Introduction to 2-Categorical Homological Algebra

Motivation

Symmetric Categorical Group = categorification of Abelian group

	category	2-category
general theory	Abelian category	Define this !!
example	Ab	SCG

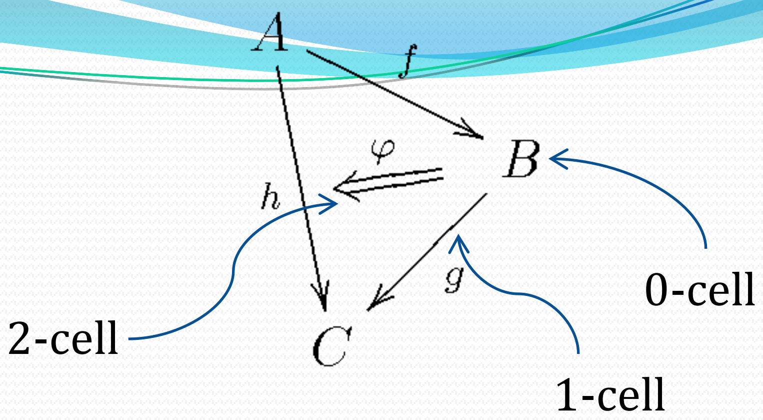
We call: **Relatively exact 2-category**

(Def 3.7 ※)

※ cf. 'Cohomology theory in 2-categories' in *Theory and Appl. Categ.*

2-category (strict)

= ‘category with 2-cells’



$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "h"'] & \arrow[d, phantom, "\overset{\varphi}{\Leftarrow}"] & B \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

S : 2-category

Assumption

$^\forall$2-cell in **S** is invertible

Duality

‘Dual’ = invert the direction of 1-cells

e.g.

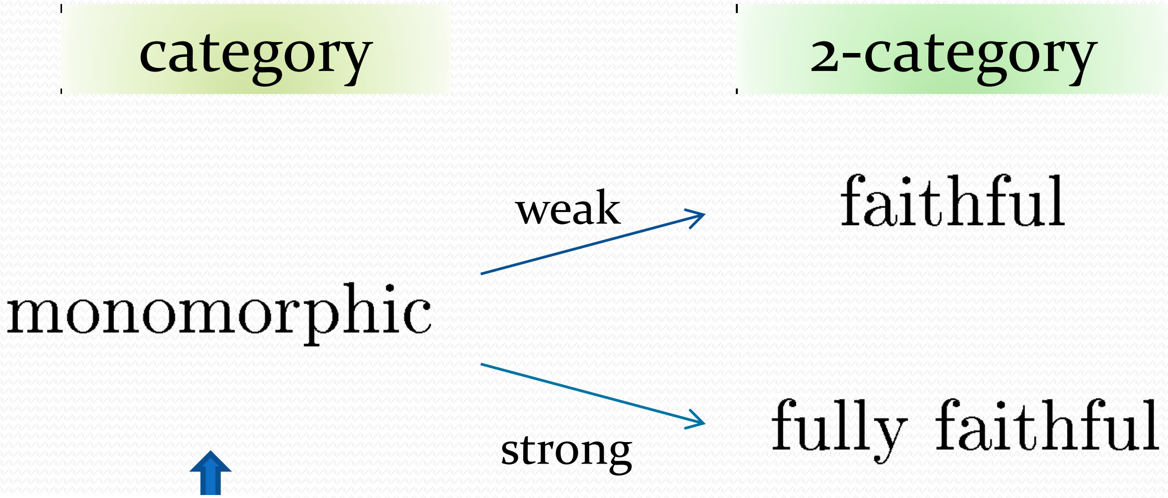
faithful ↔ cofaithful

kernel ↔ cokernel

pullback ↔ pushout

2-categorical notions

① properties of 1-cells



category

2-category

monomorphic

weak

faithful

strong

fully faithful

Definition

$$A \xrightarrow{f} B \ : \ \text{1-cell in } \mathbf{S}$$

f : (fully) faithful

$$\overset{\text{def}}{\iff} \quad f \circ - \ : \ \mathbf{S}(X, A) \to \mathbf{S}(X, B)$$

is (fully) faithful for any 0-cell X

category

functor

2-categorical notions

② Bilimits

(cf. Borceux, *Handbook of categorical algebra*)

e.g. pullback $(A_1 \times_B A_2, f_1', f_1', \eta)$

of

$$\begin{tikzcd}
A_2 \arrow[dr, "f_2"] & \\
& B \\
A_1 \arrow[ur, "f_1"'] &
\end{tikzcd}$$

is the universal one among

$$\begin{tikzcd}
& A_2 \arrow[dr, "f_2"] & \\
X \arrow[ur, "x_1"] \arrow[dr, "x_2"'] & \Downarrow \xi & B \\
& A_1 \arrow[ur, "f_1"'] &
\end{tikzcd}$$

i.e. for any (X, x_1, x_2, ξ), there exists a factorization (x, ξ_1, ξ_2)

$$\begin{tikzcd}
& {} & A_2 \arrow[dr, "f_2"] & \\
X \arrow[r, "g"] \arrow[urr, bend left, "g_1"] \arrow[drr, bend right, "g_2"'] & A_1 \times_B A_2 \arrow[ur, "f_1'"] \arrow[dr, "f_2'"'] \arrow[ul, Rightarrow, "\xi_1"] \arrow[dl, Rightarrow, "\xi_2"'] & \Downarrow \eta & B \\
& {} & A_1 \arrow[ur, "f_1"'] &
\end{tikzcd}$$

compatible with ξ and η

$$\begin{tikzcd}
g \circ f_1' \circ f_2 \arrow[r, Rightarrow, "\xi_1 \circ f_2"] \arrow[d, Rightarrow, "g \circ \xi"'] \arrow[dr, phantom, "\circlearrowright"] & g_1 \circ f_2 \arrow[d, Rightarrow, "\eta"] \\
g \circ f_2' \circ f_1 \arrow[r, Rightarrow, "\xi_2 \circ f_1"'] & g_2 \circ f_1
\end{tikzcd}$$

uniquely up to a unique 2-cell.

Symmetric Categorical Group

= symmetric 2-group = Picard category

Definition

A symmetric categorical group

$= $ a symmetric monoidal category $(\mathbb{G}, \otimes, 0)$

category

$$\otimes : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$$

tensor functor

unit object

such that

(i) any morphism $A \xrightarrow{f} B$ has an inverse

(ii) any object A has an 'inverse' w.r.t. $\otimes$

i.e. ${}^{\forall}A \in \mathrm{Ob}(\mathbb{G}),\ {}^{\exists}A^{*} \in \mathrm{Ob}(\mathbb{G}),\ {}^{\exists}\eta_A : 0 \to A \otimes A^{*}$.

2-category **SCG**

Symmetric Categorical Group

SCG

0-cell $=$ symmetric categorical group $\mathbb{G}, \mathbb{H}, \ldots$

1-cell $=$ monoidal functor $F : \mathbb{G} \to \mathbb{H}, \ldots$

2-cell $=$ monoidal transformation

$$\begin{tikzcd}
\mathbb{G} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathbb{H}
\arrow[Rightarrow, from=U, to=D, "\alpha"']
\end{tikzcd} , \ldots$$

Locally **SCG** 2-category

Locally SCG 2-category

analog of additive category

$\mathbf{S}$: locally SCG

$\overset{\text{def}}{\Longleftrightarrow}$

(A1) For every 0-cells A, B, Hom-category $\mathbf{S}(A, B)$ belongs to SCG, and $\quad \mathrm{Hom} = \mathbf{S}(-,-) : \mathbf{S} \times \mathbf{S} \to \mathrm{SCG} \quad$ is a 2-functor.

(A2) $\mathbf{S}$ has a zero object.

(A3) For any A, B, their product and coproduct exist.

Relatively exact 2-category

Relativel exact 2-category

analog of
Abelian category

$\mathbf{S}$: relatively exact 2-category

$\overset{\text{def}}{\Longleftrightarrow}$

(A) $\mathbf{S}$ is locally SCG.

(B1) For any 1-cell f, $\mathrm{Ker}(f)$ and $\mathrm{Cok}(f)$ exist.

(B2) For any 1-cell f,

$$f\ :\ \text{faithful} \Leftrightarrow f = \ker(\mathrm{cok}(f))$$

$$f\ :\ \text{cofaithful} \Leftrightarrow f = \mathrm{cok}(\ker(f)).$$

(Co-)Kernel

※ ‘2-(co)kernel’ or ‘bi(co-)kernel’ would be a more appropriate term

Kernel ⟷ Cokernel
dual

Kernel

f : 1-cell in $\mathbf{S}$

Kernel $(\mathrm{Ker}(f), \ker(f), \varepsilon_f)$

$=$ the universal one among those (K, k, ε)

$$\begin{tikzcd}
K \arrow[dr, "k"] \arrow[drr, bend left, "0"] & & \\
 & A \arrow[r, "f"'] \arrow[from=1-1, to=2-3, phantom, "\Uparrow \varepsilon"] & B \\
\mathrm{Ker}(f) \arrow[uu, "\underline{k}"] \arrow[ur, "\ker(f)"'] \arrow[urr, bend right, "0"'] \arrow[from=1-1, to=2-2, phantom, shift right=4ex, "\Downarrow \underline{\varepsilon}"] \arrow[from=2-2, to=3-3, phantom, "\Downarrow \varepsilon_f"] & &
\end{tikzcd}$$

i.e. for any (K, k, ε), there exists a factorization $(\underline{k}, \underline{\varepsilon})$ compatible with ε and ε_f, uniquely up to a unique 2-cell.

Properties of 1-cells in **S**

$\mathbf{S}$: relatively exact 2-category

f : 1-cell in $\mathbf{S}$

f : faithful $\iff$ $f = \ker(\mathrm{cok}(f))$

f : cofaithful $\iff$ $f = \mathrm{cok}(\ker(f))$

f : fully faithful $\iff$ $\mathrm{Ker}(f) = 0$

f : fully cofaithful $\iff$ $\mathrm{Cok}(f) = 0$

f : equivalence $\iff$ f : faithful & fully cofaithful

$\iff$ f : cofaithful & fully faithful

Factorization in S

Recall

Any morphism f in Abelian category $\mathcal{C}$ admits a (monomorphic, epimorphic)-factorization:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\text{epim.}"'] & \arrow[d, phantom, "\circlearrowright"] & B \\
& \mathrm{Coim}(f) \cong \mathrm{Im}(f) \arrow[ur, "\text{monom.}"'] &
\end{tikzcd}$$

$\mathbf{S}$: relatively exact 2-category

Any 1-cell f in $\mathbf{S}$ admit

(faithful, fully cofaithful)-factorization

dual

(fully faithful, cofaithful)-factorization

Factorization in **S**

In fact,

(faithful, fully cofaithful)-factorization

= Image factorization

(fully faithful, cofaithful)-factorization

= Coimage factorization

Caution

In '2-categorical homological algebra',

$$\text{Image} \quad \not\simeq \quad \text{Coimage}$$

Complex

$=$ sequence $+$ compatibility

Definition

A sequence

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[rr, bend left=60, "0", ""{name=T1, below}] & A^{n-1} \arrow[r, "d^{n-1}"'] \arrow[rr, bend right=60, "0"', ""{name=B1, above}] \arrow[Rightarrow, to=T1, "\delta^{n-1}"'] & A^{n} \arrow[r, "d^{n}"'] \arrow[rr, bend left=60, "0", ""{name=T2, below}] \arrow[Rightarrow, to=B1, "\delta^{n}"] & A^{n+1} \arrow[r, "d^{n+1}"'] \arrow[rr, bend right=60, "0"', ""{name=B2, above}] \arrow[Rightarrow, to=T2, "\delta^{n+1}"'] & A^{n+2} \arrow[r] \arrow[Rightarrow, to=B2] & \cdots
\end{tikzcd}$$

is a *complex* if it satisfies

$$d^{n+1} \circ \delta^{n} = \delta^{n+1} \circ d^{n-1}$$

for any $n \in \mathbb{N}$.

$$\begin{tikzcd}
d^{n+1} \circ d^{n} \circ d^{n-1} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, phantom, "\circlearrowright"] & d^{n+1} \circ 0 \arrow[d, Rightarrow, "\mathrm{can}"] \\
0 \circ d^{n-1} \arrow[r, Rightarrow, "\mathrm{can}"'] & 0
\end{tikzcd}$$

Want to define cohomology

Cohomology (1)

$$\begin{tikzcd}
 & & & H'^n(A^\bullet) & & & \\
 & & \mathrm{Ker}(d^n) \arrow[ur] \arrow[dr] \arrow[d, phantom, "\Downarrow"] & & & & \\
A^\bullet : \arrow[r] & A^{n-1} \arrow[rr, "d^{n-1}"'] \arrow[ur, dashed, "\underline{d^{n-1}}"] & {} & A^n \arrow[rr, "d^n"] \arrow[dr] & {} & A^{n+1} \arrow[r] & {} \\
 & & & & \mathrm{Cok}(d^{n-1}) \arrow[ur, dashed, "\overline{d^n}"'] \arrow[u, phantom, "\Uparrow"] & & \\
 & & & H'^n(A^\bullet) \arrow[ur] & & &
\end{tikzcd}$$

1^{st} naive Definition

$$H'^n(A^\bullet) := \mathrm{Cok}(\underline{d^{n-1}})$$

or $\mathrm{Ker}(\overline{d^n})$

equivalent

NO!

Can we obtain long cohomology sequence?

Bad point

2-cells $\delta^\bullet$ are not taken into account.

Suitable for sequences, but not for complexes.

Equivalence (1)

$$H'^{\,n}_1(A^\bullet) := \operatorname{Cok}(\underline{d^{n-1}}), \qquad H'^{\,n}_2(A^\bullet) := \operatorname{Ker}(\overline{d^n})$$

$$\begin{tikzcd}
 & & & {}^\exists w : H'^{\,n}_1(A^\bullet) \arrow[rr] & & H'^{\,n}_2(A^\bullet) \arrow[d, tail] & \text{equivalence} & \\
 & & & \operatorname{Ker}(d^n) \arrow[u, two heads] \arrow[dr, tail] \arrow[d, phantom, "\Downarrow"] & \Downarrow & \operatorname{Cok}(d^{n-1}) \arrow[dr, dashed, "\overline{d^n}"] \arrow[d, phantom, "\Downarrow"] & & \\
A^\bullet : & {} \arrow[r] & A^{n-1} \arrow[ur, dashed, "\underline{d^{n-1}}"] \arrow[rr, "d^{n-1}"'] & & A^n \arrow[ur, two heads] \arrow[rr, "d^n"'] & & A^{n+1} \arrow[r] & {}
\end{tikzcd} \tag{5.22}$$

$$\boxed{H'^{\,n}(A^\bullet) := H'^{\,n}_1(A^\bullet) \simeq H'^{\,n}_2(A^\bullet)}$$

Relative (Co-)Kernel

$= \boxed{\text{kernel}} + \boxed{\text{compatibility}}$

Relative kernel

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[rr, bend left=60, "0"] \arrow[rr, bend left=35, phantom, "\Uparrow\varphi"] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

: diagram in $\mathbf{S}$

Relative kernel $(\mathrm{Ker}(f,\varphi), \ker(f,\varphi), \varepsilon_{(f,\varphi)})$

$=$ the universal one among those (K, k, ε)

$$\begin{tikzcd}
K \arrow[dr, "k"] \arrow[drr, bend left=30, "0"] \arrow[drr, bend left=10, phantom, "\Uparrow\varepsilon"] & & & \\
& A \arrow[r, "f"'] \arrow[rr, bend left=60, "0"] \arrow[rr, bend left=35, phantom, "\Uparrow\varphi"] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

satisfying $\varphi \circ k = g \circ \varepsilon$

Relation: Kernel and Relative kernel

- $\mathrm{Ker}(f) = \mathrm{Ker}(f, \mathrm{can})$

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[rr, bend left=50, "0"] & B \arrow[r, "0"'] \arrow[u, phantom, "\Uparrow \mathrm{can}"] & C
\end{tikzcd}$$

- $\mathrm{ker}(f, \varphi)$ factors through $\mathrm{Ker}(f)$:

$$\begin{tikzcd}
\mathrm{Ker}(f,\varphi) \arrow[dr, "k(f,\varphi)"] \arrow[drr, bend left, "0"] \arrow[dd, dashed] & \Uparrow & & \\
\Nearrow & A \arrow[r, "f"'] \arrow[d, phantom, "\Downarrow \varepsilon_f"] & B \arrow[r, "g"'] & C \\
\mathrm{Ker}(f) \arrow[ur, "\mathrm{ker}(f)"'] \arrow[urr, bend right, "0"'] & & &
\end{tikzcd}$$

fully faithful
(3.33)

- Relative kernel = 'two-step kernel' (3.20)

$$\begin{tikzcd}
 & & \mathrm{Ker}(g) \arrow[dr] \arrow[d, phantom, "\Downarrow"] & & \\
\mathrm{Ker}(f,\varphi) \arrow[d, phantom, sloped, "\cong"] & A \arrow[ur, "\underline{f}"] \arrow[rr, "f"'] \arrow[rrr, bend right=50, "0"'] & {} & B \arrow[r, "g"'] \arrow[d, phantom, "\varphi\Downarrow"] & C \\
\mathrm{Ker}(\underline{f}) \arrow[ur] & & & {} &
\end{tikzcd}$$

Cohomology (2)

$A^\bullet = (A^n, d^n, \delta^n)$: complex

$$[Z^n(A^\bullet), z^n, \zeta^n] := \mathrm{Ker}(d^n, \delta^{n+1})$$

$$[Q^n(A^\bullet), q^n, \rho^n] := \mathrm{Cok}(d^{n-1}, \delta^{n-1})$$

Dual

$$\begin{tikzcd}
 & & & Z^n(A^\bullet) \arrow[dr, "z^n"] \arrow[rrr, phantom, ""] \arrow[drrr, "0"'] \arrow[d, Rightarrow, "\nu^{n,1}"] \arrow[dr, Rightarrow, shift right=2, "\rho^n"'] & & Q^n(A^\bullet) \arrow[dr, "\ell^n"] \arrow[drrr, bend left, "0"] \arrow[d, Rightarrow, "\mu^{n,1}"] & & & \\
A^{n-2} \arrow[urrr, bend left, "0"] \arrow[rr, "d^{n-2}", "d^{n-2}"'] \arrow[rrrr, bend right=40, "0"'] & & A^{n-1} \arrow[ur, "k^n"] \arrow[urrr, "0"] \arrow[ul, Rightarrow, "\nu^{n,2}"'] \arrow[rr, "d^{n-1}"] \arrow[rrrr, bend right=40, "0"'] \arrow[d, Rightarrow, "\delta^{n-1}"'] & & A^n \arrow[ur, "q^n"'] \arrow[ur, Rightarrow, shift left=2, "\zeta^n"] \arrow[rr, "d^n"] \arrow[rrrr, bend right=40, "0"'] \arrow[d, Rightarrow, "\delta^n"] & & A^{n+1} \arrow[ur, Rightarrow, "\mu^{n,2}"] \arrow[rr, "d^{n+1}"] \arrow[d, Rightarrow, "\delta^{n+1}"] & & A^{n+2} \\
 & & {} & & {} & & {} & &
\end{tikzcd}$$

$$H_1^n(A^\bullet) := \mathrm{Cok}(k^n, \nu^{n,2})$$

$$H_2^n(A^\bullet) := \mathrm{Ker}(\ell^n, \mu^{n,2})$$

Theorem

$$H_1^n(A^\bullet) \simeq H_2^n(A^\bullet)$$

Equivalence (2)

$$\boxed{H^n(A^\bullet) := H^n_1(A^\bullet) \simeq H^n_2(A^\bullet)} \qquad (5.24)$$

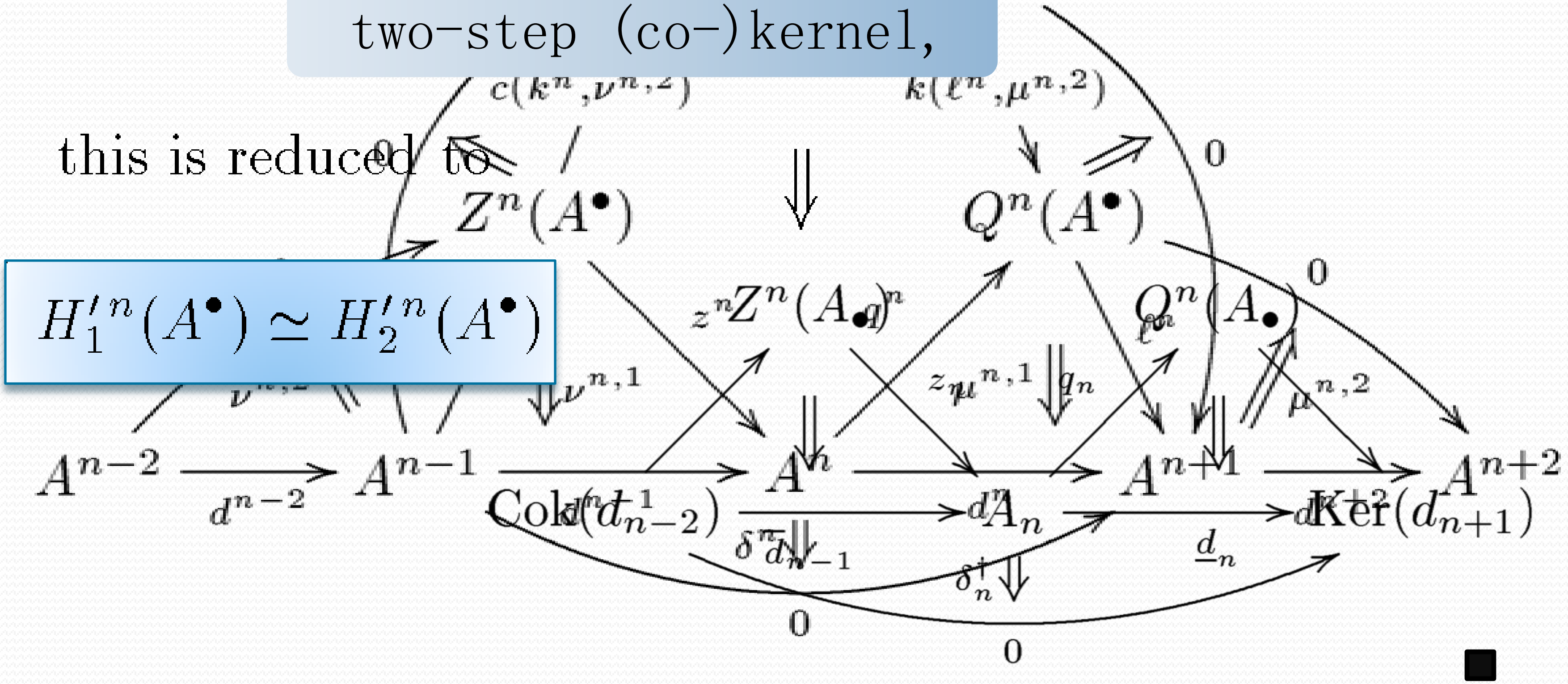
∵) Since a relative (co-)kernel is a

two-step (co-)kernel,

$\exists w : H^n_1(A^\bullet) \longrightarrow H^n_2(A^\bullet)$ equivalence

this is reduced to

$$\boxed{H'^n_1(A^\bullet) \simeq H'^n_2(A^\bullet)}$$



■

Relative 2-exactness

$$\begin{tikzcd}
 & & {} & & {} & & \\
A^\bullet : & \cdots \arrow[r] \arrow[rr, bend left=60, "0"] & A^{n-1} \arrow[r, "d^{n-1}"'] \arrow[u, Rightarrow, "\delta^{n-1}"'] \arrow[rr, bend right=60, "0"'] & A^n \arrow[r, "d^n"'] \arrow[rr, bend left=60, "0"] \arrow[d, Rightarrow, "\delta^n"] & A^{n+1} \arrow[r, "d^{n+1}"'] \arrow[u, Rightarrow, "\delta^{n+1}"'] \arrow[rr, bend right=60, "0"'] & A^{n+2} \arrow[r] \arrow[d, Rightarrow] & \cdots \\
 & & & {} & & {} &
\end{tikzcd}$$

Definition

- A sequence $A^\bullet$ is *2-exact* at A^n

  $$\overset{\text{def}}{\iff} \quad {H'}^n(A^\bullet) = 0.$$

- A sequence $A^\bullet$ is *relatively 2-exact* at A^n

  $$\overset{\text{def}}{\iff} \quad H^n(A^\bullet) = 0.$$

Snake Lemma (6.18)

Lemma

$$\begin{tikzcd}
 & A^1 \arrow[r, "d_A^1"] \arrow[rr, bend left=40, "0"] \arrow[d, "f^1"'] \arrow[dr, phantom, "\lambda^1 \Downarrow"] & A^2 \arrow[r, "d_A^2"] \arrow[u, phantom, "\Uparrow \delta_A^2"] \arrow[d, "f^2"] \arrow[dr, phantom, "\Downarrow \lambda^2"] & A^3 \arrow[r, "0"] \arrow[d, "f^3"] & 0 \\
0 \arrow[r, "0"'] & B^1 \arrow[r, "d_B^1"] \arrow[rr, bend right=40, "0"'] & B^2 \arrow[r, "d_B^2"] \arrow[d, phantom, "\delta_B^2 \Downarrow"] & B^3 & \\
 & & {} & &
\end{tikzcd}$$

relatively 2-exact at A^2, A^3, B^1, B^2

$$\Downarrow$$

$$\begin{tikzcd}
\mathrm{Ker}(f^1) \arrow[r, "\underline{d}_A^1"'] \arrow[rr, bend left=30, "0"] & \mathrm{Ker}(f^2) \arrow[r, "\underline{d}_A^2"'] \arrow[u, phantom, "\Uparrow \underline{\delta}_A^2"] \arrow[rr, bend right=30, "0"'] & \mathrm{Ker}(f^3) \arrow[r, "d"'] \arrow[d, phantom, "\Downarrow \alpha"] \arrow[rr, bend left=30, "0"] & \mathrm{Cok}(f^1) \arrow[r, "\overline{d}_B^1"] \arrow[u, phantom, "\Uparrow \beta"] \arrow[rr, bend right=30, "0"'] & \mathrm{Cok}(f^2) \arrow[r, "\overline{d}_B^2"] \arrow[d, phantom, "\overline{\delta}_B^2 \Downarrow"] & \mathrm{Cok}(f^3) \\
 & & {} & & {} &
\end{tikzcd}$$

2-exact sequence

Statement

Theorem

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r, "f^\bullet"'] \arrow[rr, bend left=50, "0", ""{name=U, below}] & B^\bullet \arrow[r, "g^\bullet"'] \arrow[Rightarrow, to=U, "\varphi^\bullet"'] & C^\bullet \arrow[r] & 0
\end{tikzcd}$$

: extension

$$\Downarrow$$

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[rr, bend left=40, "0", ""{name=A, below}] & H^n(B^\bullet) \arrow[r] \arrow[rr, bend right=40, "0"', ""{name=B, above}] \arrow[Rightarrow, to=A] & H^n(C^\bullet) \arrow[r] \arrow[rr, bend left=40, "0", ""{name=C, below}] \arrow[Rightarrow, to=B] & H^{n+1}(A^\bullet) \arrow[r] \arrow[rr, bend right=40, "0"', ""{name=D, above}] \arrow[Rightarrow, to=C] & H^{n+1}(B^\bullet) \arrow[r] \arrow[Rightarrow, to=D] & \cdots
\end{tikzcd}$$

: 2-exact sequence

Extension

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r, "f^\bullet"'] \arrow[rr, bend left=40, "0"] & B^\bullet \arrow[r, "g^\bullet"'] \arrow[u, phantom, "\Uparrow \varphi^\bullet"] & C^\bullet \arrow[r] & 0
\end{tikzcd}$$

Definition

Extension

$\overset{\text{def}}{\Longleftrightarrow}$ $A^\bullet = (A^n, d_A^n, \delta_A^n)_n$, $B^\bullet = (B^n, d_B^n, \delta_B^n)_n$, $C^\bullet = (C^n, d_C^n, \delta_C^n)_n$,

$f^\bullet = (f^n : A^n \to B^n, \lambda^n : f^{n+1} \circ d_A^n \Longrightarrow d_B^n \circ f^n)_n$,

$g^\bullet = (g^n : A^n \to B^n, \kappa^n : g^{n+1} \circ d_B^n \Longrightarrow d_C^n \circ g^n)_n$,

$\varphi^\bullet = (\varphi^n : g^n \circ f^n \Rightarrow 0)$, compatible with $\kappa^\bullet$ and $\lambda^\bullet$

such that $\forall n \in \mathbb{N}$,

$$\begin{tikzcd}
0 \arrow[r] & A^n \arrow[r, "f^n"'] \arrow[rr, bend left=40, "0"] & B^n \arrow[r, "g^n"'] \arrow[u, phantom, "\Uparrow \varphi^n"] & C^n \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
& 0 & \\
A^n \arrow[r, "f^n"] \arrow[rr, bend left=40] \arrow[d, "d_A^n"'] & B^n \arrow[r, "g^n"] \arrow[d, "d_B^n" description] \arrow[u, phantom, "\Uparrow \varphi^n"] & C^n \arrow[d, "d_C^n"] \\
A^{n+1} \arrow[r, "f^{n+1}"] \arrow[rr, bend right=40] \arrow[ur, phantom, "\overset{\lambda^n}{\Longrightarrow}"] & B^{n+1} \arrow[r, "g^{n+1}"] \arrow[d, phantom, "\varphi^{n+1} \Downarrow"] \arrow[ur, phantom, "\overset{\kappa^n}{\Longrightarrow}"] & C^{n+1} \\
& 0 &
\end{tikzcd}$$

is relatively 2-exact.

Reduction (1)

$A^\bullet$: complex

$$\Rightarrow \quad \begin{tikzcd}[column sep=large, row sep=huge]
 & & Z^{n+1}(A^\bullet) \arrow[dr, "z^{n+1}"] \arrow[drr, bend left, "0"] \arrow[d, phantom, "\nu^{n+1,1}\ \overset{\|}{\Downarrow}"] & & \\
A^{n-1} \arrow[r, "d^{n-1}"] \arrow[urr, bend left, "0"] \arrow[drr, bend right, "0"'] \arrow[ur, phantom, "\nu^{n+1,2}\Uparrow"] \arrow[dr, phantom, "\rho^n\Downarrow"'] & A^n \arrow[rr, "d^n"] \arrow[ur, "k^{n+1}"] \arrow[dr, "q^n"'] & \Uparrow \quad \Downarrow & A^{n+1} \arrow[r, "d^{n+1}"] \arrow[ur, phantom, "\zeta^{n+1}\Uparrow"'] \arrow[dr, phantom, "\mu^{n,2}\Downarrow"] & A^{n+2} \\
 & & Q^n(A^\bullet) \arrow[uu, "x_n"', crossing over] \arrow[ur, "\ell^n"'] \arrow[urr, bend right, "0"'] \arrow[u, phantom, "\Uparrow\mu^{n,1}"] & &
\end{tikzcd}$$

$$\boxed{\mathrm{Ker}(x_n) = H^n(A^\bullet),\ \mathrm{Cok}(x_n) = H^{n+1}(A^\bullet)}$$

(6.15)

Reduction (2)

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r, "f^\bullet"'] \arrow[rr, bend left=45, "0"] \arrow[rr, phantom, bend left=20, "\Uparrow \varphi^\bullet"] & B^\bullet \arrow[r, "g^\bullet"'] & C^\bullet \arrow[r] & 0
\end{tikzcd} : \text{extension}$$

$$\Rightarrow \quad \begin{tikzcd}
 & Q^n(A^\bullet) \arrow[r] \arrow[d, "x_A^n"'] \arrow[rr, bend left=30, "0"] \arrow[rr, phantom, bend left=15, "\Uparrow"] \arrow[dr, phantom, "\Downarrow"] & Q^n(B^\bullet) \arrow[r] \arrow[d, "x_B^n"'] \arrow[dr, phantom, "\Downarrow"] & Q^n(C^\bullet) \arrow[r, "0"] \arrow[d, "x_C^n"'] & 0 \\
0 \arrow[r, "0"'] & Z^{n+1}(A^\bullet) \arrow[r] \arrow[rr, bend right=30, "0"'] \arrow[rr, phantom, bend right=15, "\Downarrow"] & Z^{n+1}(B^\bullet) \arrow[r] & Z^{n+1}(C^\bullet) &
\end{tikzcd}$$

Relatively 2-exact at ●

Proof of the Theorem

By Reduction (1) and Reduction (2) ,

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r, "f^\bullet"'] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow\varphi^\bullet"] & B^\bullet \arrow[r, "g^\bullet"'] & C^\bullet \arrow[r] & 0
\end{tikzcd}$$: extension

$\Rightarrow$

Relatively 2-exact at ●

$$\begin{tikzcd}
 & Q^n(A^\bullet) \arrow[r] \arrow[d, "x^n_A"'] \arrow[rr, bend left=40, "0"] \arrow[rr, phantom, bend left=20, "\Uparrow"] \arrow[dr, phantom, "\Downarrow"] & Q^n(B^\bullet) \arrow[r] \arrow[d, "x^n_B"'] \arrow[dr, phantom, "\Downarrow"] & Q^n(C^\bullet) \arrow[r] \arrow[d, "x^n_C"] & 0 \\
0 \arrow[r] & Z^{n+1}(A^\bullet) \arrow[r] \arrow[rr, bend right=40, "0"'] \arrow[rr, phantom, bend right=20, "\Downarrow"] & Z^{n+1}(B^\bullet) \arrow[r] & Z^{n+1}(C^\bullet) &
\end{tikzcd}$$

$$\mathrm{Ker}(x^n_A) = H^n(A^\bullet)$$

$$\mathrm{Cok}(x^n_A) = H^{n+1}(A^\bullet)$$

So the theorem follows from Snake Lemma !

∎

↓ Not Use

(Co-)Image

(fully faithful, cofaithful)-factorization

= Image factorization

$\mathrm{Im}(f) := \mathrm{Ker}(\mathrm{cok}(f))$

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, dashed, "i(f)"'] & & B \arrow[r, "\mathrm{cok}(f)"] & \mathrm{Cok}(f) \\
& \mathrm{Im}(f) \arrow[ur, "\mathrm{ker}(\mathrm{cok}(f))"'] \arrow[u, phantom, "\Uparrow\iota"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "i(f)"'] & & B \\
& \mathrm{Im}(f) \arrow[ur, "\mathrm{ker}(\mathrm{cok}(f))"'] \arrow[u, phantom, "\Uparrow\iota"] &
\end{tikzcd}$$

faithful

Proposition (4.3, 4.12)

$i(f)$: fully cofaithful

Image Factorization

fully cofaithful

faithful

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "i(f)"'] & & B \\
& \mathrm{Im}(f) \arrow[ur, "\ker(\mathrm{cok}(f))"'] \arrow[u, phantom, "\Uparrow \iota"] &
\end{tikzcd}$$

- $\text{faithful} + \text{fully cofaithful} = \text{equivalence}$
- Lifting property:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "i"'] & & B \\
& I \arrow[ur, "m"'] \arrow[u, phantom, "\Uparrow \eta"] &
\end{tikzcd}$$

i : fully cofaithful

m : faithful

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "i(f)"'] & I \arrow[d, "m"] \\
\mathrm{Im}(f) \arrow[r, "\ker(\mathrm{cok}(f))"'] \arrow[ur, "t"] \arrow[ur, phantom, shift left=4, "\Nwarrow \zeta_i"] \arrow[ur, phantom, shift right=4, "\Searrow \zeta_m"'] & B
\end{tikzcd}$$

$$\Longrightarrow \quad {}^\exists t : \mathrm{Im}(f) \to I,\ {}^\exists \zeta_i,\ {}^\exists \zeta_m$$

(unique up to a unique 2-cell)

such that

$$\eta \cdot (m \circ \zeta_i) = \iota \cdot (\zeta_m \circ i(f))$$