the curvature coupling must be of the form

$$E = RK(\phi, \bar{\phi})$$

So one can compute the Kahler potential for branes.

properties

- Invariant under toric dualities.
- Makes universal SUGRA predictions
- Contains D-branes
- Can derive Kahler potential for single brane: a 2D cone geometry.
- Consistent with plane wave limit (universality manifest in limit).