FROZEN

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I have **two goals** today.

The narrower goal is to tell you some properties of **frozen singularities** in M-theory and F-theory.

This involves a fun relation between **Kodaira's classification of singular elliptic fibrations** and **commuting triples in simply-laced groups.**

I know that's completely greek to most of you.

Don't worry, I won't continue in this manner...

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Don't worry, I won't continue in this manner...

at least during the first third of the talk!

The broader goal is to tell you that the **consistency of string/M/F theory** is supported by various **mathematical accidents**, of which the narrower goal is one.

What do I mean by a **mathematical accident**, to start with?

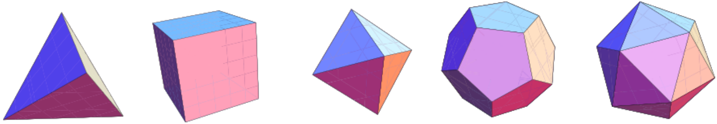
Consider the following simple question:

- Pick $n \geq 3$.
- Construct a convex polyhedron, using only regular n-gons of the same size.
- When is this possible?

Of course we know the answer ...

It's only possible with

- $n = 3$ (with 4, 8 or 20 faces)
- $n = 4$ (with 6 faces), or
- $n = 5$ (with 12 faces)!



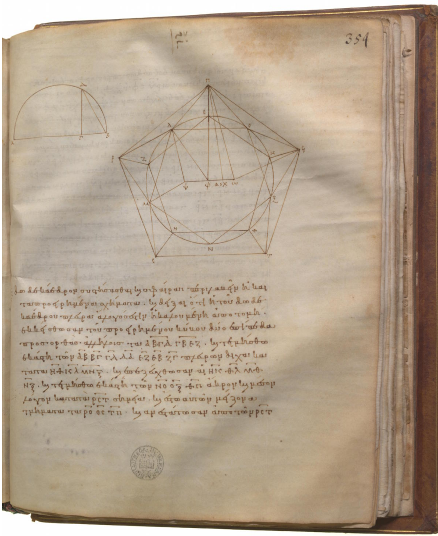
That there is no other regular polyhedron is
the **final entry** of Euclid's **Elements of Geometry**.

Yes it's a logical consequence, but to me it feels like an **accident**.

Here is a page from a manuscript of Euclid's Elements, made in 888AD in Constantinople.

It is amazing that we can have a look at the digitized version for free.

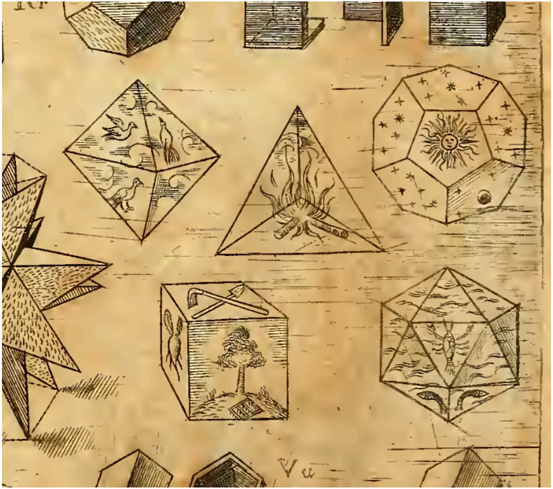
Available at `http://www.claymath.org/euclids-elements`.



This mathematical accident fascinated many physicists
(= natural philosophers) in the past.

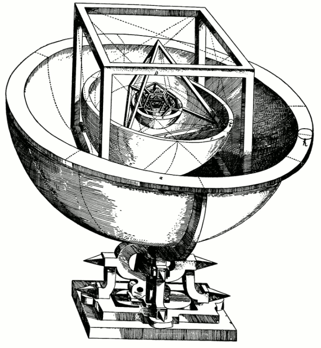
Ancient Greeks thought of a correspondence
between **five elements of the universe** and **five regular solids**:

This is Kepler's depiction of this correspondence,
in his **Harmonice Mundi**:



Again, freely available online at
https://archive.org/details/ioanniskepplerih00kepl

Kepler also tried to be more quantitative:
He thought that a regular polyhedron would fit between each consecutive planetary orbits.



This drawing is from his **Mysterium Cosmographicum** (taken from Wikipedia)

Let's check. He put the polyhedra in the order

from inside to the outside.
The ratios of the circumradius and the inradius are, respectively,

$$\sqrt{3}, \quad \sqrt{15-6\sqrt{5}}, \quad \sqrt{15-6\sqrt{5}}, \quad 3, \quad \sqrt{3}$$

which are numerically

$$1.73, \quad 1.26, \quad 1.26, \quad 3, \quad 1.73.$$

The distances from the sun to Mercury, Venus, Earth, Mars, Jupiter and Saturn are

$$57.9, \quad 108.2, \quad 149.6, \quad 227.9, \quad 778.6, \quad 1433.5$$

times 10^6 km. Taking the ratio, we get

$$1.87, \quad 1.38, \quad 1.52, \quad 3.41, \quad 1.84$$

Not terribly bad.

By now, we know neither of the two sets of ideas was right.

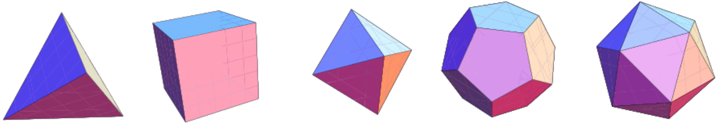
In fact they turned out to be **bullsh*ts**.

Nonetheless, platonic solids still fascinate me.

So, I just explained the prototypical example of mathematical accidents.

- Pick $n \geq 3$.
- Construct a polyhedron using only regular n-gons of the same size.
- When is this possible?

There are just a finite number of solutions,



each of which is full of character, and quite interesting!

Let me give another example. The structure of the question is similar.

Stated in theoretical particle physics jargon, it goes as follows:

- Consider **10d supersymmetric** theory,
- consisting of the **gravity field**, the **gauge field** with gauge group G and their superpartners.
- When is this possible (in the sense that the model is anomaly free)?

There are just two solutions:

$$G = E_8 \times E_8 \text{ or } G = \mathbf{SO(32)}.$$

(Note to mathematicians in the audience:
when I refer to a Lie group, in fact I'm referring to a Lie algebra.)

Let me restate this in a way understandable to mathematicians:

- Take a 12-dimensional space X, and consider a G-bundle P over it. Let R, F be the curvature 2-forms of TX and P, respectively.
- Consider the 12-form

$$I = \frac{1}{2}\hat{A}(R)\,\mathrm{tr}_{\mathfrak{g}}\, e^{F}\Big|_{12} - \frac{1}{128}\left[\frac{31}{360}\,\mathrm{tr}\, R^6 - \frac{7}{135}\,\mathrm{tr}\, R^4\,\mathrm{tr}\, R^2 + \frac{1}{162}(\mathrm{tr}\, R^2)^3\right]$$

- For which semisimple group G does this always factorize as (4-form) $\times$ (8-form) ?

There are just two solutions:

$$G = E_8 \times E_8 \text{ or } G = \mathbf{SO(32)}.$$

So this is a **mathematical accident**.

For those in the audience who is more inclined to physics,
let me explain (very remote, possible) significance of this question
to the real world, before getting further.

There are four fundamental forces in nature :

- **Gravitational force.**
  Things go down, since Earth pulls them.
- **Electromagnetic force.**
  The source of most of interactions around us.
- **'Weak' force.**
  Mediates β decay. $\sim$50 decays of K^{40} per sec. per kg
- **'Strong' force.**
  Binds quarks into protons and neutrons.

They are described by the following mathematical objects:

Gravitational force.	Metric.
Electromagnetic force.	$\mathbf{U(1)}$ gauge field.
'Weak' force.	$\mathbf{SU(2)}$ gauge field.
'Strong' force.	$\mathbf{SU(3)}$ gauge field.

The gauge groups in principle can be any of
$\mathbf{U(1)}$, $\mathbf{SU}(N)$, $\mathbf{SO}(N)$, $\mathbf{Sp}(N)$, G_2, F_4, E_6, E_7, E_8,
and their combinations.

A basic, unanswered question in physics is why we have
$G = \mathbf{U(1)} \times \mathbf{SU(2)} \times \mathbf{SU(3)}$, rather than something else.

If the universe is **10d** instead of **4d**, and
if we furthermore have **supersymmetry**,
the mathematical accident we just discussed forces us to have either $G = E_8 \times E_8$ or $G = \mathbf{SO(32)}$.

If only a similar miracle happens in 4d, we would understand why we have $\mathbf{U(1)} \times \mathbf{SU(2)} \times \mathbf{SU(3)}$!

By the way, the list of all compact Lie groups itself,
is a **mathematical accident** to me.

- Consider a Lie algebra over $\mathbb{R}$
- which has a positive-definite inner product,
  (to have a positive-definite kinetic term for the gauge fields)
- What are the possibilities?

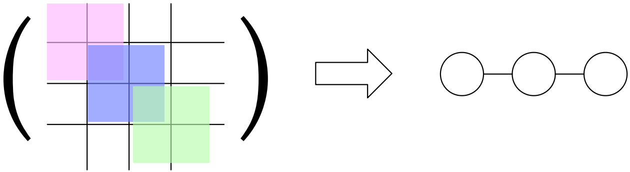
Answer:

$\mathbf{U(1)}$, $\mathbf{SU}(N)$, $\mathbf{SO}(N)$, $\mathbf{Sp}(N)$, G_2, F_4, E_6, E_7, E_8,
or combinations thereof.

The most basic case is $\mathbf{SU(2)}$:

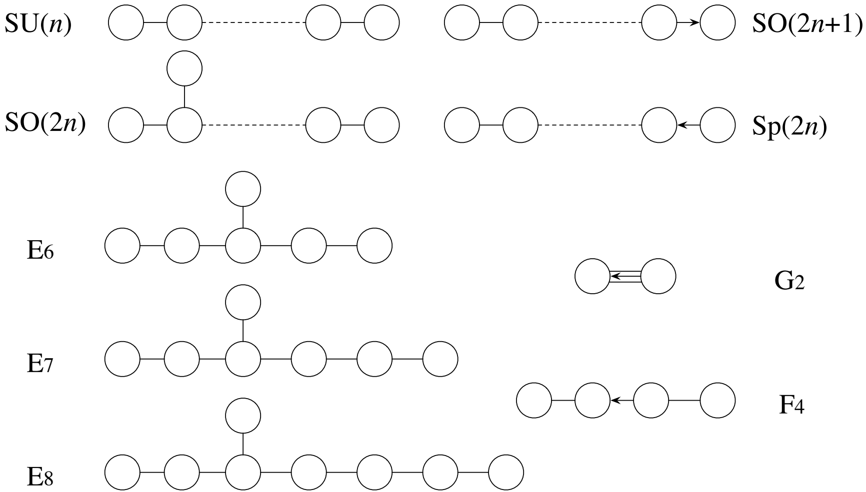
$$[J_1, J_2] = J_3, \quad [J_2, J_3] = J_1, \quad [J_3, J_1] = J_2$$

The general case is classified by how $\mathbf{SU(2)}$ embeds into them. For example, for $\mathbf{SU(4)}$:



A blob represents an $\mathbf{SU(2)}$; edges show their interrelations .

Only certain diagrams, therefore Lie algebras, are allowed:



Four infinite series and **five exceptions**. Again, an accident.

I explained that an anomaly-free 10d supersymmetric system requires either $\boldsymbol{E_8} \times \boldsymbol{E_8}$ and $\mathbf{SO(32)}$.

The same two choices, $\boldsymbol{E_8} \times \boldsymbol{E_8}$ and $\mathbf{SO(32)}$ arise in a different mathematical accident.

- Consider a 16-dimensional lattice $\boldsymbol{\Lambda} \simeq \mathbb{Z}^{\mathbf{16}} \subset \mathbb{R}^{\mathbf{16}}$.
- Suppose $\boldsymbol{v} \cdot \boldsymbol{w} \in \mathbf{2}\mathbb{Z}$ for any two vectors in $\boldsymbol{\Lambda}$, **(even)** and the unit cell has volume 1. **(self-dual)**
- When is it possible?

The answer is that $\boldsymbol{\Lambda}$ is either

- $\boldsymbol{\Lambda} =$ (root lattice of $\boldsymbol{E_8}$) $\oplus$ (root lattice of $\boldsymbol{E_8}$) , or
- (weight lattice of $\mathbf{SO(32)}$) $\subset \boldsymbol{\Lambda} \subset$ (root lattice of $\mathbf{SO(32)}$).

So there are two mathematical questions:

- **Anomaly-free 10d supergravity + gauge systems**
- **rank-16 even-self dual lattices**

that have exactly the same answer: $\boldsymbol{E_8 \times E_8}$ or $\mathbf{SO(32)}$.

Is there a deeper meaning?

There is a version of superstring theory called **heterotic string theory**.

It is consistent only when

- the spacetime is **10 dimensional**
- the gauge group is associated to a **rank-16 even self-dual lattice**

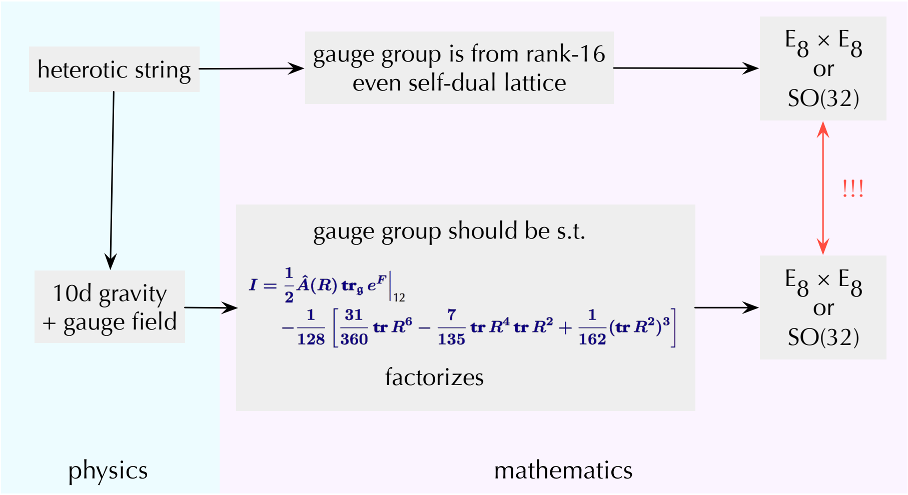
So, there are two versions:
$\boldsymbol{E_8 \times E_8}$ heterotic string, and $\mathbf{SO(32)}$ heterotic string.

A heterotic string theory automatically gives rise to a system

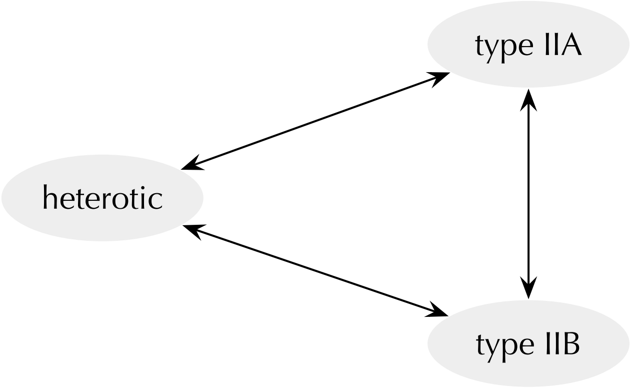
- which is **10d supersymmetric**
- and contains the **gravity field**, the **gauge field** with gauge group G and their superpartners,
- and anomaly free.

These conditions, as I already said, also lead to the choices $G = E_8 \times E_8$ or $\mathbf{SO(32)}$.

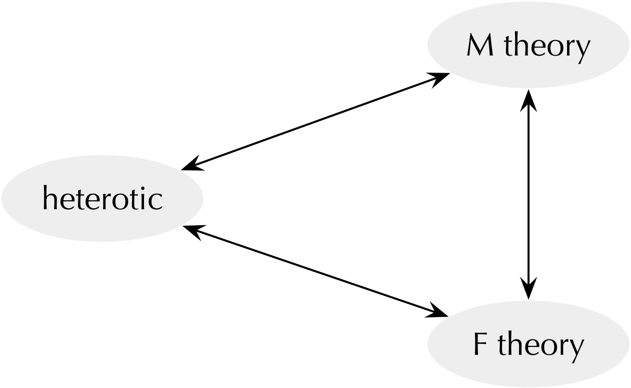
So, two mathematical accidents accidentally give the same answer.
Without this, heterotic string theory falls apart.



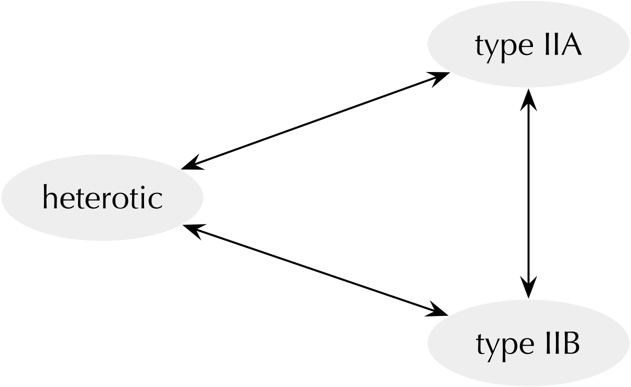
I just introduced **heterotic string**.



There are also type IIA and type IIB string,
all of which are related in various ways.

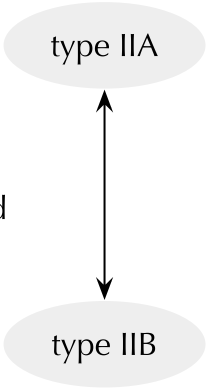


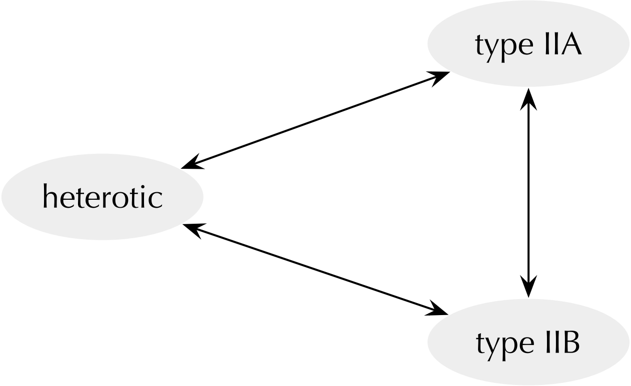
IIA has a better version called M, and IIB has a better version called F.



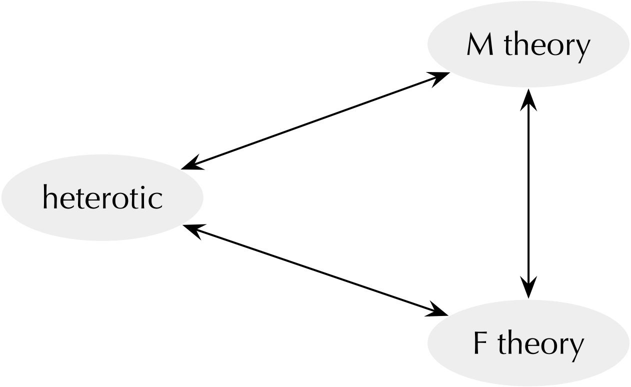
If we come back to just IIA and IIB,

Mathematical version of this
is called the mirror symmetry,
and has been intensively studied
by mathematicians.



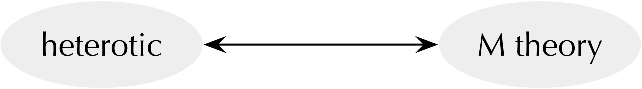


But I'd like to emphasize that all three should be equally important.



But I'd like to emphasize that all three should be equally important.

For example, consider



There are again many mathematical accidents involved.

The basic statement is that

$$\text{heterotic on } \boldsymbol{X_7 \times T^3} = \text{M on } \boldsymbol{X_7 \times K3.}$$

- Heterotic strings are 10 dimensional
- M theory is 11 dimensional
- $\boldsymbol{T^3}$ is the 3d torus
- $\boldsymbol{K3}$ is a nice compact 4-dimensional space
- $\boldsymbol{10 - 3 = 7 = 11 - 4}$

The mathematical existence of the $\boldsymbol{K3}$ itself, I say,
is a mathematical accident.

It's difficult to explain what is a $\boldsymbol{K3}$ to those who don't know already.

It's unnecessary to explain to those who already know.

So I won't explain, except that it's a very nice space.

$K3$ is named by André Weil, in honor of **Kähler**, **Kummer** and **Kodaira**,

and after the beautiful mountain K2

Let us come back to the relation

$$\text{heterotic on } X_7 \times T^3 = \text{M on } X_7 \times K3.$$

We know that heterotic strings have two versions, $E_8 \times E_8$ and $\mathbf{SO(32)}$.

How are they reflected in terms of $K3$?

Answer: $K3$ can have singularities of type $E_8 \times E_8$ or $\mathbf{SO(32)}$.

$\boldsymbol{K3}$ is real 4-dimensional, and locally looks like $\mathbb{R}^{\mathbf{4}} \simeq \mathbb{C}^{\mathbf{2}}$.

It can develop singularities of the form

$$\mathbb{C}^{\mathbf{2}}/\mathbf{\Gamma}$$

where $\mathbf{\Gamma}$ is a **finite subgroup**

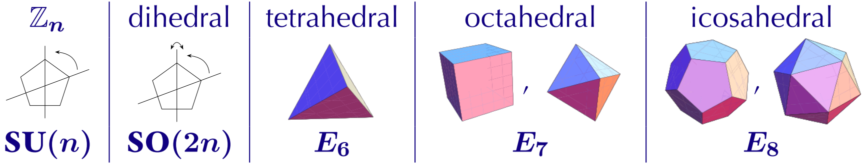
$$\mathbf{\Gamma} \subset \mathbf{SU(2)}.$$

Recall

$$\mathbf{SU(2)} \xrightarrow{\mathbf{2:1}} \mathbf{SO(3)}.$$

So such a $\mathbf{\Gamma}$ is essentially specified by a finite subgroup of $\mathbf{SO(3)}$.

The finite subgroups of $\mathbf{SO(3)}$ are exhausted by

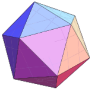


and known to correspond to the Lie groups as shown.

Regular polyhedra are somehow linked to E_6, E_7 and E_8.

Again, a **mathematical accident**.

Take the symmetry group of

, 

and consider it inside $\mathbf{SU(2)}$. The irreducible representations are

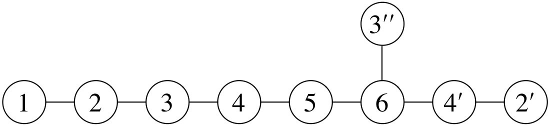
$$\mathbf{1},\ \mathbf{2},\ \mathbf{3},\ \mathbf{4},\ \mathbf{5},\ \mathbf{6};\ \mathbf{2'},\ \mathbf{4'};\ \mathbf{3''}$$

where $\mathbf{2}$ is the one coming from the $\mathbf{SU(2)}$ action on $\mathbb{C}^{\mathbf{2}}$.

Now consider the tensor product decomposition after multiplying by $\mathbf{2}$:

$$\mathbf{2} \otimes \mathbf{4} = \mathbf{3} \oplus \mathbf{5}, \qquad \mathbf{2} \otimes \mathbf{6} = \mathbf{5} \oplus \mathbf{4'} \oplus \mathbf{3''}$$

etc. Draw these relations as

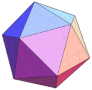


which is the Dynkin diagram of $\boldsymbol{E_8}$!

Then, in the relation

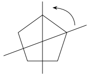
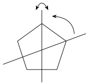
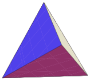
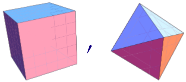
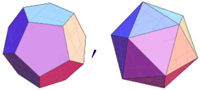
$$\text{heterotic on } X_7 \times T^3 = \text{M on } X_7 \times K3,$$

Choosing the $E_8 \times E_8$ heterotic string corresponds to choosing a $K3$ having two singularities associated to

, .

Namely, let Γ_{E_8} be the binary icosahedral group $\subset \mathbf{SU(2)}$, and the $K3$ should have two singularities of the form $\mathbb{C}^2/\Gamma_{E_8}$.

The point is that when M-theory is considered on a spacetime with a singularity locally of the form $\mathbb{C}^2/\Gamma_G$
where Γ_G is the finite subgroup of $\mathbf{SU(2)}$ according to

$\mathbb{Z}_n$	dihedral	tetrahedral	octahedral	icosahedral
				
$\mathbf{SU}(n)$	$\mathbf{SO}(2n)$	E_6	E_7	E_8

we get a gauge field with gauge group G.

So, if M theory really describes the physical world,
what ancient Greek philosophers and Kepler tried to do
might not have been completely misguided.

Or maybe we string theorists are as misguided as they were.

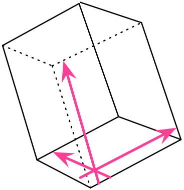
Hopefully experimenters in the future will tell.

– intermission –

We were talking about the relation

$$\text{heterotic on } \boldsymbol{T^3} = \text{M on } \boldsymbol{K3}.$$

Let us see the mapping in more detail.



On the heterotic side, the necessary data are:

- A real constant ϕ, $\mathbf{1}$ parameter.
- Flat metric on T^3. $\mathbf{6}$ parameters.
- An exact two-form B on T^3. $\mathbf{3}$ parameters.
- Flat $E_8 \times E_8$ bundle on T^3.
  Assuming that the holonomies are in the Cartan,
  $\mathbf{2 \times 3 \times 8 = 48}$ parameters.

$\mathbf{1 + 6 + 3 + 48 = 58}$ parameters in total.

On the M-theory side, we need

- hyperkähler metric on $K3$, 58 parameters.
- an exact 3-form C, 0 parameters.

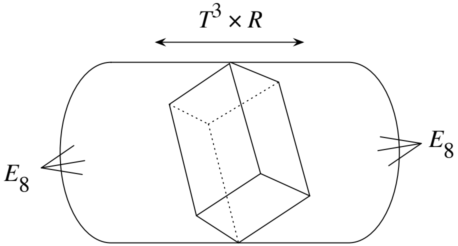
The moduli space of hk metrics on $K3$ is known to be parameterized by a symmetric space

$$\mathbb{R} \times [\mathbf{O}(3,19;\mathbb{Z}) \setminus \mathbf{O}(3,19;\mathbb{R})/\mathbf{O}(3,\mathbb{R}) \times \mathbf{O}(19,\mathbb{R})]$$

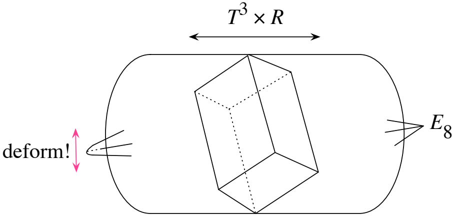
It is known that the heterotic side is the same, too.

When the $E_8 \times E_8$ bundle is trivial,
the stabilizer (or equivalently the unbroken gauge group) is
$E_8 \times E_8$ itself.

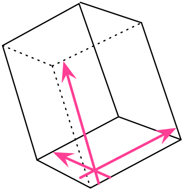
So the corresponding $K3$ should have two E_8 singularities:



Each $\boldsymbol{E_8}$ singularity has $\mathbf{3} \times \mathbf{8}$ hyperkähler deformations



giving the holonomies on the heterotic side,
which also have $\mathbf{3} \times \mathbf{8}$ parameters



So far so good. Or so it seems.

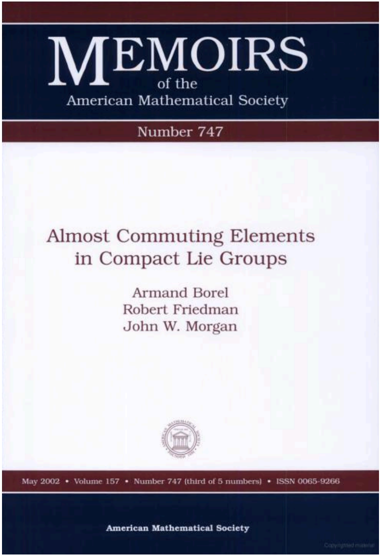
But this is too quick!

On the heterotic side, the necessary data are:

- A real constant ϕ, 1 parameter.
- Flat metric on T^3, 6 parameters.
- An exact two-form B on T^3. 3 parameters
- Flat $E_8 \times E_8$ bundle on T^3.
  Assuming that the holonomies are in the Cartan,
  $2 \times 3 \times 8 = 48$ parameters.

But **not** all flat bundles have holonomies in the Cartan.

Borel, Friedman and Morgan says in this monograph in 2002 that ...



The moduli space of flat E_8 bundles on T^3 has the form

$$\mathcal{M} = \bigsqcup_{r \in \mathbb{Q}/\mathbb{Z}} \mathcal{M}_r$$

where

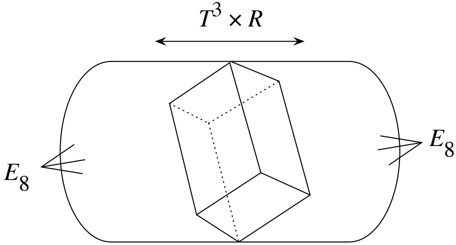
$$r = \frac{n}{d} \mod 1, \qquad d = 1,2,3,4,5,6; \quad \gcd(n,d) = 1.$$

Here r labels the Chern-Simons invariant of the bundle.

$\mathcal{M}_0$ is the component coming from three holonomies in the Cartan. The rank of the stabilizer is 8, and the maximal stabilizer is E_8. Similarly:

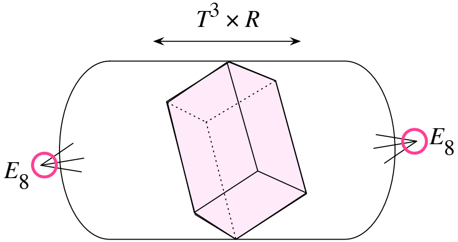
d	1	2	3	4	5	6
rank of stab.	8	4	2	1	0	0
max. stab.	$\mathfrak{e}_8$	$\mathfrak{f}_4$	$\mathfrak{g}_2$	$\mathfrak{su}(2)$	fin.	fin.

When the E_8 bundle on T^3 is in $\mathcal{M}_0$, the K3 is



What happens when the bundles are from $\mathcal{M}_r$ with $r \neq 0$?

The maximal stabilizer (or unbroken gauge group) is smaller. Correspondingly, there should be less freedom to deform the singularity. For example, for $\mathcal{M}_{r=1/6}$, you can't just deform at all.

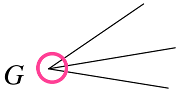


Known to be measured by

$$\int_{S^3/\Gamma_{E_8}} C = \int_{T^3} C = r.$$

[de Boer-Dijkgraaf-Hori-Keurentjes-Morgan-Morrison-Sethi, hep-th/0103170]

In general, given a singularity of the form $\mathbb{C}^2/\Gamma_G$,



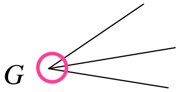
we can take

$$r := \int_{S^3/\Gamma_G} C = \frac{n}{d} \mod 1$$

where n is an integer appearing in the node of the Dynkin diagram of type G and $\gcd(n, d) = 1$.

When $r \neq 0$, this **partially freezes** the singularity, and change the gauge group appearing here.

The gauge group H that appears here



with a given r is determined as follows:

- Take a flat G bundle P over T^3 whose Chern-Simons invariant is r
- Let G_P be the stabilizer (or the unbroken gauge group) of P
- Then H is the **Langlands dual** of G_P.

In particular, H is **not** a natural subgroup of G.

For example:

When $G = \mathbf{SO(2k+8)}$, we can take $r = 1/2$.
The corresponding triple is in $\mathbf{SO(7)}$, given e.g. by

$$\mathbf{diag}(+ + + - - - -)$$
$$\mathbf{diag}(+ - - + + - -)$$
$$\mathbf{diag}(- + - + - + -)$$

The stabilizer is $\mathbf{SO(2k+1)}$, whose Langlands dual is $\mathbf{Sp(2k)}$.
This is the gauge group appearing with this singularity in M theory.

As another example,

When $G = E_7$, we can take $r = 1/2$. The corresponding triple is still the same one in $\mathbf{SO(7)}$, given e.g. by

$$\mathbf{diag}(+++----)$$
$$\mathbf{diag}(+--++--)$$
$$\mathbf{diag}(-+-+-+-)$$

In fact these three elements are in $G_2 \subset \mathbf{SO(7)}$.
And there is a maximal subgroup $G_2 \times \mathbf{Sp(6)} \subset E_7$.

The stabilizer is $\mathbf{Sp(6)}$, whose Langlands dual is $\mathbf{SO(7)}$.
This is the gauge group appearing with this singularity in M theory.

What I've told you so far was all known around the turn of the century!
In the remaining -5 minutes, let me tell you something new, from my latest paper:

v:1508.06679v1 [hep-th] 26 Aug 2015

On M and F theory's FROZEN SINGULARITIES

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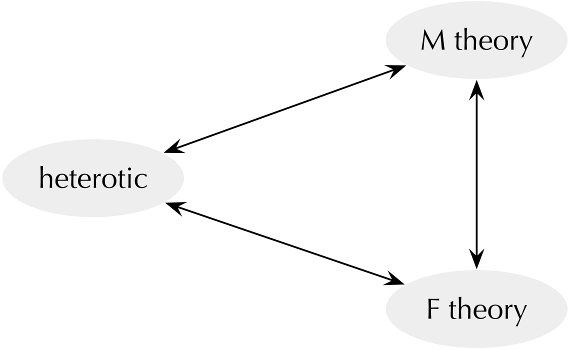
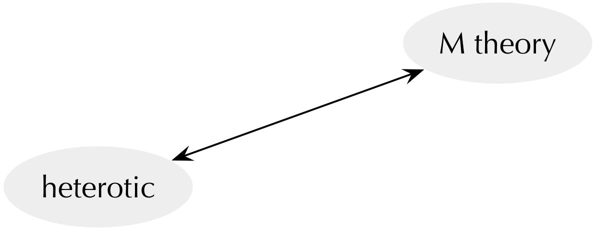
Abstract

We revisit the duality between ALE singularities in M-theory and 7-branes on a circle in F-theory. We see that a frozen M-theory singularity maps to a circle compactification involving

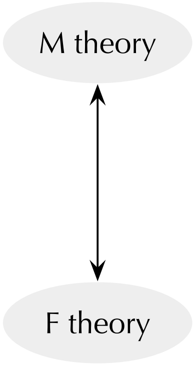
In this diagram



So far I only talked about



Let me talk a bit about



The data specifying an F-theory configuration are

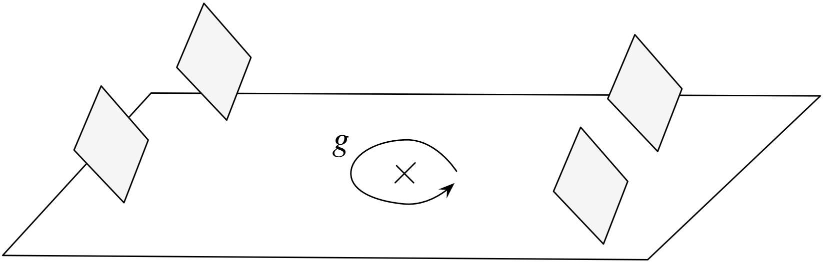
- 10d spacetime B
- a map $\tau : B \mapsto$ (upper half plane)$/\mathbf{SL}(2, \mathbb{Z})$, allowing singularities

A point on (upper half plane)$/\mathbf{SL}(2, \mathbb{Z})$ specifies a 2d torus (or equivalently an elliptic curve), so it determines an elliptic fibration

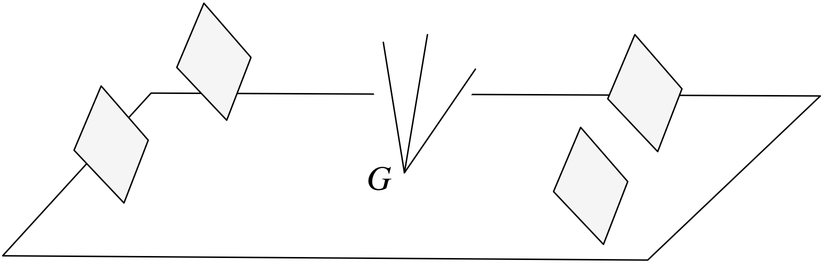
$$E \to B.$$

People often refer to this situation as "F-theory on E", but this becomes very confusing, particularly when the frozen singularities are involved. So I'll call the setup as "F-theory on B".

Anyway, there is a 4d space of the form $E \to B = \mathbb{C}$ with an $\mathbf{SL(2,\mathbb{Z})}$ monodromy g at the origin:



There's a singularity of the form $\mathbb{C}^2/\Gamma_G$ at the origin:



The correspondence between g and G was found by Kodaira.

1st example: Kodaira's $\boldsymbol{III^*}$

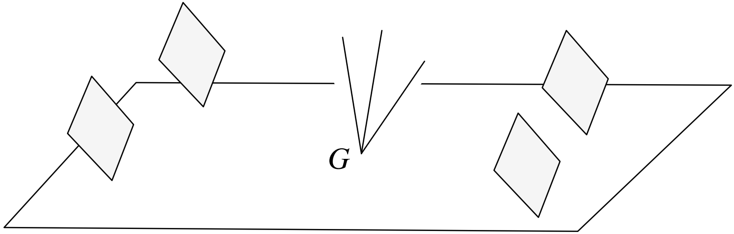
$$\boldsymbol{g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \leftrightarrow \quad G = E_7}$$

2nd example: Kodaira's $\boldsymbol{I_0^*}$

$$\boldsymbol{g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \leftrightarrow \quad G = D_4}$$

Question: consider M-theory on

$\mathbb{R}^{\mathbf{7}} \times$

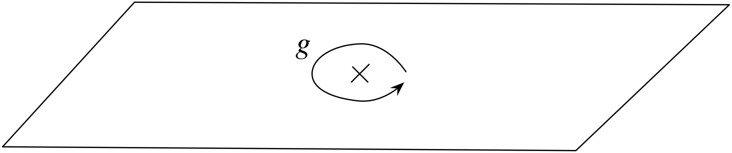


.

Note that $\mathbf{7} + \mathbf{4} = \mathbf{11}$.

What is the corresponding F-theory setup?

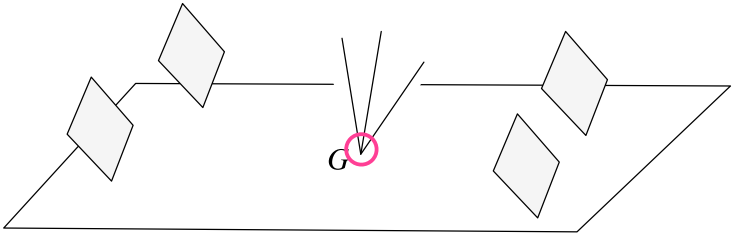
Long-known answer: F-theory on

$\mathbb{R}^7 \times S^1 \times$  .

Note that $\mathbf{7+1+2=10}$.

Question I asked in the recent paper: consider M-theory on

$\mathbb{R}^7 \times$



with non-zero

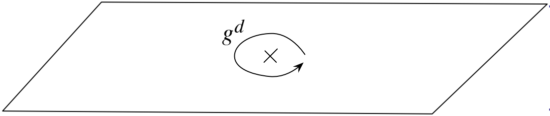
$$r = \int_{S^3/\Gamma_G} C = \frac{n}{d}$$

freezing the singularity.

What is the corresponding F-theory setup?

Answer: F-theory on $\mathbb{R}^7 \times$

$\Big[S^1 \times$



$\Big] / \mathbb{Z}_d$

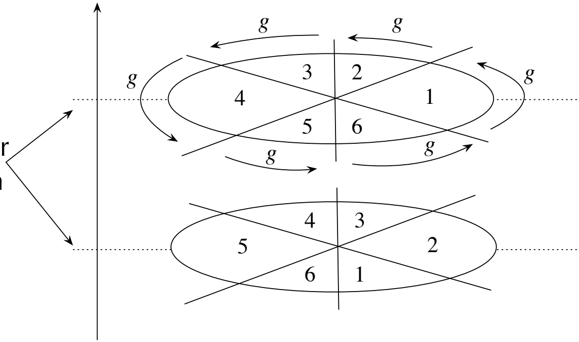
where the $\mathbb{Z}_d$ action is the combination of

- $1/d$ rotation of S^1, and
- n/d rotation of $\mathbb{C}$.

(Note that $7 + 1 + 2 = 10$.)

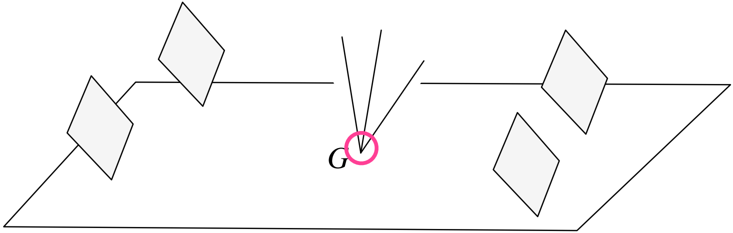
Or equivalently,

$\mathbb{R}^7 \times$ identify after 1/6 rotation



when $d = 6$, say.

An example: M-theory on

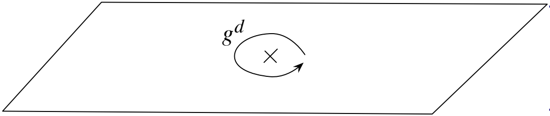
$\mathbb{R}^7 \times$ 

where $G = E_7$ and

$$r = \int_{S^3/\Gamma_G} C = \frac{1}{2}.$$

The $\mathbf{SL}(2,\mathbb{Z})$ monodromy is $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, Kodaira's III^*.

This is equivalent to F-theory on $\mathbb{R}^7 \times$

$\left[S^1 \times \right.$  $\left. \right] / \mathbb{Z}_d$

with $d = 2$, where the $\mathbb{Z}_2$ action is the combination of $1/2$ rotation of S^1 and $1/2$ rotation of $\mathbb{C}$.

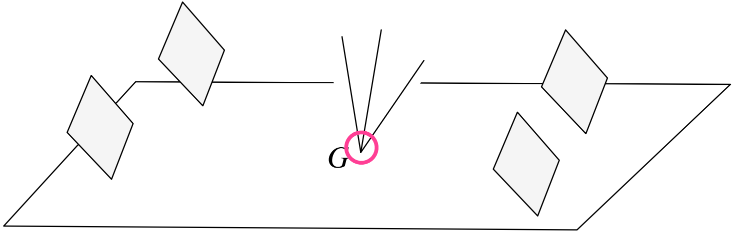
Now $g^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is Koraira's I_0^*, and the corresponding singularity is D_4.

The $1/2$ rotation acts as an outer-automorphism on D_4.

The gauge group is the invariant part, which is $\mathbf{SO(7)}$.

This is consistent with another determination of the gauge group associated to

$\mathbb{R}^7 \times$

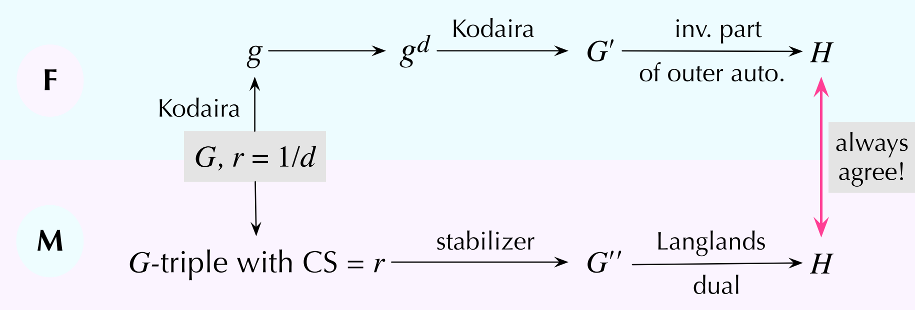


where $G = E_7$ and

$$r = \int_{S^3/\Gamma_G} C = \frac{1}{2}.$$

Namely, the triple with the Chern-Simons invariant is in G_2.
E_7 contains a maximal subgroup of the form $G_2 \times \mathbf{Sp}(6)$.
Therefore the stabilizer is $\mathbf{Sp}(6)$, whose Langlands dual is $\mathbf{SO}(7)$.

Summarizing, we have



This is again a **mathematical accident**,
without which string/M/F theory falls apart.

Big summary of the whole talk:

- **Mathematical accidents** are essential to the **consistency of string/M/F theory**.
- Many such examples are known.
- Some were recognized 30 years ago, such as the one leading to heterotic $\boldsymbol{E_8 \times E_8}$ and $\mathbf{SO(32)}$
- Some were noticed just last month, such as the one I just talked about.
- There will be more.