Generalized Indices for $\mathcal{N} = 1$ Theories in Four-Dimensions

T. Nishioka, IY - arXiv:1407.8520

Itamar Yaakov

University of Tokyo - Kavli IPMU

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Data for a QFT

A Quantum Field Theory *can* be constructed using a set of "fields" Φ and a real even functional $S[\Phi]$

- Φ could be sections of - or connections for - some bundles over a smooth "spacetime" M. They determine a (super) Hilbert space $\mathcal{H}$ associated to ∂M.
- The charges, equivalently representations, are restricted
  - Spin-statistics: odd fields sit in spinor representations of the tangent bundle.
  - Anomaly cancelation.
- A family of S's is parametrized by "coupling constants". In modern language: non-dynamical background fields - $S[\Phi, \Phi_B]$.
- S determines a linear map between $\mathcal{H}$'s associated to different components of ∂M in one of two ways
  - By determining an operator (the Hamiltonian) H and the propagator exp itH.
  - By providing a "measure" for the path integral.

Symmetries

A transformation δ on the fields Φ is said to be a symmetry if

$$\delta S\left[\Phi, \Phi_B\right] = 0.$$

Every δ determines a U_δ such that

$$[U_\delta, H] = 0.$$

Some standard QFT symmetries when $M = \mathbb{R}^d$ (a group G_{even} with algebra $\mathfrak{g}_{\text{even}}$)

1. The Lorentz or Euclidean rotation groups $(SO\left(1, d-1\right), SO(d))$. The central element of the double cover (e.g. $\text{Spin}\left(d\right)$) is denoted $\left(-1\right)^F$. The Poincare group also includes translations.
2. Global symmetries - do not act on M. Sometimes called "flavor" if they come from including duplicate fields in Φ.
3. Conformal symmetry - an extension of 1.

Supersymmetry and BPS states

An ($\mathcal{N}$ extended) supersymmetry algebra adds odd generators (must be Lorentz spinors)

$$\{Q_i, Q_j\} \subset \mathfrak{g}_{\text{even}}, \quad (-1)^F Q_i = -Q_i (-1)^F, \qquad i \in \{1 \ldots \mathcal{N}\}$$

States are paired when $Q^2 \neq 0$

$$Q^2|\Psi>= (H + \ldots)|\Psi>= \lambda|\Psi> \quad \Rightarrow \quad |\Psi>= Q\left(\frac{Q}{\lambda}|\Psi>\right).$$

Note that

$$|\Psi>= \begin{pmatrix} \text{B} \\ \text{F} \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & \bullet \\ \bullet & 0 \end{pmatrix}, (-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define a state $|\Psi>$ is said to be **BPS** if

$$Q|\Psi>= 0 \quad \Leftrightarrow \quad (H + \ldots)|\Psi>= 0.$$

The Witten Index

An "index" is a quantity you can calculate in a supersymmetric QFT defined on $\mathbb{R}_t \times M_{\text{space}}$.

- Example: Choose a "space" manifold T^{d-1}. Q is odd and Hermitian

$$Q^2 = H, \qquad Q = \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}.$$

The Witten index is[1]

$$\mathcal{I}_{\mathrm{W}} \equiv \mathrm{tr}_{\mathcal{H}}\,(-1)^F = \dim\ker M - \dim\ker M^*$$

If $[Q, X_i] = 0$, form a "refined" index

$$\mathcal{I}(\{a\}) = \mathrm{tr}_{\mathcal{H}}\left[(-1)^F\, e^{a^i X_i}\right]$$

[1] Witten (1988)

Calculating an index by deformation (localization)

Indices are deformation invariant and get contributions only from BPS ("unpaired") states

$$A = \{Q, V\}\,, \quad [Q, A] = 0 \quad \Rightarrow$$
$$\mathrm{tr}_{\mathcal{H}}\left[(-1)^F\, e^{a^i X_i}\right] = \mathrm{tr}_{\mathcal{H}}\left[(-1)^F\, e^{a^i X_i} e^{-tA}\right].$$

Specifically, can be calculated at weak coupling ($\beta \to \infty$)

$$\mathrm{tr}_{\mathcal{H}}\left[(-1)^F\, e^{a^i X_i}\right] = \mathrm{tr}_{\mathcal{H}}\left[(-1)^F\, e^{a^i X_i} e^{-\beta(H+\dots)}\right]$$

- Note: interesting deformations (a^i) parametrize the Q-cohomology.

Indices and path integrals

Path integral formula for an index of states on M_3

$$\mathrm{tr}\left[(-1)^F e^{a^i X_i} e^{-\beta(H+\ldots)}\right] = \int \mathcal{D}[\Phi] \exp\left(-S_{\{a\},\beta}[\Phi]\right)$$

- The fields Φ live on $S^1 \times M_3$.
- Supersymmetry means $\delta_Q S_{\{a\},\beta}[\Phi] = 0$. Example: $(-1)^F$ picks out the spin structure on the S^1 such that fermions are periodic.
- The a_i are coordinates on some space of supersymmetric deformations of S: metrics, background fluxes etc.

Atiyah-Bott-Berline-Vergne formula

Theorem (Atiyah and Bott - 1984, Berline and Vergne - 1982)

Let Q be an equivariant differential and α a Q-closed equivariant form on a compact manifold M, then the following holds

$$\int_M \alpha = \int_{\mathcal{K}_Q} \frac{i^*_{\mathcal{K}_Q}\alpha}{e(N_{\mathcal{K}_Q})}$$

where $\mathcal{K}_Q$ is the zero set of Q, $i^*_{\mathcal{K}_Q}$ is the pullback and $e(N_{\mathcal{K}_Q})$ is the equivariant Euler class of the normal bundle of $\mathcal{K}_Q$ in M.

- Example: Duistermaat-Heckman Formula (1982)

$$\alpha = \exp\left[i\left(H+\Omega\right)\right]$$

$$\int_M \Omega^n e^{iH} = i^n \sum_{p\in R} e^{\frac{i\pi}{4} sgn(\mathrm{Hess}(H(p)))} \frac{e^{iH(p)}}{\sqrt{det(\mathrm{Hess}(H(p)))}}$$

Localization in supergeometry

Theorem (Schwarz and Zaboronski - 1995)

Let M be a compact supermanifold with volume form dV. Let Q be an odd non-degenerate vector field on M such that

1. $div_{dV} Q = 0$ (the volume form is Q invariant)
2. Q^2 is an even compact vector field on M.

Let $\mathcal{K}_Q$ be the zero set of Q and let S be an even Q-invariant function, $\rho(p)$ is the volume density at p, and "sdet" denotes the superdeterminant (Berezinian)

$$\int_M dVe^{is} = \sum_{p\in\mathcal{K}_Q} \frac{\rho(p)e^{iS(p)}}{\sqrt{sdet(\mathrm{Hess}(S(p)))}}$$

In the DH formula

$$\int_M \Omega^n e^{iH} \to i^{-n} \int_{\Pi TM} \prod_{i=1}^{2n} dx^i d\xi^i e^{i(H(x)+\Omega_{ab}(x)\xi^a\xi^b)}$$

Localization for path integrals

Deformation

- Identify an appropriate conserved fermionic charge: Q.
- Choose V such that $\{Q, V\}$ is a positive semi-definite functional (Q should square to 0 on V).
- Deform the action by a total Q variation $S \to S + t\{Q, V\}$. The resulting path integral is independent of t!
- Add some Q closed operators (Wilson loops, defect operators).

Localization

- Take the limit $t \to \infty$.
- The measure $exp(-S)$ is very small for $\{Q, V\} \neq 0$.
- The semi-classical approximation becomes exact, but there may be many saddle points to sum over ("the zero locus").
- Integrate over the zero locus of $\{Q, V\}$ (+ small fluctuations)

Setting up QFT localization

Set up an integral with the odd symmetry Q

1. Write down a general $S\left[\Phi, \Phi_B\right]$ such that $\delta_Q S = 0$.
2. Pick background fields Φ_{B} such that $\delta_Q \Phi_B = 0$.

Some susy jargon

- **Twisting**: picking Q and $\Phi_B\left(g\right)$ such that $T_{\mathrm{EM}} \equiv dS/dg = \{Q, X\}$. Under mild assumptions, the result is a ("cohomological" or "Witten type") TQFT - changing the metric g results in

$$\frac{d}{dg}\int \mathcal{D}\left[\Phi\right]\exp\left(-S\right) = \int \mathcal{D}\left[\Phi\right] T \exp\left(-S\right) = 0.$$

- **Moduli space** - the set $\{\Phi | \delta_Q \Phi = 0\}$.
- **One loop determinant** - the function on moduli space given by $\mathrm{sdet}^{-1/2}\left[\mathrm{Hess}\left(\{Q, V\}\right)\right]$.

About the model

The (dynamical) field content

1. $U(N)$ vector multiplet (SYM) - A, λ, D
2. Some chiral multiplets - ϕ_i, ψ_i, F_i

The action functional ($S[A, \lambda, D, \phi, \psi, F]$)

- Yang Mills action - $\frac{1}{g_{YM}^2} \int \mathrm{tr}(F \wedge \star F)$
- Kinetic terms and minimal coupling - $\int \bar{\lambda} \not{D} \lambda, \int \bar{\psi} \not{D} \psi, \int D\phi \wedge \star D\phi$
- A "superpotential" which won't play a prominent role.
- Non-derivative terms in D, F.

Parameters and symmetries of the model

Some parameters are not background fields

1. The gauge group G (I took $U(N)$).
2. The representations of the matter fields (chirals).

Spacetime symmetries

1. Poincare - translations + rotations + boosts.
2. $\mathcal{N} = 1$ supersymmetry - a fermionic symmetry with one Weyl generator.

Global symmetries

1. $U(1)_R$ which does not commute with supersymmetry.
2. Some flavor symmetry group F acting on chirals.

General motivation for $\mathcal{N} = 1$ SYM and SQCD

- A lot in common with QCD and electroweak theory
  - Asymptotic freedom/strongly coupled IR theory, higgs mechanism.
  - Confinement of color, chiral symmetry breaking.
  - Instantons and monopoles.
- Many other interesting features
  - Some exact results: non-renormalization theorem, NSVZ β-function etc.
  - Interacting conformal phase.
  - Seiberg duality.
  - No "solution" a la Seiberg-Witten for $\mathcal{N} = 2$ (but some partial results).

Additional motivation

- Exact results for strongly coupled theories are hard to come by.
- Few computations for 4d $\mathcal{N} = 1$ theories using localization.
- Supersymmetric backgrounds have been worked out recently and a large class of manifolds preserving two supercharges was identified.[2]
- Existing examples like the superconformal index[3]$(S^1 \times S^3)$ and $T^2 \times S^2$[4] show that the two supercharge case is particularly nice.

[2] Dumitrescu, Festuccia, and Seiberg (2012)
[3] Assel et al (2014)
[4] Closset and Shamir (2013)

Indices and partition functions

Indices are Euclidean partition functions that can be interpreted as a supertrace over the spectrum of a theory quantized on a $d-1$ dimensional manifold (usually compact)

- The Witten index is a partition function on T^d. It counts supersymmetric ground states with signs.[5]
- The superconformal index counts local BPS operators in a CFT.[6] In 4d

$$\mathcal{I}(p,q,u) = \mathrm{Tr}_{S^3}\left((-1)^F p^{J_3+J_3'-\frac{R}{2}} q^{J_3-J_3'-\frac{R}{2}} u^{Q_f}\right)$$

  is equivalently the partition function on a Hopf surface (topologically $S^1 \times S^3$) and p, q are complex structure moduli.[7]
- The lens space index replaces S^3 by $L(r,1)$.[8]

[5] Witten (1982)

[6] Kinney et al (2005), Romelsberger (2005)

[7] Closset, Dumitrescu, Festuccia, and Komargodski (2013)

[8] Benini, Nishioka and Yamazaki (2012) Razamat and Willett (2013)

Overview

The goal: compute partition functions that represent indices for 4d $\mathcal{N} = 1$ theories

- Applicability
  - The theory must have a conserved $U(1)_R$ current.
  - The manifold should admit an appropriate metric with a holomorphic torus isometry.
  - The result is an unambiguous universal quantity which characterizes the IR CFT.[9]
- Method
  - Choose a topology and complex structure only. The metric doesn't matter![10]
  - Calculate fluctuations using the equivariant index theorem.

[9] Assel, Cassani, and Martelli (2014)

[10] Closset, Dumitrescu, Festuccia, and Komargodski (2013)

Rigid supersymmetry in curved space

New minimal supergravity couples to the $\mathcal{R}$ multiplet[11] of a 4d $\mathcal{N}=1$ theory with a conserved $U(1)_R$

- The SUGRA multiplet: $g_{\mu\nu}, \quad A_\mu^{(R)}, \quad B_{\mu\nu}, \quad \psi_\mu, \quad \tilde{\psi}_\mu$
- The $\mathcal{R}$ multiplet: $T_{\mu\nu}, \quad J_\mu^{(R)}, \ldots$

Rigid supersymmetric backgrounds solve a generalized Killing spinor equation [12]$(V \propto \star dB)$

$$\delta\psi_\mu = \left(\nabla_\mu - i\left(A_\mu - V_\mu\right) - iV^\nu\sigma_{\mu\nu}\right)\epsilon = 0\ ,$$
$$\delta\tilde{\psi}_\mu = \left(\nabla_\mu + i\left(A_\mu - V_\mu\right) + iV^\nu\bar{\sigma}_{\mu\nu}\right)\tilde{\epsilon} = 0\ ,$$

The backgrounds are complex manifolds

$$J_{\mu\nu} \equiv -\frac{2i}{|\epsilon|^2}\epsilon^\dagger\sigma_{\mu\nu}\epsilon, \qquad J^\mu{}_\rho J^\rho{}_\nu = -\delta^\mu{}_\nu$$

[11] Komargodski and Seiberg (2010)

[12] Dumitrescu, Festuccia, and Seiberg (2012)

Backgrounds with both ϵ and $\tilde{\epsilon}$

When we restrict to backgrounds preserving an ϵ and an $\tilde{\epsilon}$ we get, in addition

- two commuting complex structures

$$J_{\mu\nu} = -\frac{2i}{|\epsilon|^2}\epsilon^\dagger \sigma_{\mu\nu}\epsilon, \quad \tilde{J}_{\mu\nu} = -\frac{2i}{|\tilde{\epsilon}|^2}\tilde{\epsilon}^\dagger \bar{\sigma}_{\mu\nu}\tilde{\epsilon}, \qquad \left[J, \tilde{J}\right] = 0$$

- a complex holomorphic Killing vector

$$K^\mu = \epsilon\sigma^\mu\tilde{\epsilon} \ .$$

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \ , \qquad J^\mu_\nu K^\nu = \tilde{J}^\mu_\nu K^\nu = iK^\mu \ ,$$

- the backgrounds are torus fibrations over a Riemann surface. We'll restrict to

$$\left[K, K^\dagger\right] = 0$$

A simple class: $M_4 \simeq S^1 \times M_3$

Take M_4 to be the total space of a *principal elliptic fiber bundle* over a compact orientable Riemann surface Σ_g

$$T^2 \to M_4 \xrightarrow{\pi} \Sigma_g \ .$$

- M is actually diffeomorphic to $S^1 \times M_3$ where M_3 is a principal $U(1)$ bundle over Σ_g. The topology is determined by two numbers: the genus (g) and the degree (d).
- M is Kähler if and only if $d=0$, in which case it is diffeomorphic to $T^2 \times \Sigma_g$.
- M has interesting cohomology classes, specifically[13]

$$\mathrm{Tor}\left(H^2\left(M_4,\mathbb{Z}\right)\right) = \pi^*\left(H^2\left(\Sigma_g,\mathbb{Z}\right)\right) \simeq \mathbb{Z}_d \ .$$

[13] Teleman (1998)

Complex structure and R symmetry

The localization depends on the topological and holomorphic properties of the R symmetry line bundle L.

- The supersymmetry equations imply that L is "locked" to the canonical bundle: $L^{-2}\times\mathcal{K}_{M_4}$ is a trivial line bundle.[14]
- For most values of g,d the manifold M_4 has a canonical bundle with properties[15]

$$\mathcal{K}_{M_4}=\pi^\star\mathcal{K}_{\Sigma_g}\ ,$$

and hence

$$c_1\left(\mathcal{K}_{M_4}\right)=\pi^\star c_1\left(\mathcal{K}_{\Sigma_g}\right)=2g-2 \bmod d\in\mathbb{Z}_d\subset H^2\left(M,\mathbb{Z}\right)\ .$$

- For $g=0$ and $d\geq 3$ there is a more general possibility[16]

$$\mathcal{K}_{M_4}=\begin{cases}\text{topologically trivial} & \text{I}\ ,\\ \pi^\star\mathcal{K}_{\Sigma_g} & \text{II}\ .\end{cases}$$

[14] Dumitrescu, Festuccia, and Seiberg (2012)
[15] Hofer (1993)
[16] Nakagawa (1995)

Supersymmetry on M_4

At this point we *assume* that M admits the right type of metric to support two supercharges

- The complex Killing vector K has non-vanishing components in the fiber directions and acts freely on them.
- The supersymmetry algebra is

$$\begin{aligned} \{\delta_\epsilon, \delta_{\tilde{\epsilon}}\} &= \frac{1}{2}\delta_K \ , \\ \{\delta_\epsilon, \delta_\epsilon\} &= \{\delta_{\tilde{\epsilon}}, \delta_{\tilde{\epsilon}}\} = 0 \ , \\ &= [\delta_K, \delta_\epsilon] = 0 \ , \\ &= [\delta_K, \delta_{\tilde{\epsilon}}] = 0 \ , \end{aligned}$$

$$\delta_K \equiv \mathcal{L}_K - irK^\mu A_\mu^{(R)} - iq_{\text{flavor/gauge}} K^\mu a_\mu \ .$$

- Supersymmetric actions for vector/chiral multiplets are easy to write down. R charge quantization may be required if L is non-trivial.

Localization on M_4

We choose a supercharge Q which is a linear combination of transformation using ϵ and $\tilde{\epsilon}$

$$\{Q, Q\} = \frac{1}{2}\delta_K \ ,$$

$$\delta_K = \mathcal{L}_K - irK^\mu A_\mu^{(R)} - iq_{\text{flavor/gauge}} K^\mu a_\mu \ .$$

The localizing functionals are the curved space D terms

$$\mathcal{L}^{(\text{loc})}_{\text{gauge}} = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \lambda\sigma^\mu D_\mu\tilde{\lambda} + \tilde{\lambda}\bar{\sigma}^\mu D_\mu\lambda + D^2 \ ,$$

$$\mathcal{L}^{(\text{loc})}_{\text{matter}} = D_\mu\tilde{\phi}D^\mu\phi + \frac{1}{2}\tilde{\psi}\bar{\sigma}^\mu D_\mu\psi + \dots$$

The path integral localizes to flat connections

$$F_{\mu\nu} = 0 \ , \quad D = 0, \quad \phi = 0 \ , \quad F = 0,$$

and we'll call a linearized operator acting on fluctuations around this D_{oe}.

The partition function

$$Z_{G,r,M_{g,d}}\left(\tau_{\text{cs}},\xi_{\text{FI}},a_f\right)=\int_{\mathcal{M}^0_G(g,d)} e^{-S_{\text{classical}}(\tau_{\text{cs}},\xi_{\text{FI}})}\times Z^{g,d}_{\text{gauge}}\left(\tau_{\text{cs}}\right)Z^{g,d,r}_{\text{matter}}\left(\tau_{\text{cs}},a_f\right)$$

- Actually an integral and sum over the moduli space of flat connections $\mathcal{M}^0_G(g,d)$. Background flat connections are included: a_f.
- Dependence on the metric is through the space of complex structures τ_{CS}.
- The determinants will be computed using the equivariant index theorem

$$\text{ind}(D_{oe})=\text{tr}_{\text{Ker}D_{oe}}e^{\delta_K}-\text{tr}_{\text{Coker}D_{oe}}e^{\delta_K}\ \rightarrow\ Z_{\text{one-loop}}=\frac{\det_{\text{Coker}D_{oe}}\delta_K}{\det_{\text{Ker}D_{oe}}\delta_K}$$

Moduli space of flat connections I - $\pi_1(M_4)$

The fundamental group of $M_4(g, d)$ is described by generators

$$a_i, b_i, h, x, \qquad i \in 1, \ldots, g \ ,$$

and relations

$$[a_i, h] = [b_i, h] = [a_i, x] = [b_i, x] = [x, h] = 1 \ , \qquad \prod_{i=1}^{g} [a_i, b_i] = h^d \ .$$

- It's a central extension of $\pi_1(\Sigma_g)$ plus the decoupled generator x. For $g \neq 1$ only the h and x holonomies deform δ_K.
- For non-trivial values of h^d this implies flux on Σ_g.[17] The flux changes the bundles used in the index theorem for D_{oe}.

[17] Atiyah and Bott (1983)

Moduli space of flat connections II - $U(N)$

This is the simplest case: in the holonomy representation $\mathcal{M}^0_{g,d}$ is the set of N dim unitary representations of $\pi_1(M_4)$

- Commuting generators can be simultaneously put in the Cartan.
- $\det h^d = 1$ so the spectrum of h is discrete - the quantum number m is the flux. The effect of the degree is $m \to m \bmod d$.
- In an irrep of $\prod_{i=1}^{g} [a_i, b_i] = h^d$ the additional holonomy x must be scalar. A general representation breaks

$$U(N) \to U(N_1) \times U(N_2) \times \cdots \times U(N_p)$$

  and has p fluxes.

Gaugino zero modes

The Killing spinor equations and the eom for the gaugino are similar

$$\bar{\sigma}^{\mu}\left(\nabla_{\mu}-i\left(A_{\mu}^{(R)}+\frac{1}{2}V_{\mu}\right)\right)\epsilon=0\ ,$$

$$\bar{\sigma}^{\mu}\left(\nabla_{\mu}-i\left(A_{\mu}^{(R)}+a_{\mu}^{\text{gauge}}-\frac{3}{2}V_{\mu}\right)\right)\lambda=0\ .$$

- The background has $\chi(M_4)=\sigma(M_4)=0$ and all the gauge fields satisfy ${c_1}^2=c_2=0$ so the index theorem for the Dirac operator gives 0.
- If $V_{\mu}=0$, i.e. Kähler manifolds with $d=0$ and $M_4\simeq T^2\times\Sigma_g$, then gauginos in the same Cartan as the holonomies have an obvious zero mode: ϵ.
- Under some assumptions $d>0$ guarantees no gaugino zero modes.

Equivariant index for $d > 0$

The index is a function (density) on the abelian group of "symmetries" $\mathcal{S}$ or chemical potentials

$$\mathrm{ind}(D_{oe}) = \mathrm{tr}_{\mathrm{Ker} D_{oe}} e^{\delta_K} - \mathrm{tr}_{\mathrm{Coker} D_{oe}} e^{\delta_K} \ ,$$

which can be used to compute the one loop determinants by the rule

$$\mathrm{ind}(D_{oe}) = \sum_{\alpha} c_\alpha e^{t w_\alpha} \quad \longrightarrow \quad Z_{\text{one-loop}} = \prod_{\alpha} w_\alpha^{-c_\alpha} \ .$$

- w_α are weights in the representation in which the field sits. c_α is the multiplicity.
- $\mathcal{S}$ includes the geometric action of $\mathcal{L}_K$, dynamical/background gauge transformations, and R symmetry transformations.
- The structure of M_4 allows us to reduce to Σ_g. For a chiral, D_{oe} is the pullback of a Dirac operator on Σ_g and its index will be calculated using the Atiyah Singer index theorem (transversally elliptic version). The gauge sector is similar.

Equivariant index - $g > 1$

The computation simplifies because there are no isometries on Σ_g.

- The holonomies on the base do not deform the equivariant complex.
- We can use the usual Atiyah Singer index theorem for the Dirac operator

$$\mathrm{ind}(D_{\mathrm{Dirac}}; E) = \int_X \hat{\mathrm{A}}(TX)\,\mathrm{ch}(E) = \int_\Sigma 1 \cdot c_1(E) = \deg(E) \ .$$

The bundle on the base is geometric+gauge+R symmetry. The index and determinant are

$$\mathrm{ind}(D_{\mathrm{oe}}) = \sum_{\rho\in\mathfrak{R},n,l\in\mathbb{Z}} \left(-(r-1)\frac{\chi(\Sigma)}{2} + dl + \rho(m) \right) x^n y^{dl-(r-1)\frac{\chi(\Sigma)}{2}} u \ ,$$

$$Z^{(r,\rho)}_{\mathrm{matter}} = \prod_{n,l\in\mathbb{Z}}^{\rho\in\mathfrak{R}} \left(n + \tau d \left(l - (r-1)\frac{\chi(\Sigma)}{2d} \right) + \rho(a_w) \right)^{-(r-1)\frac{\chi(\Sigma)}{2}+dl+\rho(m)}$$

Equivariant index - $g=0$

This is the lens space index[18] for which we use the Atiyah Bott fixed point formula on $\Sigma_0 = S^2$

$$\mathrm{ind}_T(D) = \sum_{p\in F} \frac{\mathrm{tr}_{E_e(p)}t - \mathrm{tr}_{E_o(p)}t}{\det_{TX_p}(1-t)} \ .$$

The index and determinant are

$$\mathrm{ind}(D_{\mathrm{oe}}) = \sum_{\rho\in\mathfrak{R},n,l\in\mathbb{Z}} t^{-r/2}\frac{t^{(dl+\rho(m))/2} - t^{-(dl+\rho(m))/2}}{1-t^{-1}} x^n y^{dl+\rho(m)} u \ ,$$

$$Z^{(r,\rho)}_{\mathrm{matter}}(m,u) = e^{i\pi\mathcal{E}^{(r)}(\rho(m),u)}\,\Gamma(u(pq)^{r/2}q^{d-\rho(m)};q^d,pq) \begin{pmatrix} p\leftrightarrow q \\ \rho\to d-\rho \end{pmatrix}$$

- $e^{i\pi\mathcal{E}^{(r)}(\rho(m),u)}$ is an interesting zero point energy.

[18] Benini, Nishioka and Yamazaki (2012) Razamat and Willett (2013)

Equivariant index - $g = 1$

An interesting case

- $\chi(\Sigma) = 0$ implies that there is no R charge quantization for any d.
- There are isometries on the base torus, but no fixed points.
- General arguments imply that the base complex structure does not affect the partition function, but it seems like the holonomies do.

Classical contributions

Fayet-Iliopoulos terms for $U(1)$ factors exist in curved space

$$\xi \int (D - iV^\mu a_\mu) \ ,$$

- After localizing to flat connections only $K^\mu a_\mu$ contributes due to
$$V_\mu = -\frac{1}{2}\nabla^\nu J_{\nu\mu} + \kappa K_\mu \ , \qquad K^\mu \partial_\mu \kappa = 0.$$
- ξ must be quantized to keep this invariant under large gauge transformations. This may not make sense for arbitrary g, d and an arbitrary complex structure.
- The result is trivial if $V = \star dB$ for a well defined B, hence we must have a non trivial three form flux in $H^{1,2}(M_4)$.
- The expression is equivalent to a sort of smeared supersymmetric abelian Wilson loop. Is there a non-abelian analogue?

Aspects of the partition function - I

$$Z_{G,r,M_{g,d}}\left(\tau_{\text{cs}}, \xi_{\text{FI}}, a_f\right) = \frac{1}{|\mathcal{W}|}\int_{\mathcal{M}^0_G(g,d)} e^{-S_{\text{classical}}(\tau_{\text{cs}},\xi_{\text{FI}})} \times Z^{g,d}_{\text{gauge}}\left(\tau_{\text{cs}}\right) Z^{g,d,r}_{\text{matter}}\left(\tau_{\text{cs}}, a_f\right)$$

- The restriction on R charges is
$$r\,(g-1 \text{ mod } d) \in \mathbb{Z}\ .$$
This does not apply to the (usual) lens space index.
- τ_{CS} consists of the complex structure parameter for the torus fiber (τ), an additional complex number for the fibration (σ) when $g = 0$, and possibly the complex structure on the base for $g = 1$.
- $Z^{g,d,r}_{\text{matter}}\left(\tau_{\text{cs}}, a_f\right)$ and $Z^{g,d}_{\text{gauge}}\left(\tau_{\text{cs}}\right)$ are elliptic gamma (type) functions.
- An overall factor is included to account for the residual Weyl

Aspects of the partition function - II

The parameters entering the partition function are split between[19]

1. Parameters and deformations of the theory
   1. The gauge/flavor groups and the matter representations. This is where the superpotential comes in.
   2. A set of admissible Fayet-Iliopoulos terms ξ, one for each independent $U(1)$ factor in G.
   3. An element of the moduli space of flat connections on M of the flavor symmetry group F.
2. Parameters of M
   1. The genus, g, of the underlying Riemann surface and the first Chern class, d, of the circle bundle whose total space is M_3.
   2. A point in the complex structure moduli space on M admitting a holomorphic Killing vector K. This may include a discrete choice in the case $g = 0$.
   3. A choice of $W \in H^{1,2}(M)$.

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[19] In agreement with Closset et al. (2014)

Issues/caveats

The interpretation of the index is complicated by

1. Accidental symmetries may prevent us from correctly identifying the IR R charge.
2. A metric supporting the necessary holomorphic Killing vector may not exist for all g, d, τ_{CS}.

The computation itself has a few shortcomings

1. The integral over the moduli space of flat connections is complicated and involves an unresolved quantity

$$\int_{\mathcal{M}_G^0(g,d)} = \sum_{\text{partitions } N} \prod_{j=1}^{p} \left(\sum_{m_j \in 0,\ldots,dN_j-1} V\left[\mathcal{M}^g_{N_j,m_j}\right] \int_0^1 \frac{dx_j}{2\pi} \right) .$$

2. Exclusion of fermionic zero modes required some assumptions.

Applications

A few standard applications for exact calculations

1. Checking dualities: this involved a complicated calculation in the case of Seiberg duality and the superconformal index $(S^1 \times S^3)$.[20] The more intricate topology of M_4 can help check some global issues like discrete theta angles.[21] Mapping of operators would be more ambitious.
2. Holography and large N: this potentially sidesteps some of the intricacies of the moduli space of flat connections.

Some more recent applications

1. Extracting trace anomalies from supersymmetric partition functions at "high temperature".[22]
2. Constructing integrable lattice models.[23]

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[20] Spiridonov and Vartanov (2009)
[21] Razamat and Willett (2013)
[22] Di Pietro and Komargodski (2014)
[23] Yamazaki (2013)

Future directions

Extending the results to include

- Manifolds where K acts with finite isotropy groups. The same basic techniques can be used.
- Looking for supersymmetric operators/defects.

More challenging options

- Manifolds with gaugino zero modes.
- Backgrounds preserving one supercharge: localization to the instanton moduli space.

Thank you!