A $\mathbb{Z}^m$-graded generalization of the Witt Algebra and its Representations

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Contents

1 Simple $\mathbb{Z}^m$-graded Lie algebras

Here, we recall some known facts about simple $\mathbb{Z}^m$-graded Lie algebras over $\mathbb{C}$.

1.1 Definition

For $m \in \mathbb{Z}_{>0}$, let $\Lambda = \mathbb{Z}^m$ be a lattice of rank m. We consider the classes of Lie algebras $\mathfrak{g}$ with some extra conditions:

1. $\mathfrak{g}$ is **Λ-graded**, i.e., $\mathfrak{g} = \bigoplus_{\lambda \in \Lambda} \mathfrak{g}_\lambda$ s.t. i) $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$ for any $\lambda, \mu \in \Lambda$, and ii) $\dim \mathfrak{g}_\lambda < \infty$ for any $\lambda \in \Lambda$,

2. $\mathfrak{g}$ is **simple in graded sense**, i.e., i) $\dim \mathfrak{g} > 1$ and ii) there is no non-trivial proper graded ideal of $\mathfrak{g}$.

For simplicity, we call such a Lie algebra **Λ-graded simple Lie algebra**.

A natural but too naive question is
classify all Λ-graded simple Lie algebras up to isomorphism.
In fact, the classification can be too wild. Hence, one should impose some reasonable additional conditions. Let us look at some examples.

$\boxed{m=0}$
In this case, the question reduces to the classification of simple finite dimensional Lie algebras over $\mathbb{C}$, which is known by W. Killing, E. Cartan etc. since the beginning of the 20th century.

Classified by $A \sim G$. $\leftarrow$ discrete data !

$\boxed{m=1}$
Let us start from examples:

1. $\mathfrak{g} = \mathfrak{g}_0$: simple finite dimensional Lie algebra.
2. $\mathfrak{a}$: simple finite dimensional Lie algebra, $\mathfrak{g} = L(\mathfrak{a}) := \mathfrak{a} \otimes \mathbb{C}[t, t^{-1}]$ and its fixed point subalgebras.
3. For $r > 0$, let W_r be the Lie algebra of the derivations of $A = \mathbb{C}[X_1, X_2, \cdots, X_r]$.
   $W_r \curvearrowright \Omega_A = \bigoplus_{i=1}^{r} AdX_i$ by `Lie derivative`.
   $S_r \subset W_r$: the subalgebra annihilating a volume form.
   $H_{2m} \subset W_{2m}$: the subalgebra annihilating a symplectic form.
   $K_{2m+1} \subset W_{2m+1}$: the subalgebra preserving a contact form.
   These algebras are called of **Cartan type**.
4. $W = \mathbb{C}[t, t^{-1}]\dfrac{d}{dt}$: the Witt algebra.

O. Mathieu in the 80's proved that if the function $k \mapsto \dim \mathfrak{g}_k$ is bounded by a polynomial, then the above list exhausts all such algebras.

Classified by discrete data !

For higher rank case ??

1.2 General cases

Let us start from a trivial but important observation:

Let $\mathfrak{g}$ be a Λ-graded Lie algebra. Then,
the Lie algebra $\mathfrak{g}(m) := \mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$ is naturally $\Lambda \oplus \mathbb{Z}^n$-graded !

If the Lie algebra $\mathfrak{g}$ is not of the form $\mathfrak{a}(m)$ for some simple-graded $\mathfrak{a}$ and $m > 0$, $\mathfrak{g}$ is called **primitive**.

Sufficient to classify primitive simple Λ-graded algebras.

Suppose that $(*)$ dim $\mathfrak{g}_\lambda = 1$ **for any** $\lambda \in \Lambda$.

Theorem 1.1 (K.I. and O. Mathieu, in Proc. LMS (3) 106, 2013). *Let $\mathfrak{g}$ be a primitive simple Λ-graded Lie algebra satisfying $(*)$. Then, $\mathfrak{g}$ is isomorphic either to $A_1^{(1)}, A_2^{(2)}$ or to some W_π where $\pi : \Lambda \hookrightarrow \mathbb{C}^2$ is an additive map with certain condition.*

Notice that in the cases when $\mathfrak{g}$ is of type $A_1^{(1)}, A_2^{(2)}$, $\mathfrak{g}$ is $\mathbb{Z}$-graded.

Let us define W_π. Let $\mathcal{P}$ be the Poisson algebra of symbols of twisted ordinary pseudo-differential operators which is defined as follows.
For $\lambda = (a, b) \in \mathbb{C}^2$, let E_λ be the symbol of the twisted pseudo-differential operator $z^{a+1}\partial^{b+1}$ ($\partial = \frac{d}{dz}$) and set $\rho = (1, 1)$. Then, $\mathcal{P}$ is the $\mathbb{C}$-vector space with basis $\{E_\lambda\}_{\lambda \in \mathbb{C}^2}$ whose multiplicative and Poisson structures are given by

$$E_\lambda \cdot E_\mu := E_{\lambda+\mu+\rho}, \qquad \{E_\lambda, E_\mu\} = \langle \lambda + \rho, \mu + \rho \rangle E_{\lambda+\mu},$$

where $\langle \cdot, \cdot \rangle$ is a non-degenerate skew-symmetric bilinear form on $\mathbb{C}^2$:

$$\langle (a, b), (c, d) \rangle = bc - ad.$$

The Lie algebra $W_\pi \subset \mathcal{P}$ is the subalgebra with basis $\{E_\lambda\}_{\lambda \in \pi(\Lambda)}$.

Remark 1.2. 1. *W_π is simple-graded iff $\pi(\Lambda) \not\subset \mathbb{C}\rho$ and $2\rho \notin \pi(\Lambda)$.*

2. *In case $\dim \mathbb{C}\pi(\Lambda) = 1$, this W_π becomes a **`generalized Witt algebra`**: $\{E_\lambda, E_\mu\} = \langle \rho, \mu - \lambda \rangle E_{\lambda+\mu}$.*

3. *In case $\dim \mathbb{C}\pi(\Lambda) = 2$, $H^2(W_\pi, \mathbb{C}) = 0$.*

Hence, even with this strong restriction, the classification involves a continuous parameter !

Remark 1.3. *The classification problem of simple* Λ*-graded Lie algebras with the conditions* $\dim \mathfrak{g}_\lambda \leq 1$ *is still open !*

N.B. The classification of all possible $\mathbb{Z}^n$-gradation on a given Lie algebra is even non-trivial. (Good example is to find the $\mathbb{Z}$-graded structure of twisted loop algebra of type $A_2^{(2)}$.)

2 Representations of W_π

Here, we consider the representations with bounded multiplicity.

2.1 Witt algebra I

Let $\mathbf{W} = \mathbb{C}[t, t^{-1}]\frac{d}{dt}$ be the Witt algebra. For $m \in \mathbb{Z}$, set $L_m = -t^{m+1}\frac{d}{dt}$. It is clear that $[L_m, L_n] = (m-n)L_{m+n}$.

In 1985, Kaplansky and Santharoubane [KS] classified all $\mathbb{Z}$-graded W-module $M = \bigoplus_m M_m$ such that $\dim M_m = 1$. Here are examples:

1. For $(u, \delta) \in \mathbb{C}/\mathbb{Z} \times \mathbb{C}$, $\Omega_u^\delta := \bigoplus_{x \in u} \mathbb{C} e_x^\delta$ with
$$L_m.e_x^\delta := (m\delta + x)e_{x+m}^\delta.$$
2. The *A-family* $(A_{a,b})_{(a,b)\in\mathbb{C}^2}$. Here, $A_{a,b}$ is the $\mathbf{W}$-module with basis $\{e_n^A\}_{n\in\mathbb{Z}}$ and the action given by the formula:
$$L_m.e_n^A := \begin{cases} (m+n)e_{m+n}^A & n \neq 0, \\ (am^2 + bm)e_m^A & n = 0. \end{cases}$$
3. The *B-family* $(B_{p,q})_{(p,q)\in\mathbb{C}^2}$. Here, $B_{p,q}$ is the $\mathbf{W}$-module with basis $\{e_n^B\}_{n\in\mathbb{Z}}$ and the action gien by the formula:
$$L_m.e_n^B := \begin{cases} ne_{m+n}^B & m+n \neq 0, \\ (pm^2 + qm)e_0^B & m+n = 0. \end{cases}$$

Set $\overline{A} := A/\mathbb{C}$. We remark that there are two exact sequences:
$$0 \longrightarrow \overline{A} \longrightarrow A_{a,b} \longrightarrow \mathbb{C} \longrightarrow 0,$$
$$0 \longrightarrow \mathbb{C} \longrightarrow B_{a,b} \longrightarrow \overline{A} \longrightarrow 0.$$

These exact sequences do not split, except for $(a,b)=(0,0)$. Therefore, the A-family is a deformation of $\Omega^1_0 \cong A_{0,1}$ and the B-family is a deformation of $\Omega^0_0 \cong B_{0,1}$. Except for the pevious two isomorphisms and the obvious $A_{0,0} \cong B_{0,0} \cong \overline{A} \oplus \mathbb{C}$, there are some repetitions in the previous list due to the following isomorphisms:

1. the de Rham differential $d : \Omega^0_u \longrightarrow \Omega^1_u$, if $u \not\equiv 0 \mod \mathbb{Z}$,

2. $A_{\lambda a, \lambda b} \cong A_{a,b}$ and $B_{\lambda a, \lambda b} \cong B_{a,b}$ for $\lambda \in \mathbb{C}^*$.

There is no other isomorphism in the class $\mathcal{S}$ beside those described above. From now on, we will consider $(a,b) \neq (0,0)$ as a projective coordinate, and the indecomposable modules in the AB-families are now parametrizes by $\mathbb{P}^1$.

The classification of $\mathbf{W}$-modules of the class $\mathcal{S}$ has been achieved by I. Kaplansky et L. J. Santharoubane.

Theorem 2.1. *Let M be a $\mathbf{W}$-module of the class $\mathcal{S}$.*

1. *If M is irreducible, then there exists $(u,\delta) \in \mathbb{C}/\mathbb{Z} \times \mathbb{C}$, with $(u,\delta) \neq (0,0)$ or $(0,1)$, such that $M \cong \Omega^\delta_u$.*

2. *If M is reducible and indecomposable, then M is isomorphic to either A_ξ or B_ξ for some $\xi \in \mathbb{P}^1$.*

3. *Otherwise, M is isomorphic to $\overline{A} \oplus \mathbb{C}$.*

2.2 Witt algebra II

Here, we show that the three family of $\mathbf{W}$-modules introduced in the previous subsection can be realized in terms of the Poisson algebra $\mathcal{P}$ and its deformation.

Fix $\alpha \in \mathbb{C}^2$ s.t. $\langle \rho, \alpha \rangle \neq 0$. A key fact is that $\mathbf{W}$ can be realized as a subalgebra of $\mathcal{P}$: $\mathbf{W} \cong \bigoplus_m \mathbb{C}E_{m\alpha}; L_m \mapsto -\dfrac{1}{\langle \rho, \alpha \rangle} E_{m\alpha}$. Hence, we identify $\mathbf{W}$ with $\bigoplus_m \mathbb{C}E_{m\alpha}$.

First of all, let us realize Ω^δ_u. Let $\mu \in \mathbb{C}$ be a representative of $u \in \mathbb{C}/\mathbb{Z}$. Then, it is clear that the subspace of $\mathcal{P}$

$$\mathcal{T}_{\mu\alpha-(\delta+1)\rho} := \bigoplus_{n \in \mathbb{Z}} \mathbb{C}E_{(n+\mu)\alpha-(\delta+1)\rho}$$

is a $\mathbf{W}$-submodule isomoprhic to Ω_u^δ. Indeed, we have

$$\begin{aligned}\{L_m, E_{(n+\mu)\alpha-(\delta+1)\rho}\} =& -\frac{1}{\langle \rho, \alpha\rangle}\langle m\alpha+\rho, (n+\mu)\alpha-\delta\rho\rangle E_{(m+n+\mu)\alpha-(\delta+1)\rho} \\ =&(m\delta-(\mu+n))E_{(m+n+\mu)\alpha-(\delta+1)\rho}.\end{aligned}$$

To realize A, B-familly, we need some preparation.

For $\xi \in \mathbb{C}^2$, let δ_ξ be the derivation of $\mathcal{P}$ defined by

$$\delta_\xi(E_\lambda) := \{\log E_\xi, E_\lambda\} = \langle \xi+\rho, \lambda+\rho\rangle E_{\lambda-\rho}.$$

Let $\pi^{ab} : \mathcal{P} \twoheadrightarrow \mathcal{P}/\{\mathcal{P},\mathcal{P}\} \cong \mathbb{C}$ be the canonical projection and set

$$\kappa : \mathcal{P}\times\mathcal{P} \longrightarrow \mathbb{C}; \qquad (X, Y) \longmapsto \pi^{ab}(X\cdot Y).$$

It can be checked that for any $\xi \in \mathbb{C}^2$, we have

$$\kappa(\delta_\xi(X), Y) + \kappa(X, \delta_\xi(Y)) = 0.$$

The Lie algebra $\mathcal{P}_\xi$ is the vector space $\mathcal{P}\oplus\mathbb{C}c$ with its Lie bracket $[\cdot,\cdot]$ is given by

$$\begin{aligned}&[X, Y] = \{X, Y\} + \kappa(X, \delta_\xi(Y))c,\\ &[c, \mathcal{P}_\xi] = 0.\end{aligned}$$

Remark that $\mathcal{P}_\xi = [\mathcal{P}_\xi, \mathcal{P}_\xi]\oplus\mathbb{C}E_{-2\rho}$. For $\eta \in \mathbb{C}^2$, we define the derivation $\tilde{\delta}_\eta$ of $\mathcal{P}_\xi$ by

$$\tilde{\delta}_\eta(E_\lambda) = \begin{cases}\langle \eta+\rho, \lambda+\rho\rangle E_{\lambda-\rho} & \text{if } \lambda \neq -\rho,\\ \langle \eta+\rho, \xi+\rho\rangle c & \text{if } \lambda = -\rho.\end{cases}$$

Set $\mathcal{P}_{\xi,\eta} = \mathcal{P}_\xi \ltimes \mathbb{C}\tilde{\delta}_\eta$. It can be checked that $\mathbf{W}$ acts on $\mathcal{P}_{\xi,\eta}$, $[\mathcal{P}_\xi, \mathcal{P}_\xi]\ltimes\mathbb{C}\tilde{\delta}_\eta$ and $\mathbb{C}E_{-\rho}$, hence on the subquotient

$$\overline{\mathcal{P}}_{\xi,\eta} := [\mathcal{P}_\xi, \mathcal{P}_\xi]\ltimes\mathbb{C}\tilde{\delta}_\eta/\mathbb{C}E_{-2\rho}.$$

For $(a, b), (p, q) \in \mathbb{C}^2$, set $\eta = b\alpha-(a+1)\rho$ and $\xi = q\alpha-(p+1)\rho$. Then, it can be verified that the $\mathbf{W}$-submodule

$$\bigoplus_{m\neq 0}\mathbb{C}E_{m\alpha-\rho}\oplus\mathbb{C}\tilde{\delta}_\eta \subset \overline{\mathcal{P}}_{\xi,\eta}$$

is isomoprhic to $A_{a,b}$ and

$$\bigoplus_{m\neq 0}\mathbb{C}E_{m\alpha-2\rho}\oplus\mathbb{C}c \subset \overline{\mathcal{P}}_{\xi,\eta}$$

is isomorphic to $B_{p,q}$.

2.3 W_π

What should be notice for W_π is that since it is realized as a Lie subalgebra of $\mathcal{P}$, it can be checked that W_π acts on $\mathcal{P}_{\xi,\eta}$, $[\mathcal{P}_\xi, \mathcal{P}_\xi] \ltimes \mathbb{C}\tilde{\delta}_\eta$ and $\mathbb{C}E_{-\rho}$, hence on $\overline{\mathcal{P}}_{\xi,\eta}$ of the previous subsection. Hence, with the same recipe, one can get indecomposable Λ-graded multiplicity free W_π-modules $\mathcal{M} = \bigoplus_{\lambda \in \pi(\Lambda)} \mathcal{M}_\lambda$. In fact, we can show that above constructions exhaust all such W_π-modules ! The only thing what one should work carefully is that the result depend on π, namely, whether $\pi(\Lambda)$ contains ρ or not.

Next, we will show that, for W_π, there are intermediate modules with arbitrary homogenous components of any dimension $d \geq 3$.

Set $V = \mathbb{C}^2$. The symplectic structure on V induces a Lie bracket on SV defined by the requirement

$$[\alpha^m, \beta^n] = nm < \alpha|\beta > \alpha^{m-1}\beta^{n-1}$$

for all α, $\beta \in V$ and any n, $m \in \mathbb{Z}_{\geq 0}$. Since $[S^nV, S^mV] \subset S^{n+m-2}V$, it follows that S^2V is a Lie subalgebra and each component S^nV is a S^2V-module. Indeed S^2V is isomorphic with $\mathfrak{sl}(2)$ and S^nV is the irreducible $\mathfrak{sl}(2)$-module of dimension $n+1$.

Since $\mathcal{P}$ is a Poisson algebra, it will be convenient to denote by $\mathcal{P}_-$ the underlying Lie algebra and by $\mathcal{P}_+$ the underlying commutative algebra. Set:

$$\mathcal{P}^{ext} = \mathcal{P}_- \ltimes \mathcal{P}_+ \otimes S^2V$$

Clearly $\mathcal{P}^{ext}$ has a structure of Lie algebra, and for any n, $\mathcal{P}_+ \otimes S^nV$ is a $\mathcal{P}^{ext}$-module. Define a map $c : \mathcal{P}_- \to \mathcal{P}_+ \otimes S^2V$ by the formula:

$$c(L_\lambda) = 1/2 L_{\lambda-\rho} \otimes \lambda(\lambda+\rho)$$

and for $X \in \mathcal{P}_-$ set $j(X) = X + c(X)$.

Lemma 2.2. *The map $j : \mathcal{H}_- \to \mathcal{H}^{ext}$ is a Lie algebra morphism, i.e. the map c satisfies the Maurer Cartan equation*

$$c([X,Y]) = X.c(Y) - Y.c(X) + [c(X), c(Y)]$$

for any X, $Y \in \mathcal{P}_-$.

For any $n \geq 0$, $\mathcal{P}_+ \otimes S^nV$ is naturally a $\mathcal{P}^{ext}$-module. Then $\mathcal{M}^n := j_*\mathcal{P}_+ \otimes S^nV$ is a $\mathbb{C}^2$-graded $\mathcal{P}_-$-module, whose all homogenous components have dimension n. Given $\beta \in \mathbb{C}^2/\pi(\Lambda)$, set

$$\mathcal{M}^n(\beta) = \bigoplus_{\mu \in \beta} \mathcal{M}^n_\mu$$

It follows that $\mathcal{M}^n(\beta)$ is a graded W_π-module whose all non-zero components have dimension n. Recall that we assume that W_π is not a generalized Witt algebra, i.e, we assume that $\pi(\Lambda)$ does not lie in a complex line.

Lemma 2.3. *For any $n \geq 3$, the W_π-module $\mathcal{M}^n(\beta)$ is irreducible. Moreover, given two distinct $\pi(\Lambda)$-cosets $\beta \neq \beta'$, the W_π-modules $\mathcal{M}^n(\beta)$ and $\mathcal{M}^n(\beta')$ are not isomorphic.*

Conjecture 2.4. *Any irreducible W_π-module of the intermediate series has all its homogenous components of dimension ≤ 1 or it is isomorphic to $\mathcal{M}^n(\beta)$ $(n \geq 3)$ or $\overline{\mathcal{P}}(\beta)$ for some $\beta \in \mathbb{C}^2/\pi(\Lambda)$.*

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