Hyperscaling-violating Lifshitz hydrodynamics from black holes

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Introduction

The holography gives the correspondences between black holes and fluids.

[Son-Starinets, 02] [Policastro-Son-Starinets, 02] [Bhattacharyya-Hubeny-Minwalla-Rangamani, 07]

Lifshitz spacetimes give holographic description of the Lifshitz scaling invariant theories.

[Son, 08] [Balasubramanian-McGreevy, 08]

[Kachru-Liu-Mulligan, 08] [Taylor, 08]

$$ds^2 = -r^{2z}dt^2 + r^2\left(dx^i\right)^2 + \frac{dr^2}{r^2}$$

Boundary theories in the Lifshitz spacetimes have the Newton-Cartan geometry as a background.

[Christensen-Hartong-Obers-Rollier, 14]

[Hartong-Kiritsis-Obers, 14, 14, 15]

Here we would like to see that

Fluid/gravity correspondence for the Lifshitz spacetime give a holographic description of fluids in Newton-Cartan background.

Fluid mechanics

Fluid variables

ρ: particle number density

$\mathcal{E}$: energy density

P: Pressure

v^i: velocity field

Fluid equations

Continuity equation (for ρ, v^i)

Energy conservation equation (for $\rho,\ \mathcal{E},\ v^i$)

Navier-Stokes equation (for $\rho,\ P,\ v^i$)

Relativistic fluid

Fluid equation is unified to conservation law of stress-energy tensor $T^{\mu\nu}$

$$T^{tt} \sim \rho \qquad T^{ti} = T^{it} = (\rho + P)v^i \qquad T^{ij} = P\delta_{ij} + \tau^{ij}$$

Conservation law: $D_\mu T^{\mu\nu} = 0$

Viscous stress tensor τ^{ij} is related to other fluid variables (constitutive relation)

AdS/CFT correspondence

Correspondence between classical solutions in anti-de Sitter spacetime (AdS) and conformal field theories

Classical solutions of matter fields in anti-de Sitter spacetime or asymptotically AdS (classical) solution of gravity

$\Updownarrow$ correspondence

(Quantum) states in conformal field theory

GKPW relation

In gravity side, solutions are parametrized by boundary conditions.

The boundary condition at the boundary is related to CFT;
Dirichlet condition ⇒ source for operators
Neumann condition ⇒ vacuum expectation value of operators

Source: metric $\Rightarrow$ operator: stress-energy tensor

Stress-energy tensor in CFT corresponds to the Brown-York tensor, which is expressed in terms of the extrinsic curvature at the boundary.

Einstein equation in gravity side

Equations which determine propagation from the bounary

Equations which give constraints on the boundary conditions

Constraint equations gives conservation law for the stress-energy tensor in conformal field theory.

In order to see the correspondence between fluids in CFT and black holes in AdS, we have to calculate the stress-energy tensor and its conservation law.

Stress-energy tensor can be calculated by using the extrinsic curvature at the boundary in the gravity side.

Conservation law for stress-energy tensor is obtained from parts of the Einstein equations which give constraints on the boundary conditions.

Asymptotically AdS solution for fluids

In CFT side, matters behaves as a fluid at finite temperature.

In AdS side, black holes appear at finite temperature.

⇒ Finite temperature state in CFT corresponds to Black holes in AdS.

For fluids

Energy density ($\sim$ temperature) depends on position x^μ

⇒ Horizon radius ($\sim$ temperature) of BH depends on x^μ

Fluids have flow ⇒ introduce (x^μ-dependent) boost to BH geometry

The modified black hole solution describes fluids in dual field theory

Solution can be calculated by using expansion for long wavelength.

Scaling symmetry of AdS

The metric of (pure) AdS can be written as

$$ds^2 = -r^2 dt^2 + \sum_i r^2 \left(dx^i\right)^2 + \frac{dr^2}{r^2}$$

The metric has scaling symmetry

$$t \to c\, t \qquad x^i \to c\, x^i \qquad r \to c^{-1} r$$

This symmetry corresponds to the scaling symmetry of CFT

$$t \to c\, t \qquad x^i \to c\, x^i$$

Generalization of AdS/CFT for Lifshitz

The Lifshitz scaling symmetry is an anisotropic scaling symmetry.

$$t \to c^z t \qquad\qquad x^i \to c\, x^i$$

Lifshitz spacetime is given by

$$ds^2 = -r^{2z} dt^2 + \sum_i r^2 \left(dx^i\right)^2 + \frac{dr^2}{r^2}$$

The metric has Lifshitz scaling symmetry

$$t \to c^z t \qquad\qquad x^i \to c\, x^i \qquad\qquad r \to c^{-1} r$$

Lifshitz theory (field theory side) is non-relativistic.

Boundary of Lifshitz spacetime

For the Lifshitz spacetime

$$ds^2 = -r^{2z}dt^2 + \sum_i r^2 (dx^i)^2 + \frac{dr^2}{r^2}$$

the induced metric on the boundary is not well defined (singular).

Induced metric on the boundary ($r \to \infty$) of Lifshitz spacetime

$$ds^2 = -r^{2z}dt^2 + r^2 \sum_i (dx^i)^2 \to -r^{2z}dt^2 + \mathcal{O}(r^2)$$

Inverse induced metric on the boundary of Lifshitz spacetime

$$g^{\mu\nu}\partial_\mu\partial_\nu = -r^{-2z}\partial_t^2 + r^{-2}\sum_i \partial_i^2 \to r^{-2}\sum_i \partial_i^2$$

The leading terms are invariant under Galilean boost $x^i \to x^i - v^i t$.

Corresponding field theory for Lifshitz spacetime is non-relativistic.

Fluid gravity correspondence in AdS

Modified black hole solution in AdS spacetime $\Longleftrightarrow$ Relativistic conformal fluid

$\Uparrow$

Source term for stress-energy tensor is metric

Fluid gravity correspondence in Lifshitz spacetime

Modified black hole solution in Lifshitz spacetime $\Longleftrightarrow$ Non-relativistic fluid?

$\Uparrow$

What is the source term for fluid variables?

Newton-Cartan gravity is non-relativistic theory of gravity

Field theory side of Lifshitz spacetime has Newton-Cartan theory as background

We calculate the stress-energy tensor via holography and show that
it is non-relativistic fluid in Newton-Cartan background

Newton-Cartan geometry

Nonrelativistic theory of gravity

Described by Newton-Cartan data $\left(\tau_\mu\,, h^{\mu\nu}, \bar{v}^\mu, \tilde{A}_\mu\right)$

$$\left.\begin{array}{l}\text{1-form } \tau \text{ defines time direction.}\\[2ex] h^{\mu\nu} \text{ is inverse metric on time slice.}\end{array}\right\}\quad \text{Galilei data}$$

Galilei data $(\tau_\mu, h^{\mu\nu})$ satisifies

$$\tau_\mu h^{\mu\nu} = 0$$

Galilei data is constant. For covariant derivative D_μ

$$D_\mu \tau_\nu = 0 \qquad\qquad D_\rho h^{\mu\nu} = 0$$

Galilei connection is not unique.

In order to define connection, we have to introduce $\bar{v}^\mu$ and $\tilde{A}_\mu$.

Conservation law for Newton-Cartan theory

In relativistic theory, geometry is described by metric $g_{\mu\nu}$

Associated operator = stress-energy tensor $T^{\mu\nu}$

Conservation law is simply expressed only in terms of stress-energy tensor

The Newton-Cartan theory has Newton-Cartan data $\left(\tau_{\mu}\,, h^{\mu\nu}, \bar{v}^{\mu}, \tilde{A}_{\mu}\right)$

Associated operators are

Energy density (flow): $\mathcal{E}^{\mu}$

Viscous stress tensor: $\mathcal{T}_{\mu\nu}$

Momentum density: $\mathcal{P}_{\mu}$

Mass current: J^{μ}

The conservation law is expressed in terms of these 4 quantities.

Gravity theory for Lifshitz

Einstein gravity + massive vector field

⇒ No analytic black hole solution is known

Einstein gravity + massless vector (gauge) field + dilaton (scalar) ⇐ We consider this model

We consider analytic asymptotically Lifshitz black hole solution.

introduce x^μ-dependence to the horizon radius

introduce x^μ-dependent Galilean boost

introduce x^μ-dependence to other parameters in the solution for massless vector and dilaton

then, calculate correction terms in long wavelength expansion

Stress-energy tensor for Lifshtiz fluid/gravity

We obtain the gravitational solution for long wavelength expansion.

We calculate the stress-energy tensor in field theory from the solution.

We also have charge associated to the massless vector (gauge) field.

→ This corresponds to mass (particle number) density.

The gravity solution for fluids contains some parameters:
boost parameter $v^i(x)$, horizon radius $r_0(x)$, charge density $a(x)$.

→ They are related to fluid variables:
velocity fields, energy density, mass density

The stress-energy tensor can be expressed in terms of the fluid variables.

→ takes a similar form to that for fluids

Asymmetric (relativistic stress-energy tensor is symmetric)

Can be decomposed into energy density (flow), momentum density and viscous stress tensor

Conservation law for Lifshtiz fluid/gravity

Einstein equation in gravity side

Equations which determine propagation from the bounary

Equations which give constraints on the boundary conditions

The gravity solution for fluids contains some parameters:
velocity field $v^i(x)$, horizon radius $r_0(x)$, charge density $a(x)$.

They must satisfy constraints

Conservation law for the stress-energy tensor

The equations of motion for massless vector has same structure.

Conservation law for the charge (= mass density)

The conservation law takes the same form to that in Newton-Cartan theory.

Fluid equation and gauge field

An equation from gravity theory is slightly different from the Navier-Stokes eq.

An equation from gravity theory (one of conservation law)

$$\partial_i P - \partial_j(\eta\sigma_{ij}) = \mathcal{F}_{i\mu} J^\mu$$

The Navier-Stokes equation (equation for fluids)

$$0 = \partial_i P + \rho\partial_t v^i + \rho v^j \partial_j v^i - \partial_j(\eta\sigma_{ij})$$

In Newton-cartan theory, if the gauge field $\tilde{A}$ for $v^i = 0$,
it transforms under a symmetry "Milne boost" as

$$\tilde{A} \to \hat{A} = \tilde{A} + v^i dx^i - \frac{1}{2} v^2 dt$$

If we identify gauge field $\mathcal{A}$ ($\mathcal{F} = d\mathcal{A}$) with that in Newton-Cartan theory $\hat{A}$
the above equation becomes same to the Navier-Stokes equation.

The fluid is non-relativistic fluids in Newton-Cartan background

Summary

- We have studied fluid/gravity correspondence for Lifshitz spacetime
- We calculate a modified black hole solution which describes fluids in dual field theory side.
- Conservation law takes the same form to that in the Newton-Cartan theory
- We calculate the stress-energy tensor and conservation law from the modified black hole solution. They takes same form to those for fluids but the Navier-Stokes equation is slightly different.
- If we identify the gauge field as that in the Newton-Cartan theory, the Navier-Stokes equation (from black hole solution) agrees with that for the standard fluids.
- We also calculated entropy density (current) and it satisfies the thermodynamic relation and 2nd law.

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Einstein-Maxwell-Dilaton model

The action

$$S = \frac{1}{16\pi G}\int d^{d+1}x\,\sqrt{-g}\left(R - 2\Lambda - \frac{1}{4}\mathrm{e}^{\lambda\phi}F^2 - \frac{1}{2}(\partial\phi)^2\right)$$

Lifshitz spacetime solution

$$ds^2 = -r^{2z}dt^2 + \frac{dr^2}{r^2} + \sum_i r^2\left(dx^i\right)^2$$

$$A = ar^{z+d-1}dt \qquad e^{\lambda\phi} = \mu r^{2(1-d)}$$

$$\lambda^2 = 2\frac{d-1}{z-1}$$

$$\Lambda = -\frac{(z+d-1)(z+d-2)}{2}$$

$$\mu a^2 = \frac{2(z-1)}{z+d-1}$$

The metric has Lifshitz scaling symmetry

$$t \to c^z t \qquad x^i \to c\, x^i \qquad r \to c^{-1} r$$

The gauge field A and dilaton ϕ breaks the scaling symmetry

Hydrodynamic ansatz

The black hole solution in the Eddington-Finkelstein coordinate

$$ds^2 = -r^{2z} f(r) dt^2 + 2r\, dr\, dt + r^2 \left(dx^i\right)^2$$

$$A = a r^{z+d-1} f(r) dt - a r^2 dr \qquad e^{\lambda\phi} = \mu r^{2(1-d)}$$

$$f(r) = 1 - \frac{r_0^{z+d-1}}{r^{z+d-1}}$$

Introduce Galilean boost ($x^i \to x^i - v^i t$) and x^μ-dependence

$$ds^2 = -r^{2z} f(r) dt^2 + 2r\, dr\, dt + r^2 \left(dx^i - v^i(x) dt\right)^2$$

$$A = a(x) r^{z+d-1} f(r) dt - a(x) r^2 dr + \mathcal{A}_i(x)\left(dx^i - v^i(x) dt\right)$$

$$e^{\lambda\phi} = \mu(x) r^{2(1-d)} \qquad f(r) = 1 - \frac{r_0^{z+d-1}(x)}{r^{z+d-1}}$$

This is not a solition of EOM $\Rightarrow$ We have to introduce correction terms

Derivative expansion

We introduce the correction terms

$$ds^2 = -r^{2z} f(r) dt^2 + 2r\, dr\, dt + r^2 \left(dx^i - v^i(x) dt\right)^2 + h_{\mu\nu} dx^\mu dx^\nu$$

$$A = a(x) r^{z+d-1} f(r) dt - a(x) r^2 dr + \mathcal{A}_i(x) \left(dx^i - v^i(x) dt\right) + a_\mu dx^\mu$$

$$e^{\lambda\phi} = \mu(x) r^{2(1-d)} e^{\lambda\varphi}$$

Hydrodynamic regime: variation of x^μ-dependence is very slow.

$\Rightarrow$ derivative expansion (with respect to ∂_μ)

$$r_0(x) = r_0(0) + x^\mu \partial_\mu r_0(0) + \cdots$$

$$v^i(x) = v(0) + x^\mu \partial_\mu v^i(0) + \cdots$$

$$a(x) = a(0) + x^\mu \partial_\mu a(0) + \cdots$$

Derivative expansion

We introduce the correction terms

$$ds^2 = -r^{2z}f(r)dt^2 + 2r\,dr\,dt + r^2\big(dx^i - v^i(x)dt\big)^2 + h_{\mu\nu}dx^\mu dx^\nu$$

$$A = a(x)r^{z+d-1}f(r)dt - a(x)r^2dr + \mathcal{A}_i(x)\big(dx^i - v^i(x)dt\big) + a_\mu dx^\mu$$

$$e^{\lambda\phi} = \mu(x)r^{2(1-d)}e^{\lambda\varphi}$$

First order EOM

Correction terms $\psi = (h_{\mu\nu}, a_\mu, \varphi)$ are treated as first order in ∂_μ.

$$\psi''(r) + F_1(r)\psi'(r) + F_2\psi(r) = F_3(\partial_\mu r_0, \partial_\mu v^i, \cdots)$$

Linear terms of
correction terms $h_{\mu\nu}, a_\mu, \varphi$

Linear terms of
source terms $\partial_\mu r_0, \partial_\mu v^i, \partial_\mu a, \cdots$

We solve these linear inhomogeneous differential equations

First order solution (for $d = 4, z = 2$)

$$ds^2 = -r^4 f(r)dt^2 + 2r\,dr\,dt + r^2\left(dx^i - v^i dt\right)^2$$
$$+\frac{2}{3}r^2\partial_i v^i dt^2 - r^2 F(r)\sigma_{ij}\left(dx^i - v^i dt\right)\left(dx^j - v^j dt\right)$$

$$A = a\left(r^5 f(r) - \frac{1}{3}r^3\partial_i v^i\right)dt - ar^2 dr + \mathcal{A}_i\left(dx^i - v^i dt\right)$$

$$e^{\lambda\phi} = \mu r^{2(1-d)} \qquad f(r) = 1 - \frac{r_0^5}{r^5}$$

$$F(r) = \int dr \frac{r^3 - r_0^3}{r(r^5 - r_0^5)} \qquad \sigma_{ij} = \partial_i v^j + \partial_j v^i - \frac{2}{3}\delta_{ij}\partial_k v^k$$

It must satisfy the following constraints

$$0 = \partial_t a + v^i \partial_i a - a\partial_i v^i$$

$$0 = \partial_t r_0 + v^i \partial_i r_0 + \frac{1}{3}r_0 \partial_i v^i \qquad \mu(x)a^2(x) = \frac{2(z-1)}{z+d-1}$$

$$0 = \partial_t \mathcal{A}_i + v^j \partial_j \mathcal{A}_i + \mathcal{A}_j \partial_i v^j + a\partial_i r_0^5$$

Boundary stress-energy tensor for Lifshitz

Induced metric in terms of the vielbein

$$\gamma_{\mu\nu} = -r^{2z} f \tau_\mu \tau_\nu + r^2 e^a_\mu e^a_\nu \qquad \gamma^{\mu\nu} = -r^{-2z} f^{-1} \hat{v}^\mu \hat{v}^\nu + r^{-2} e^\mu_a e^\nu_a$$

Gauge field in this frame: $\hat{A}_0 = \hat{v}^\mu A_\mu \qquad \hat{A}_a = e^\mu_a A_\mu$

Variation of the action in these variables

$$\delta S = \int d^d x \left(-\hat{S}^0_\mu \delta \hat{v}^\mu + \hat{S}^a_\mu \delta e^\mu_a + \hat{J}^0 \delta \hat{A}_0 + \hat{J}^a \delta \hat{A}_a + \mathcal{O}_\phi \delta \phi \right)$$

The stress-energy tensor and current are defined by

$$\hat{T}^\mu{}_\nu = \hat{S}^0_\nu \hat{v}^\mu - \hat{S}^a_\nu e^\mu_a \qquad J^\mu = \hat{J}^0 \hat{v}^\mu + \hat{J}^a e^\mu_a$$

The stress-energy tensor is related to Brown-York tensor $T^\mu{}_\nu$ as

$$\hat{T}^\mu{}_\nu = T^\mu{}_\nu + J^\mu A_\nu \qquad \text{where} \qquad T^{\mu\nu} = \frac{1}{8\pi G}\left(\gamma^{\mu\nu} K - K^{\mu\nu}\right)$$

$\hat{T}^\mu{}_\nu$ is asymmetric tensor

$K_{\mu\nu}$: extrinsic curvature

Stress-energy tensor for Lifshitz fluid

We introduce the counter term

$$S_{\mathrm{ct}} = \int d^4x\sqrt{-\gamma}\left[-(5+z) + \frac{z+d-1}{2}e^{\lambda\phi}\gamma^{\mu\nu}A_\mu A_\nu\right]$$

The renormalized stress-energy tensor

$$\tilde{T}^0{}_0 = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[-\frac{3}{2}r_0^{z+3} - \frac{z-1}{a}v^i\mathcal{A}_i\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^i{}_0 = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[-\frac{z+3}{2}r_0^{z+3}v^i + \frac{z(z+3)}{4(z-1)}r_0^{2z}\partial_i r_0 - \frac{z-1}{a}v^iv^j\mathcal{A}_j \right.$$
$$\left. + \frac{1}{2}r_0^3\sigma_{ij}v^j\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^0{}_i = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\frac{z-1}{a}\mathcal{A}_i + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^i{}_j = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[\frac{z}{2}r_0^{z+3} - \frac{1}{2}r_0^3\sigma_{ij} + \frac{z-1}{a}v^i\mathcal{A}_j\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

Stress-energy tensor for Lifshitz fluid

Since the volume form behaves as

$$\sqrt{-\gamma} \sim r^{z+3}$$

The leading terms of the following stress-energy tensor gives regular contributions

$$\tilde{T}^0{}_0 = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[-\frac{3}{2}r_0^{z+3} - \frac{z-1}{a}v^i\mathcal{A}_i\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^i{}_0 = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[-\frac{z+3}{2}r_0^{z+3}v^i + \frac{z(z+3)}{4(z-1)}r_0^{2z}\partial_i r_0 - \frac{z-1}{a}v^i v^j\mathcal{A}_j + \frac{1}{2}r_0^3\sigma_{ij}v^j\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^0{}_i = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\frac{z-1}{a}\mathcal{A}_i + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^i{}_j = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[\frac{z}{2}r_0^{z+3} - \frac{1}{2}r_0^3\sigma_{ij} + \frac{z-1}{a}v^i\mathcal{A}_j\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

Constraints and conservation law

The following constraints in EOM gives the conservation law on the boundary

$$n^{\mu}\gamma^{\nu\rho}R_{\mu\nu} = 8\pi G n^{\mu}\gamma^{\nu\rho}T_{\mu\nu} \qquad n_{\nu}\nabla_{\mu}\left(e^{\lambda\phi}F^{\mu\nu}\right) = 0$$

This should agree with the conservation law for $\tilde{T}^{\mu}{}_{\nu}$.

The constraints do not agree with $D_{\mu}\tilde{T}^{\mu}{}_{\nu} = 0$,
but agree with the conservation law in the Newton-Cartan theory.

$$D_{\mu}\mathcal{E}^{\mu} = -\frac{1}{2}(D^{\mu}\hat{v}^{\nu} + D^{\nu}\hat{v}^{\mu})\mathcal{T}_{\mu\nu}$$

$$D_{\mu}\mathcal{T}^{\mu}{}_{i} = \hat{v}^{\mu}D_{i}\mathcal{P}_{\mu} - D_{\mu}(\hat{v}^{\mu}\mathcal{P}_{i})$$

$$D_{\mu}\mathcal{J}^{\mu} = 0 \qquad \text{where} \qquad \mathcal{J}^{\mu} = \frac{1}{a}\hat{v}^{\mu}$$

Energy flow $\mathcal{E}^{\mu}$, momentum density $\mathcal{P}_{\mu}$ and stress tensor $\mathcal{T}^{\mu}{}_{\nu}$ are defined as

$$\mathcal{E}^{\mu} = -\tilde{T}^{\mu}{}_{\nu}\hat{v}^{\nu} \qquad \mathcal{P}_{\mu} = \tilde{T}^{\rho}{}_{\nu}\tau_{\rho}P^{\nu}_{\mu} \qquad \mathcal{T}^{\mu}{}_{\nu} = \tilde{T}^{\rho}{}_{\sigma}P^{\mu}_{\rho}P^{\sigma}_{\nu}$$

$$\hat{v}^{\mu} = \left(1, v^{i}\right)$$

$P^{\mu}_{\nu} = e^{\mu}_{a}e^{a}_{\nu}$: Projection to spatial direction

Newton-Cartan geometry

Non-relativistic theory of gravity

Described by Newton-Cartan data $\left(\tau_\mu\,, h^{\mu\nu}, \bar{v}^\mu, \tilde{A}_\mu\right)$

1-form τ defines time direction.

$h^{\mu\nu}$ is inverse metric on time slice.

Galilei data

Galilei data $(\tau_\mu, h^{\mu\nu})$ satisifies

$$\tau_\mu h^{\mu\nu} = 0$$

Galilei data is constant. For covariant derivative D_μ

$$D_\mu \tau_\nu = 0 \qquad\qquad D_\rho h^{\mu\nu} = 0$$

Galilei connection is not unique.

In order to define connection, we have to introduce $\bar{v}^\mu$ and $\tilde{A}_\mu$.

Timelike unit vector $\bar{v}^\mu$

$$\tau_\mu \bar{v}^\mu = 1$$

Induced metric on time slice $h_{\mu\nu}$ is defined by Galilei data and $\hat{v}^\mu$

$$\bar{v}^\mu h_{\mu\nu} = 0 \qquad\qquad h_{\mu\rho} h^{\rho\nu} = \delta_\mu^\nu - \tau_\mu \bar{v}^\nu$$

To write down connection $\Gamma^\rho_{\mu\nu}$, we introduce $\bar{v}^\mu$ and $\tilde{F} = d\tilde{A}$

$$\Gamma^\rho_{\mu\nu} = \bar{v}^\rho \partial_\mu \tau_{\ \nu} + \frac{1}{2} h^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right) + h^{\rho\sigma} n_{(\mu} \tilde{F}_{\nu)\sigma}$$

In general, Newton-Cartan theory has torsion

$$T^\rho_{\mu\nu} = \bar{v}^\mu\left(\partial_\mu \tau_\nu - \partial_\nu \tau_\mu\right)$$

Conservation law in Newton-Cartan theory

In the Newton-Cartan theory variation of the action is

$$\delta S = \int d^d x \sqrt{-\gamma} [J^\mu \delta \tilde{A}_\mu - \mathcal{P}_\mu \delta \bar{v}^\mu - \mathcal{E}^\mu \delta \tau_\mu - \frac{1}{2} \mathcal{T}_{\mu\nu} \delta h^{\mu\nu}]$$

Under the infinitesimal coordinate transformation ξ^μ (for torsionless case)

$$\delta_\xi \tau_\mu = D_\mu (\xi^\nu \tau_\nu) \qquad \delta_\xi h^{\mu\nu} = h^{\mu\rho} D_\rho \xi^\nu + h^{\nu\rho} D_\rho \xi^\mu$$

$$\delta_\xi \hat{v}^\mu = \xi^\rho D_\rho \bar{v}^\nu - \hat{v}^\rho D_\rho \xi^\nu \qquad \delta_\xi \tilde{A}_\mu = -\tilde{F}_{\mu\nu} \xi^\nu$$

Since the covariant derivative of $\hat{v}^\mu$ does not vanish,
the conservation law is different from ordinary cases.

$$D_\mu \mathcal{E}^\mu = -\tilde{F}_{\mu\nu} \bar{v}^\mu J^\nu - \frac{1}{2} (D^\mu \bar{v}^\nu + D^\nu \bar{v}^\mu) \mathcal{T}_{\mu\nu}$$

$$D_\mu \mathcal{T}^\mu{}_\nu = \bar{v}^\mu D_\nu \mathcal{P}_\mu - D_\mu (\bar{v}^\mu \mathcal{P}_\nu) - \tilde{F}_{\mu\nu} J^\mu$$

Fluid variables and fluid equations

Energy flow, momentum density and stress tensor takes the following form

$$\mathcal{E}^0 = \mathcal{E} \qquad \mathcal{E}^i = \mathcal{E} v^i - \kappa \partial_i T$$

$$\mathcal{P}_i = q\mathcal{A}_i \qquad \mathcal{T}_{ij} = P\delta_{ij} - \eta\sigma_{ij}$$

The energy density, pressure and charge density are

$$\mathcal{E} = \frac{3}{16\pi G} r_0^{z+3} \quad P = \frac{z}{16\pi G} r_0^{z+3} \qquad q = \frac{z-1}{a} \qquad T = \frac{z+3}{4\pi} r_0^z$$

The shear viscosity and thermal conductivity are

$$\eta = \frac{1}{16\pi G} r_0^3 \qquad \kappa = \frac{1}{8G(z-1)} r_0^{z+1}$$

Bulk viscosity vanishes.

Energy density and pressure satisfy the Lifshitz scaling condition

$$z\mathcal{E} = (d-1)P$$

Fluid variables and fluid equations

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$$\mathcal{E}^0 = \mathcal{E} \qquad \mathcal{E}^i = \mathcal{E} v^i - \kappa \partial_i T$$

$$\mathcal{P}_i = q\mathcal{A}_i \qquad \mathcal{T}_{ij} = P\delta_{ij} - \eta\sigma_{ij}$$

The conservation law gives the fluid equations

$$D_\mu \mathcal{E}^\mu = -\frac{1}{2}(D^\mu \hat{v}^\nu + D^\nu \hat{v}^\mu)\mathcal{T}_{\mu\nu} \qquad D_\mu \mathcal{J}^\mu = 0$$

$$D_\mu \mathcal{T}^\mu{}_i = \hat{v}^\mu D_i \mathcal{P}_\mu - D_\mu(\hat{v}^\mu \mathcal{P}_i)$$

The fluid equation is expressed as

$$0 = \partial_t \mathcal{E} + v^i \partial_i \mathcal{E} + (\mathcal{E} + P)\partial_i v^i - \frac{1}{2}\eta\sigma_{ij}\sigma_{ij} - \partial_i(\kappa \partial_i T)$$

$$0 = \partial_i P + q\partial_t \mathcal{A}_i + q v^j \partial_j \mathcal{A}_i + q\mathcal{A}_j \partial_i v^j - \partial_j\left(\eta\sigma_{ij}\right)$$

$$0 = \partial_t q + \partial_i\left(q v^i\right)$$

Non-relativistic fluid equations

The fluid equations from the Lifshitz black hole are

$$0 = \partial_t \mathcal{E} + v^i \partial_i \mathcal{E} + (\mathcal{E} + P)\partial_i v^i - \frac{1}{2}\eta\sigma_{ij}\sigma_{ij} - \partial_i(\kappa\partial_i T)$$

$$0 = \partial_i P + q\partial_t \mathcal{A}_i + qv^j\partial_j\mathcal{A}_i + q\mathcal{A}_j\partial_i v^j - \partial_j(\eta\sigma_{ij})$$

$$0 = \partial_t q + \partial_i(qv^i)$$

$$\partial_i P - \partial_j(\eta\sigma_{ij}) = \mathcal{F}_{i\mu}J^\mu$$

Ordinary fluid equations are

$$0 = \partial_t \mathcal{E} + v^i \partial_i \mathcal{E} + (\mathcal{E} + P)\partial_i v^i - \frac{1}{2}\eta\sigma_{ij}\sigma_{ij} - \partial_i(\kappa\partial_i T)$$

$$0 = \partial_i P + \rho\partial_t v^i + \rho v^j\partial_j v^i - \partial_j(\eta\sigma_{ij})$$

$$0 = \partial_t \rho + \partial_i(\rho v^i)$$

Energy conservation (1st line) and continuity equation (3rd line) agree.
But the Navier-Stokes does not agree.

Fluid equations in Newton-Cartan theory

The Newton-Cartan theory gives ordinary fluid equations.

Torsionless Newton-Cartan connection is invariant under Milne boost

$$\bar{v}^\mu = (1,0) \to (1, v^i) \qquad \tilde{A} \to \hat{A} = \tilde{A} + v^i dx^i - \frac{1}{2} v^2 dt$$

Then, the Navier-Stokes equation with external gauge field

$$\partial_i P + \rho \partial_t v^i + \rho v^j \partial_j v^i - \partial_j \left(\eta \sigma_{ij} \right) = \tilde{F}_{i\mu} J^\mu$$

is expressed as

$$\partial_i P - \partial_j \left(\eta \sigma_{ij} \right) = \hat{F}_{i\mu} J^\mu \qquad \text{where} \qquad J^\mu = \rho \hat{v}^\mu$$

Our result agrees with the Navier-Stokes equation in terms of $\hat{A}$.

In our conservation equations, vielbein is not $\bar{v}^\mu = (1,0)$ but $\hat{v}^\mu = (1, v^i)$

Navier-Stokes in non-zero Newton potential

The Navier-Stokes equation from Lifshitz black hole is

$$\partial_i P - \partial_j\left(\eta\sigma_{ij}\right) = \mathcal{F}_{i\mu}J^{\mu} \qquad \text{where} \qquad \mathcal{A} = \mathcal{A}_i(dx^i - v^i dt)$$

The Navier-Stokes equation in the Newton-Cartan theory is

$$\partial_i P - \partial_j\left(\eta\sigma_{ij}\right) = \hat{F}_{i\mu}J^{\mu} \qquad \text{where} \qquad \hat{A} = \tilde{A} + v^i dx^i - \frac{1}{2}v^2 dt$$

If we identify $\mathcal{A}$ with $\hat{A}$,

$$\tilde{A}_t = -\frac{1}{2}v^2 \qquad \tilde{A}_i = 0$$

Since the gauge field in the Newton-Cartan theory is generalization of Newton gravity, $\tilde{A} \neq 0$ implies non-zero Newton potential.

Since we took Eddington-Finkelstein coordinates,
the gauge field is not singular even if $A_t \neq 0$, at $r = r_0$.

General background gauge field

If we impose regularity condition only at the future horizon, we can take

$$\mathcal{A} = v^i dx^i - \frac{1}{2} v^2 dt + \tilde{\mathcal{A}}_\mu dx^\mu$$

We generalize $\mathcal{A}_t$ to arbitrary

$$ds^2 = -r^{2z} f(r) dt^2 + 2r\, dr\, dt + r^2 \left(dx^i - v^i(x) dt\right)^2$$

$$e^{\lambda\phi} = \mu(x) r^{2(1-d)} \qquad f(r) = 1 - \frac{r_0^{z+d-1}(x)}{r^{z+d-1}}$$

$$A = a(x) r^{z+d-1} f(r) dt - a(x) r^2 dr + \mathcal{A}_i(x)\left(dx^i - v^i(x) dt\right)$$

$$A = a(x) r^{z+d-1} f(r) dt - a(x) r^2 dr + \mathcal{A}_t(x) dt + \mathcal{A}_i(x) dx^i$$

The solution does not change but constraint for $\mathcal{A} = v^i dx^i - \frac{1}{2} v^2 dt$

$$\partial_i P - \partial_j\left(\eta \sigma_{ij}\right) = \mathcal{F}_{i\mu} J^\mu = -q \partial_t v^i - q v^j \partial_j v^i$$

gives the non-relativistic Navier-Stokes equation.

The stress-energy tensor becomes

$$\tilde{T}^0{}_0 = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[-\frac{3}{2}r_0^{z+3} - \frac{z-1}{a}v^i\mathcal{A}_i\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^i{}_0 = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[-\frac{z+3}{2}r_0^{z+3}v^i + \frac{z(z+3)}{4(z-1)}r_0^{2z}\partial_i r_0 + \frac{z-1}{a}v^i\mathcal{A}_t + \frac{1}{2}r_0^3\sigma_{ij}v^j\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^0{}_i = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\frac{z-1}{a}\mathcal{A}_i + \mathcal{O}\left(r^{-(z+4)}\right)$$

$$\tilde{T}^i{}_j = \frac{1}{8\pi G}\frac{1}{r^{z+3}}\left[\left(zr_0^{z+3} - \frac{z-1}{a}\left(\mathcal{A}_t + v^k\mathcal{A}_k\right)\right)\delta_{ij} - \frac{1}{2}r_0^3\sigma_{ij} + \frac{z-1}{a}v^i\mathcal{A}_j\right] + \mathcal{O}\left(r^{-(z+4)}\right)$$

$\tilde{T}^{\mu}{}_{\nu}$ cannot be identified with the fluid stress-energy tensor.

We define new stress-energy tensor $T^{\mu}{}_{\nu}$ as

$$\tilde{T}^{\mu}{}_{\nu} = T^{\mu}{}_{\nu} + J^{\mu}\mathcal{A}_{\nu} - \delta^{\mu}{}_{\nu}J^{\rho}\mathcal{A}_{\rho}$$

Energy density $\mathcal{E}^{\mu}$, pressure P and stress tensor $\mathcal{T}^{i}{}_{j}$ does not change but momentum density vanishes $\mathcal{P}_i = 0$, for new $T^{\mu}{}_{\nu}$.

Now, we take $F_{\mu\nu}J^{\nu}$ term in the conservation into account.

$$D_{\mu}\mathcal{E}^{\mu} = -\mathcal{F}_{\mu\nu}\hat{v}^{\mu}J^{\nu} - \frac{1}{2}(D^{\mu}\hat{v}^{\nu} + D^{\nu}\hat{v}^{\mu})\mathcal{T}_{\mu\nu}$$

$$D_{\mu}\mathcal{T}^{\mu}{}_{\nu} = \hat{v}^{\mu}D_{\nu}\mathcal{P}_{\mu} - D_{\mu}(\hat{v}^{\mu}\mathcal{P}_{\nu}) - \mathcal{F}_{\mu\nu}J^{\mu}$$

Then, these equation gives the non-relativistic fluid equations

$$0 = \partial_t\mathcal{E} + v^i\partial_i\mathcal{E} + (\mathcal{E} + P)\partial_i v^i - \frac{1}{2}\eta\sigma_{ij}\sigma_{ij} - \partial_i(\kappa\partial_i T)$$

$$0 = \partial_i P + q\partial_t v^i + qv^j\partial_j v^i - \partial_j\left(\eta\sigma_{ij}\right) - \tilde{\mathcal{F}}_{i\mu}J^{\mu}$$

$$0 = \partial_t q + \partial_i\left(qv^i\right)$$

We define another stress-energy tensor $\bar{T}^{\mu}{}_{\nu}$ as

$$\bar{T}^{\mu}{}_{\nu} = \tilde{T}^{\mu}{}_{\nu} + \delta^{\mu}{}_{\nu} J^{\rho} \mathcal{A}_{\rho}$$

The stress-energy tensor is expressed in terms of fluid variables as

$$\bar{T}^{0}{}_{0} = -\left(\mathcal{E} + \frac{1}{2} q v^2 - q \tilde{\mathcal{A}}_t \right)$$

$$\bar{T}^{i}{}_{0} = -\left(\mathcal{E} + P + \frac{1}{2} q v^2 - q \tilde{\mathcal{A}}_t \right) v^i + \eta \sigma_{ij} v^j + \kappa \partial_i T$$

$$\bar{T}^{0}{}_{i} = q v^i + q \tilde{\mathcal{A}}_i$$

$$\bar{T}^{i}{}_{j} = \delta_{ij} - \eta \sigma_{ij} + q v^i v^j + q v^i \tilde{\mathcal{A}}_j$$

The bulk constraint equations can be rewritten as

$$\partial_\mu \bar{T}^\mu{}_\nu + J^\mu \partial_\nu \tilde{\mathcal{A}}_\mu = 0$$

$$\partial_\mu J^\mu = 0$$

In terms of the fluid variables, conservation of $\bar{T}^\mu{}_\nu$ is expressed as

$$\begin{aligned} 0 = \partial_t &\left(\mathcal{E} + \frac{1}{2} q v^2 - q \tilde{\mathcal{A}}_t \right) \\ &+ \partial_i \left[\left(\mathcal{E} + P + \frac{1}{2} q v^2 - q \tilde{\mathcal{A}}_t \right) v^i - \eta \sigma_{ij} v^j - \kappa \partial_i T \right] \\ &+ q \partial_t \tilde{\mathcal{A}}_t + q v^i \partial_t \tilde{\mathcal{A}}_i \end{aligned}$$

$$0 = \partial_i P + q \partial_t v^i + q v^j \partial_j v^i - \partial_j \left(\eta \sigma_{ij} \right) - \tilde{\mathcal{F}}_{i\mu} J^\mu$$

Holographic entropy current

Entropy current J_S^μ $\Leftarrow$ Volume form on time slice at the horizon

$$\epsilon_{\mu_1\cdots\mu_d} J_S^{\mu_1} dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_d}$$

In terms of the normal vector to the horizon $n_\mu = \partial_\mu \mathcal{S}$ where $\mathcal{S} = r - r_0(x)$.

$$J_S^\mu = \frac{\sqrt{h}}{4G} \frac{n^\mu}{n^0}$$

The entropy current for Lifshitz black hole is calculated as

$$J_S^0 = \frac{1}{4G} r_0^3 \qquad J_S^i = \frac{1}{4G} r_0^3 v^i - \frac{z}{8G(z-1)} r_0^z \partial_i r_0$$

Thermodynamic relation

Energy flow, pressure and temperature

$$\mathcal{E}^0 = \frac{1}{16\pi G} r_0^{z+3} \qquad \mathcal{E}^i = \frac{1}{16\pi G}\left(r_0^{z+3} v^i - \frac{z(z+3)}{2(z-1)} r_0^{2z} \partial_i r_0 \right)$$

$$P = \frac{z}{16\pi G} r_0^{z+3} \qquad T = \frac{z+3}{4\pi} r_0^z$$

Entropy current

$$J_S^0 = \frac{1}{4G} r_0^3 \qquad J_S^i = \frac{1}{4G} r_0^3 v^i - \frac{z}{8G(z-1)} r_0^z \partial_i r_0$$

They satisfy the thermodynamic relation

$$T J_S^\mu = -\tilde{T}^\mu{}_\nu \hat{v}^\nu + P \hat{v}^\nu = \mathcal{E}^\mu + P \hat{v}^\mu$$

Second law of thermodynamics

Divergence of the entropy current becomes

$$\partial_\mu J_S^\mu = \frac{1}{4G}\left[\partial_t r_0^3 + \partial_i\left(r_0^3 v^i - \frac{z}{2(z-1)} r_0^z \partial_i r_0\right)\right]$$

Since fluid equaitons give $0 = \partial_t \mathcal{E} + v^i \partial_i \mathcal{E} + (\mathcal{E} + P)\partial_i v^i - \frac{1}{2}\eta\sigma_{ij}\sigma_{ij} - \partial_i(\kappa\partial_i T)$

$$0 = 3\partial_t r_0^{z+3} + 3v^i \partial_i r_0^{z+3} + (z+3) r_0^{z+3}\partial_i v^i$$
$$-\sigma_{ij}\sigma_{ij} - \frac{z(z+3)}{2(z-1)}\partial_i(r_0^{2z}\partial_i r_0)$$

divergence of the entropy current becomes

$$\partial_\mu J_S^\mu = 2\pi \frac{\eta}{r_0^z}\sigma_{ij}\sigma_{ij} + \frac{z^2}{8G(z-1)} r_0^{z-1}(\partial_i r_0)^2 \geq 0$$

Divergence of the entropy current is non-negative

⇒ 2nd law of thermodynamics

Kovtun-Son-Starinets bound

Entropy density and shear viscosity satisfy the following condition

$$\frac{\eta}{s} \geq \frac{1}{4\pi}$$

In our case, the entropy density is

$$s = J_S^0 = \frac{1}{4G} r_0^3$$

Shear viscosity is

$$\eta = \frac{1}{16\pi G} r_0^3$$

In the fluid/gravity model of Lifshitz, it saturate the KSS bound

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

Conclusion

- We have considered fluid/gravity correspondence for the Lifshitz black hole geometry.
- Naïve ansatz gives the stress-energy tensor which satisfy the conservation law of the Newton-Cartan theory.
- Energy conservation and continuity equations agree with those for ordinary non-relativistic fluids.
- The Navier-Stokes equation is different from ordinary one, but in terms of the gauge field, it agrees with that in the Newton-Cartan theory.
- If we take $\mathcal{A} = v^i dx^i - \frac{1}{2} v^2 dt$, the constraints agrees with the ordinary non-relativistic fluid equations.
- Entropy current is defined from the horizon area and satisfies the local thermodynamic relation and second law.

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