MODELING DYNAMIC PHASES IN STELLAR EVOLUTION USING MULTIDIMENSIONAL HYDRODYNAMICS SIMULATIONS

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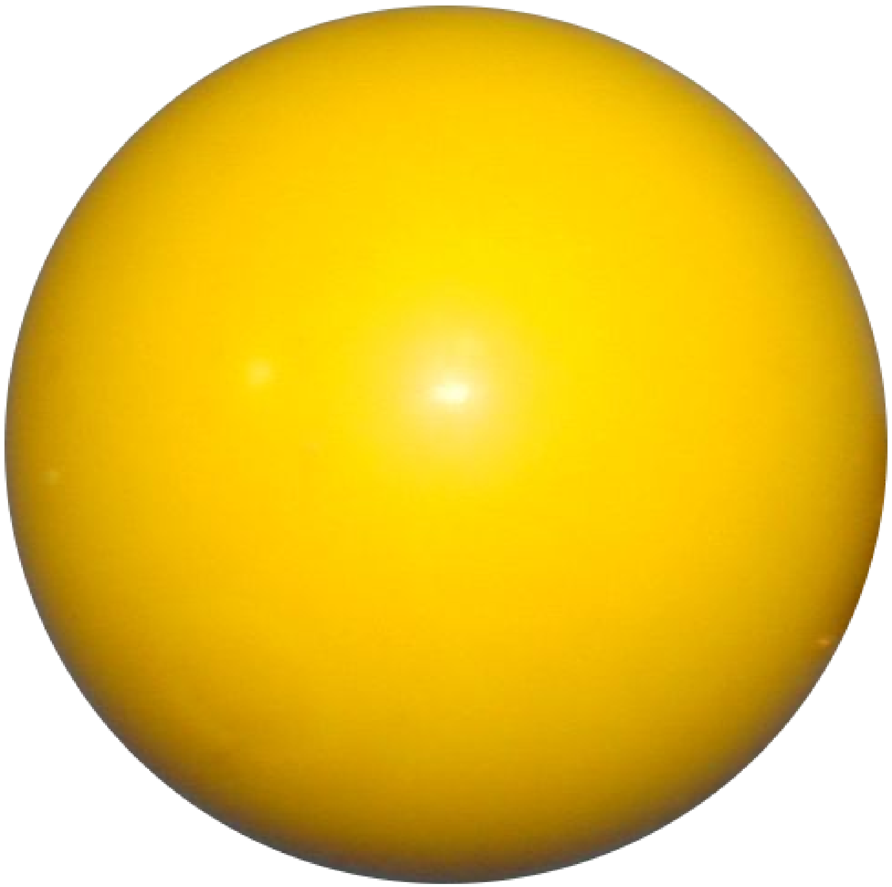
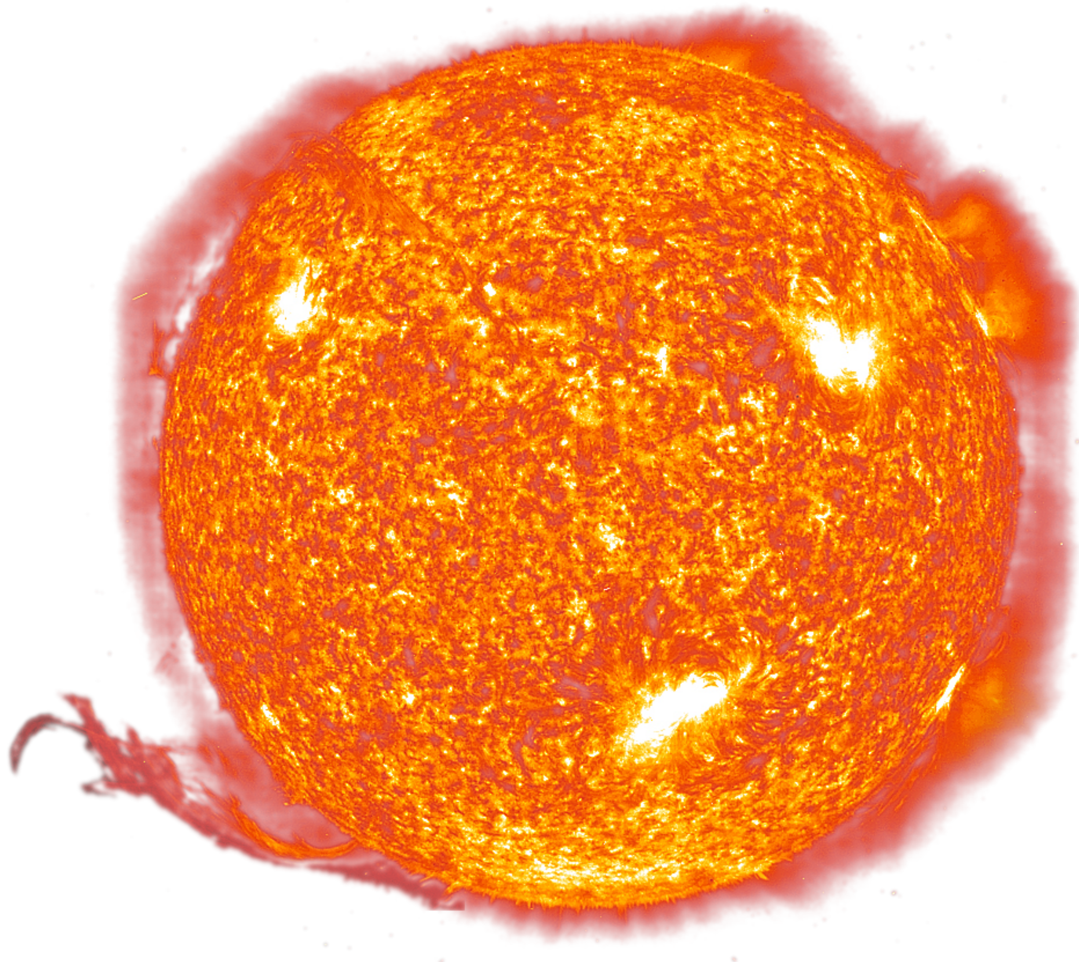
MOTIVATION

CLASSIC STELLAR EVOLUTION (SE)

- one-dimensional, spherically symmetric
- sequence of hydrostatic profiles
- evolution driven by nuclear burning
- parametrized treatment of
  convection, convective boundary mixing, all kinds of instabilities, …
- rotation only possible for certain classes of profiles



- spherical symmetry
- no dynamical effects
- turbulence model with free parameters



- no enforced symmetry
- full equations of fluid dynamics
- turbulence from first principles

A MATTER OF TIMESCALES

- hydrodynamics occurs on the *free-fall timescale*
  $\tau_{\mathrm{ff},\odot} = 27\,\mathrm{min}$
- thermal structure changes on the *Kelvin-Helmholtz timescale*
  $\tau_{\mathrm{KH},\odot} = 2 \times 10^7\,\mathrm{a}$
- nuclear burning occurs on the *nuclear timescale*
  $\tau_{\mathrm{nuc},\odot} = 10^{11}\,\mathrm{a} \approx 10^3\,\tau_{\mathrm{KH}} \approx 10^{15}\,\tau_{\mathrm{ff}}$

no hydrodynamics simulation over significant part of stellar lifetime

SO WHAT CAN WE DO?

- simulate only a small fraction of an evolutionary phase
  - to see if the 3D structure is consistent with SE code
  - to adjust prescriptions in SE code
- simulate phases that are reasonably short
  - short-lived instabilities
    improve treatment in SE code
  - very late burning stages
    3D progenitors for supernovae?

LOW MACH NUMBER HYDRODYNAMICS

WHAT?

Mach number $M = \frac{u}{c} = \frac{\text{fluid velocity}}{\text{speed of sound}}$

WHY?

Flows in the stellar interior are usually at low Mach numbers.

speed of sound $c = \sqrt{\gamma \frac{p}{\rho}} \propto \sqrt{\frac{T}{\mu}}$

EULER EQUATIONS

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial y} + \frac{\partial \mathbf{H}(\mathbf{U})}{\partial z} = 0$$

$$\mathbf{U} = (\rho, \rho u, \rho v, \rho w, \rho E, \rho X_i)^T$$

$$\mathbf{F} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ u(\rho E + p) \\ \rho u X_i \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v w \\ v(\rho E + p) \\ \rho v X_i \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} \rho w \\ \rho u w \\ \rho v w \\ \rho w^2 + p \\ w(\rho E + p) \\ \rho w X_i \end{pmatrix}$$

NONDIMENSIONALIZATION

Replace all quantities with unitless number times reference quantity

$$\rho = \hat{\rho}\,\rho_{\mathrm{r}}\,,$$

$$p = \hat{p}\,p_{\mathrm{r}}\,,$$

$$\dots$$

NONDIMENSIONAL EULER EQUATIONS

$$\frac{\partial \hat{\mathbf{U}}}{\partial \hat{t}} + \frac{\partial \hat{\mathbf{F}}(\hat{\mathbf{U}})}{\partial \hat{x}} + \frac{\partial \hat{\mathbf{G}}(\hat{\mathbf{U}})}{\partial \hat{y}} + \frac{\partial \hat{\mathbf{H}}(\hat{\mathbf{U}})}{\partial \hat{z}} = 0$$

$\hat{\mathbf{U}} = (\hat{\rho}, \hat{\rho}\hat{u}, \hat{\rho}\hat{v}, \hat{\rho}\hat{w}, \hat{\rho}\hat{E}, \hat{\rho}X_i)^T$, Mach number: $M_\mathrm{r} = \frac{u_r}{c_r}$

$$\hat{\mathbf{F}} = \begin{pmatrix} \hat{\rho}\hat{u} \\ \hat{\rho}\hat{u}^2 + \frac{\hat{p}}{M_\mathrm{r}^2} \\ \hat{\rho}\hat{u}\hat{v} \\ \hat{\rho}\hat{u}\hat{w} \\ \hat{u}(\hat{\rho}\hat{E} + \hat{p}) \\ \hat{\rho}\hat{u}X_i \end{pmatrix}, \quad \hat{\mathbf{G}} = \begin{pmatrix} \hat{\rho}\hat{v} \\ \hat{\rho}\hat{u}\hat{v} \\ \hat{\rho}\hat{v}^2 + \frac{\hat{p}}{M_\mathrm{r}^2} \\ \hat{\rho}\hat{v}\hat{w} \\ \hat{v}(\hat{\rho}\hat{E} + \hat{p}) \\ \hat{\rho}\hat{v}X_i \end{pmatrix}, \quad \hat{\mathbf{H}} = \begin{pmatrix} \hat{\rho}\hat{w} \\ \hat{\rho}\hat{u}\hat{w} \\ \hat{\rho}\hat{v}\hat{w} \\ \hat{\rho}\hat{w}^2 + \frac{\hat{p}}{M_\mathrm{r}^2} \\ \hat{w}(\hat{\rho}\hat{E} + \hat{p}) \\ \hat{\rho}\hat{w}X_i \end{pmatrix}$$

THE ROE SCHEME

Flux at interface

$$\mathbf{F}_{i+1/2} = \frac{1}{2}\Big(\mathbf{F}(\mathbf{U}^{L}_{i+1/2}) + \mathbf{F}(\mathbf{U}^{R}_{i+1/2}) - |A_{\mathrm{roe}}|(\mathbf{U}^{R}_{i+1/2} - \mathbf{U}^{L}_{i+1/2})\Big)$$

physical flux upwind term

A_{roe}: flux Jacobian $\left(\dfrac{\partial \mathbf{F}}{\partial \mathbf{U}}\right)$ at Roe average state

ASYMPTOTIC ANALYSIS

in primitive variables $\mathbf{V} = (\rho, u, v, w, p, X)^T$

flux Jacobian $\frac{\partial \mathbf{F}}{\partial \mathbf{U}}$

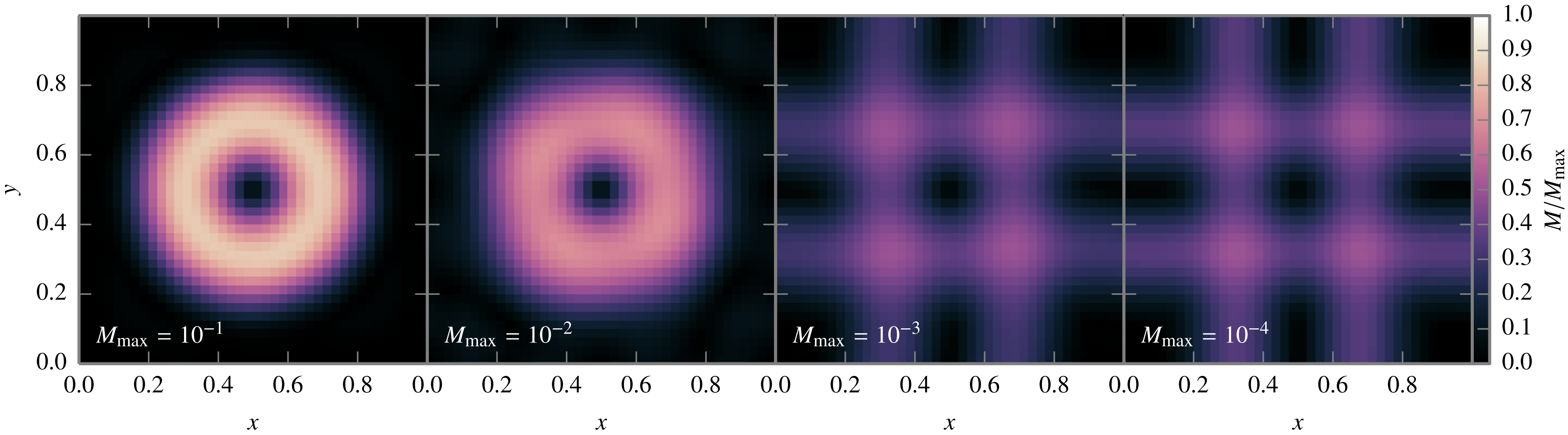
$$\begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & 0 & 0 \\ 0 & \mathcal{O}(1) & 0 & 0 & \mathcal{O}(\frac{1}{M_\mathrm{r}^2}) & 0 \\ 0 & 0 & \mathcal{O}(1) & 0 & \mathcal{O}(\frac{1}{M_\mathrm{r}^2}) & 0 \\ 0 & 0 & 0 & \mathcal{O}(1) & \mathcal{O}(\frac{1}{M_\mathrm{r}^2}) & 0 \\ 0 & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{O}(1) \end{pmatrix}$$

Roe matrix

$$\begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(M_\mathrm{r}) & \mathcal{O}(M_\mathrm{r}) & \mathcal{O}(M_\mathrm{r}) & \mathcal{O}(\frac{1}{M_\mathrm{r}}) & 0 \\ 0 & \mathcal{O}(\frac{1}{M_\mathrm{r}}) & \mathcal{O}(\frac{1}{M_\mathrm{r}}) & \mathcal{O}(\frac{1}{M_\mathrm{r}}) & \mathcal{O}(\frac{1}{M_\mathrm{r}}) & 0 \\ 0 & \mathcal{O}(\frac{1}{M_\mathrm{r}}) & \mathcal{O}(\frac{1}{M_\mathrm{r}}) & \mathcal{O}(\frac{1}{M_\mathrm{r}}) & \mathcal{O}(\frac{1}{M_\mathrm{r}}) & 0 \\ 0 & \mathcal{O}(\frac{1}{M_\mathrm{r}}) & \mathcal{O}(\frac{1}{M_\mathrm{r}}) & \mathcal{O}(\frac{1}{M_\mathrm{r}}) & \mathcal{O}(\frac{1}{M_\mathrm{r}}) & 0 \\ 0 & \mathcal{O}(M_\mathrm{r}) & \mathcal{O}(M_\mathrm{r}) & \mathcal{O}(M_\mathrm{r}) & \mathcal{O}(\frac{1}{M_\mathrm{r}}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{O}(1) \end{pmatrix}$$

GRESHO VORTEX

STANDARD ROE SCHEME


$M_{\max} = 10^{-1}$
$M_{\max} = 10^{-2}$
$M_{\max} = 10^{-3}$
$M_{\max} = 10^{-4}$
y
x
$M/M_{\max}$
0.0 0.2 0.4 0.6 0.8
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

PRECONDITIONED ROE SCHEME

(Miczek+, 2015)

$$\mathbf{F}_{i+1/2} = \frac{1}{2}\left(\mathbf{F}(\mathbf{U}^L_{i+1/2}) + \mathbf{F}(\mathbf{U}^R_{i+1/2}) - (P^{-1}|PA|)_{\mathrm{roe}}(\mathbf{U}^R_{i+1/2} - \mathbf{U}^L_{i+1/2})\right)$$

$$P_{\mathbf{V}} = \begin{pmatrix} 1 & n_x\frac{\rho\delta M_{\mathrm{r}}}{c} & n_y\frac{\rho\delta M_{\mathrm{r}}}{c} & n_z\frac{\rho\delta M_{\mathrm{r}}}{c} & 0 & 0 \\ 0 & 1 & 0 & 0 & -n_x\frac{\delta}{\rho c M_{\mathrm{r}}} & 0 \\ 0 & 0 & 1 & 0 & -n_y\frac{\delta}{\rho c M_{\mathrm{r}}} & 0 \\ 0 & 0 & 0 & 1 & -n_z\frac{\delta}{\rho c M_{\mathrm{r}}} & 0 \\ 0 & n_x\rho c\delta M_{\mathrm{r}} & n_y\rho c\delta M_{\mathrm{r}} & n_z\rho c\delta M_{\mathrm{r}} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\delta = \frac{1}{\min(1, \max(M, M_{\mathrm{cut}}))} - 1$$

ASYMPTOTIC ANALYSIS

PRECONDITIONED ROE SCHEME (MICZEK+, 2015)

in primitive variables $\mathbf{V} = (\rho, u, v, w, p, X)^T$

flux Jacobian $\qquad$ $(\boldsymbol{P}^{-1}|\boldsymbol{PA}|)_{\mathbf{roe}}$

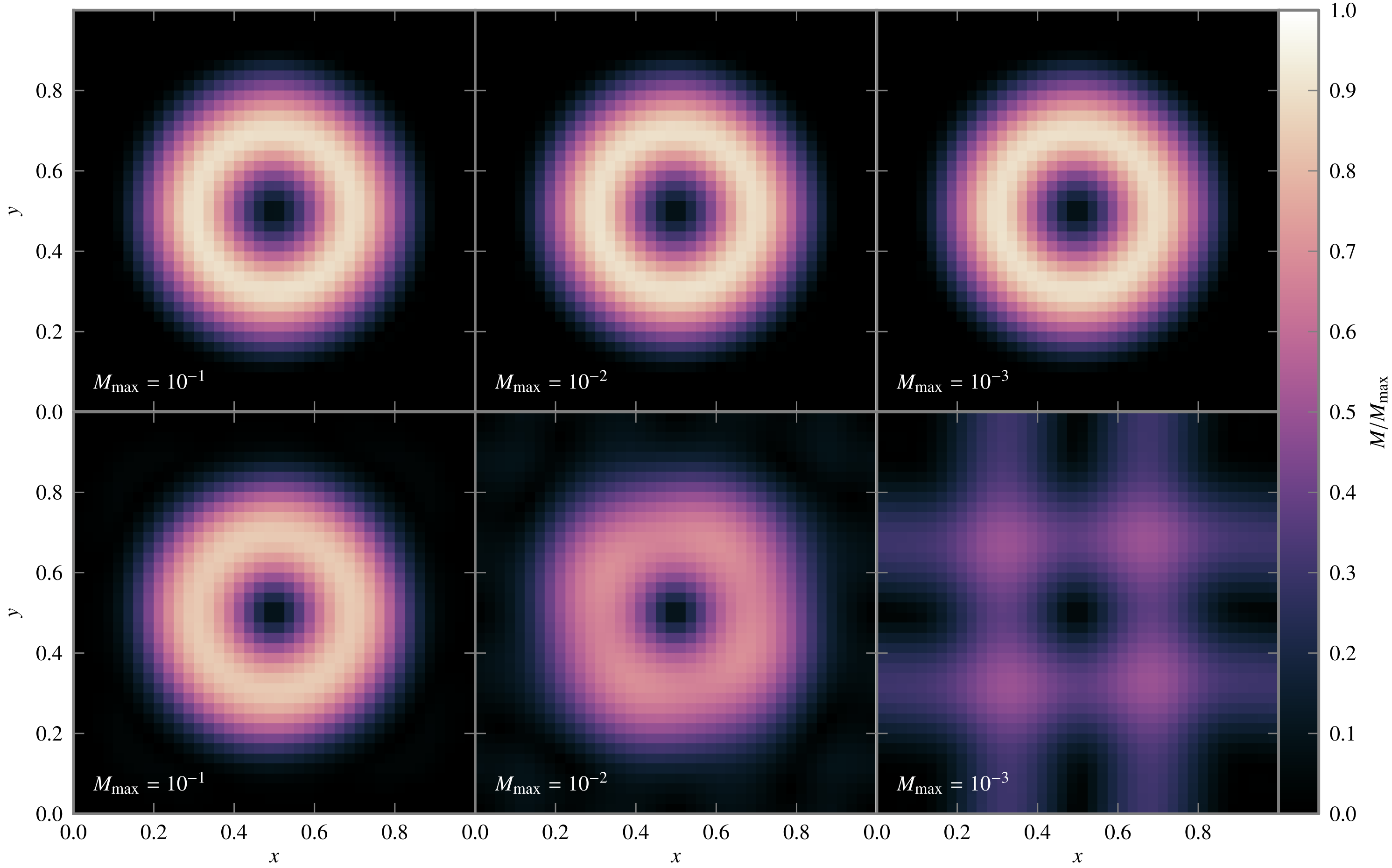
$$\begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & 0 & 0 \\ 0 & \mathcal{O}(1) & 0 & 0 & \mathcal{O}(\frac{1}{M_\mathrm{r}^2}) & 0 \\ 0 & 0 & \mathcal{O}(1) & 0 & \mathcal{O}(\frac{1}{M_\mathrm{r}^2}) & 0 \\ 0 & 0 & 0 & \mathcal{O}(1) & \mathcal{O}(\frac{1}{M_\mathrm{r}^2}) & 0 \\ 0 & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{O}(1) \end{pmatrix} \qquad \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & 0 \\ 0 & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(\frac{1}{M_\mathrm{r}^2}) & 0 \\ 0 & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(\frac{1}{M_\mathrm{r}^2}) & 0 \\ 0 & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(\frac{1}{M_\mathrm{r}^2}) & 0 \\ 0 & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{O}(1) \end{pmatrix}$$

GRESHO VORTEX

Mach	10^{-1}	10^{-2}	10^{-3}

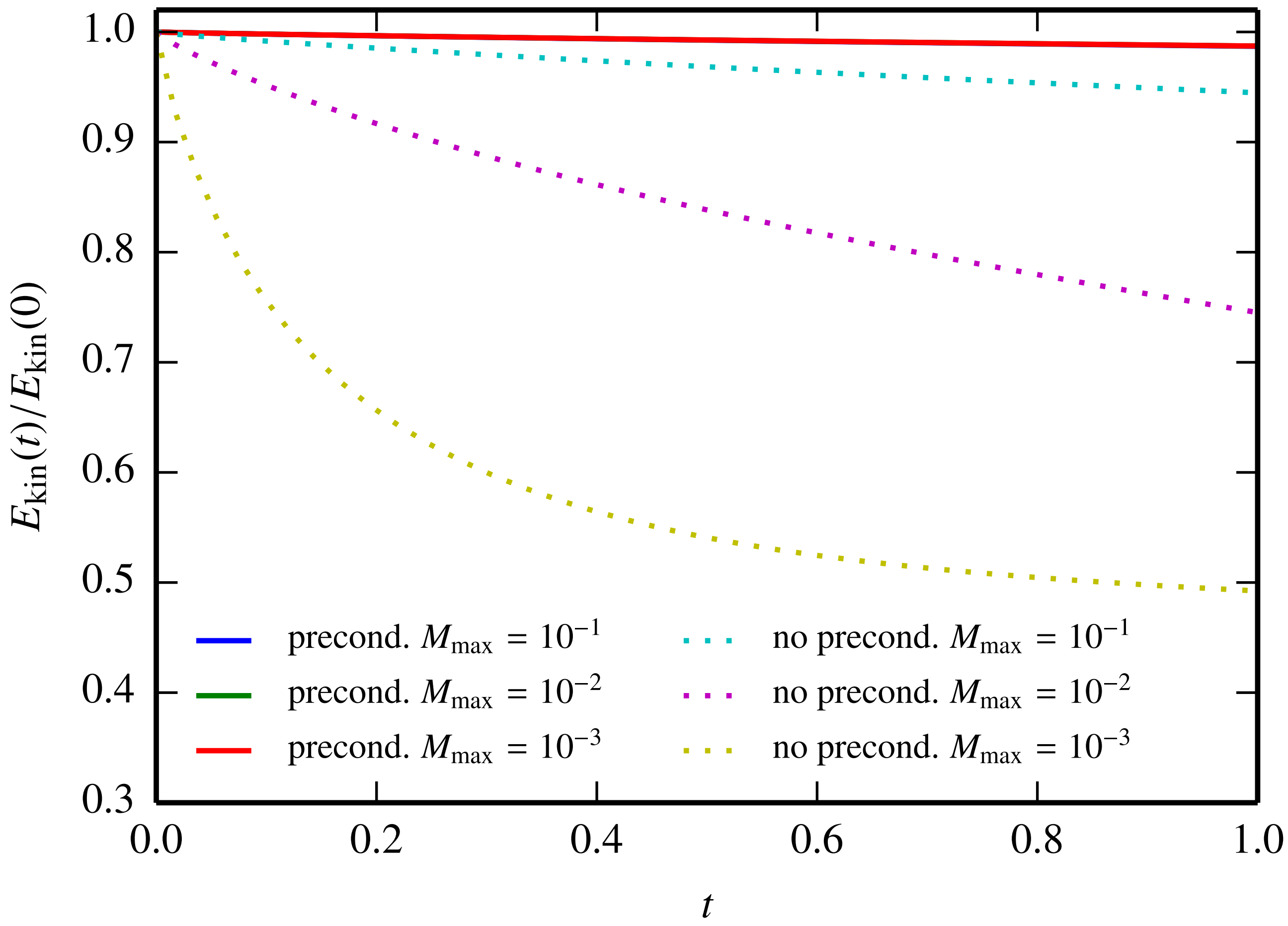
prec. Roe

Roe

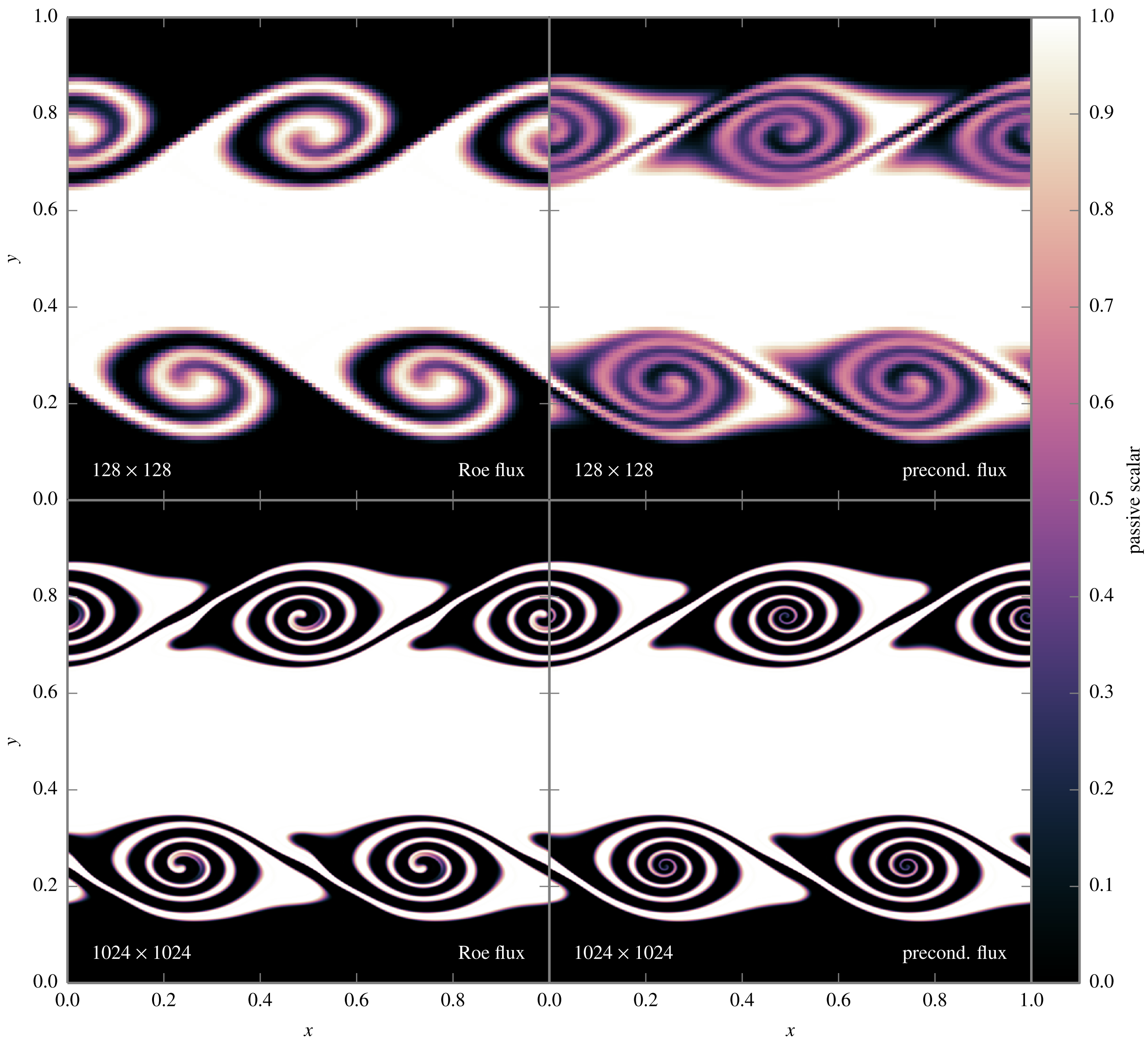


GRESHO VORTEX

KINETIC ENERGY



KELVIN–HELMHOLTZ INSTABILITY



OTHER APPROACHES

- modify underlying equations
- e.g. anelastic approximation, Maestro, …

works well for flows with only low Mach numbers

intermediate Mach numbers ($\sim 10^{-1}$) or mixed case needs the full Euler equations

THE TOOL

SEVEN-LEAGUE HYDRO (SLH) CODE

F. MICZEK, F. K. RÖPKE, P. V. F. EDELMANN

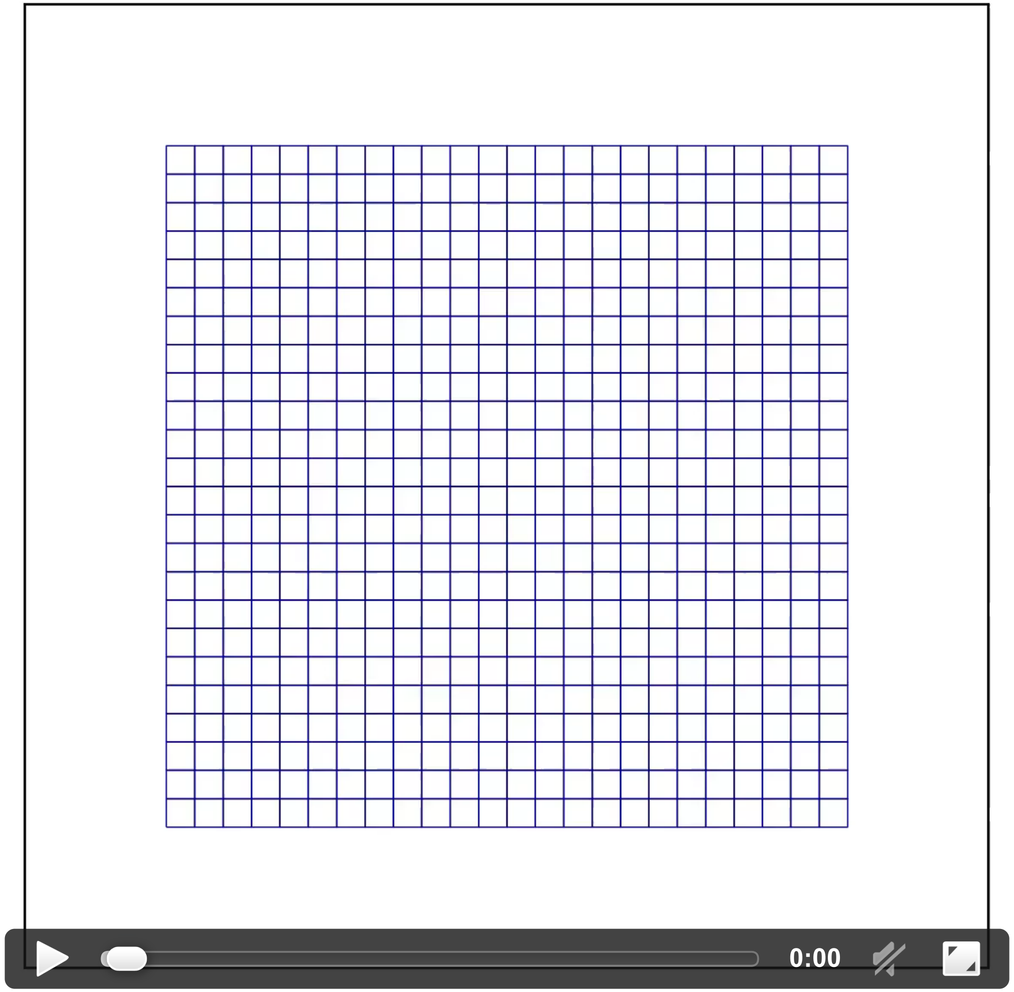
ALEJANDRO BOLAÑOS, ARON MICHEL, JONAS BERBERICH, FLORIAN LACH

FEATURES

- solves the compressible Euler equations in 1-, 2-, 3-D
- explicit and implicit time integration
- flux preconditioning to ensure correct behavior at low Mach numbers
- other low Mach number schemes (e.g. AUSM^{+}-up)
- works for low and high Mach numbers on the same grid
- hybrid (MPI, OpenMP) parallelization (works up to 100 000 cores)
- several solvers for the linear system:
  BiCGSTAB, GMRES, Multigrid, (direct)
- arbitrary curvilinear meshes
  using a rectangular computational mesh
- gravity solver (monopole, Multigrid)
- radiation in the diffusion limit
- general equation of state
- general nuclear reaction network

THE GRID

- Cartesian grids are badly adapted to spherical stars
- Spherical grids have singularities (center, axis)
- Map Cartesian computational grid to curvilinear grid
- Code stays simple, geometry encoded in *metric* terms



IMPLICIT HYDRODYNAMICS

explicit	**implicit**
time step constraint for stability $\Delta t_{\text{explicit}} \leq \mathrm{CFL}\,\frac{\Delta x}{\lvert u+c \rvert} \overset{u \ll c}{\approx} \mathrm{CFL}\,\frac{\Delta x}{c}$ sound crosses one cell per step	time step constraint for accuracy $\Delta t_{\text{implicit}} \leq \mathrm{CFL}\,\frac{\Delta x}{\lvert u \rvert}$ fluid crosses one cell per step

- Implicit time steps are larger by a factor of $1/M$.
- At each step a non-linear system has to be solved using Newton–Raphson.
- We need iterative linear solvers to invert the huge Jacobian.
- In SLH implicit time-stepping is more efficient for $M \lesssim 0.1$.

COMPUTATIONAL EFFORT

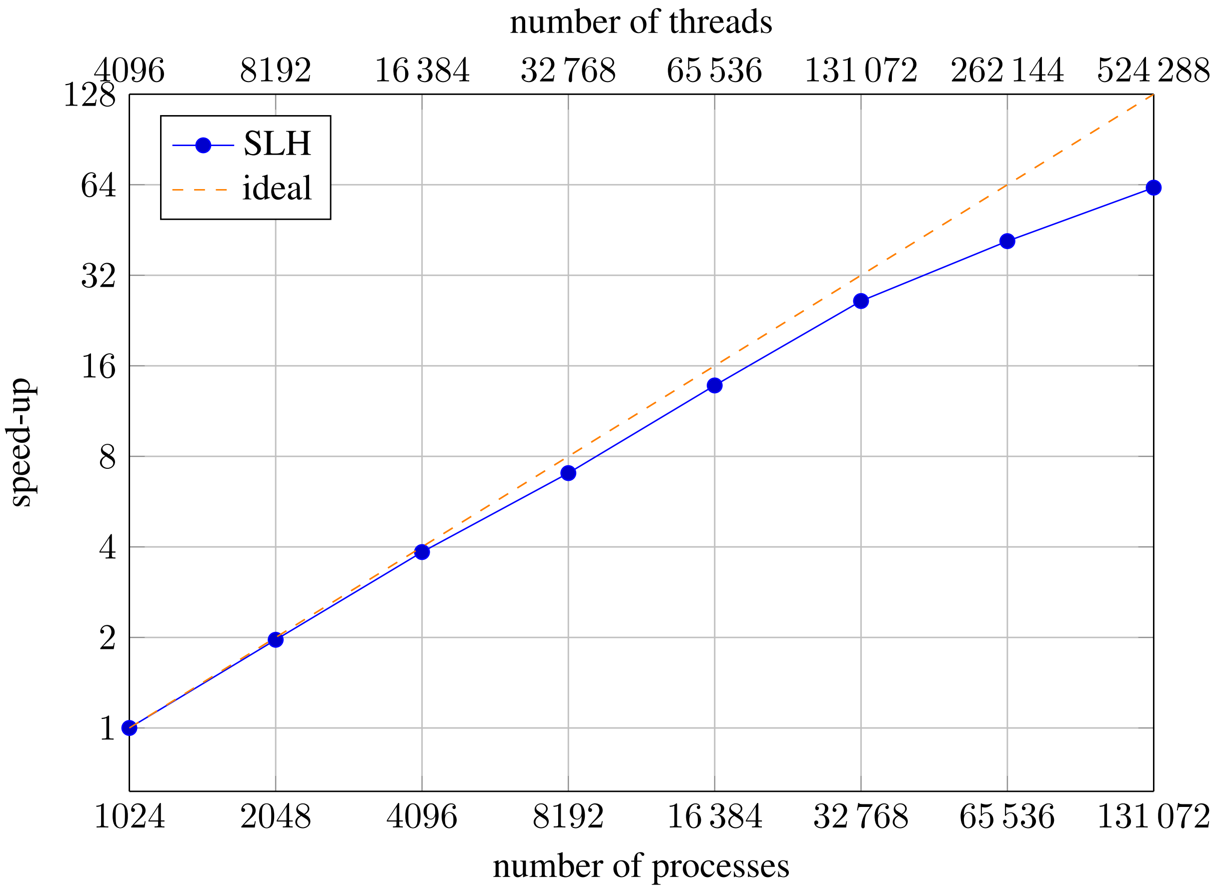
- size of matrix: $n \times n = (5N_xN_yN_z)^2$ ($n \approx 4 \times 10^{10}$ for 2048^3 grid)
- non-zero entries: $13 \times 5^2 \times N_xN_yN_z$ ($\approx 3 \times 10^{12}$ for 2048^3 grid)
- density of Jacobian: $13/(N_xN_yN_z)$ ($\approx 1.5 \times 10^{-7}\,\%$ for 2048^3 grid)
- storage of sparse Jacobian in memory: 21 TiB (2048^3 grid)
- Iterative solvers, Krylov subspace methods

SCALING ON LARGE HPC SYSTEMS

JUQUEEN

Jülich Supercomputing Center, Jülich, Germany

458 752 cores IBM PowerPC® A2, 1.6 GHz

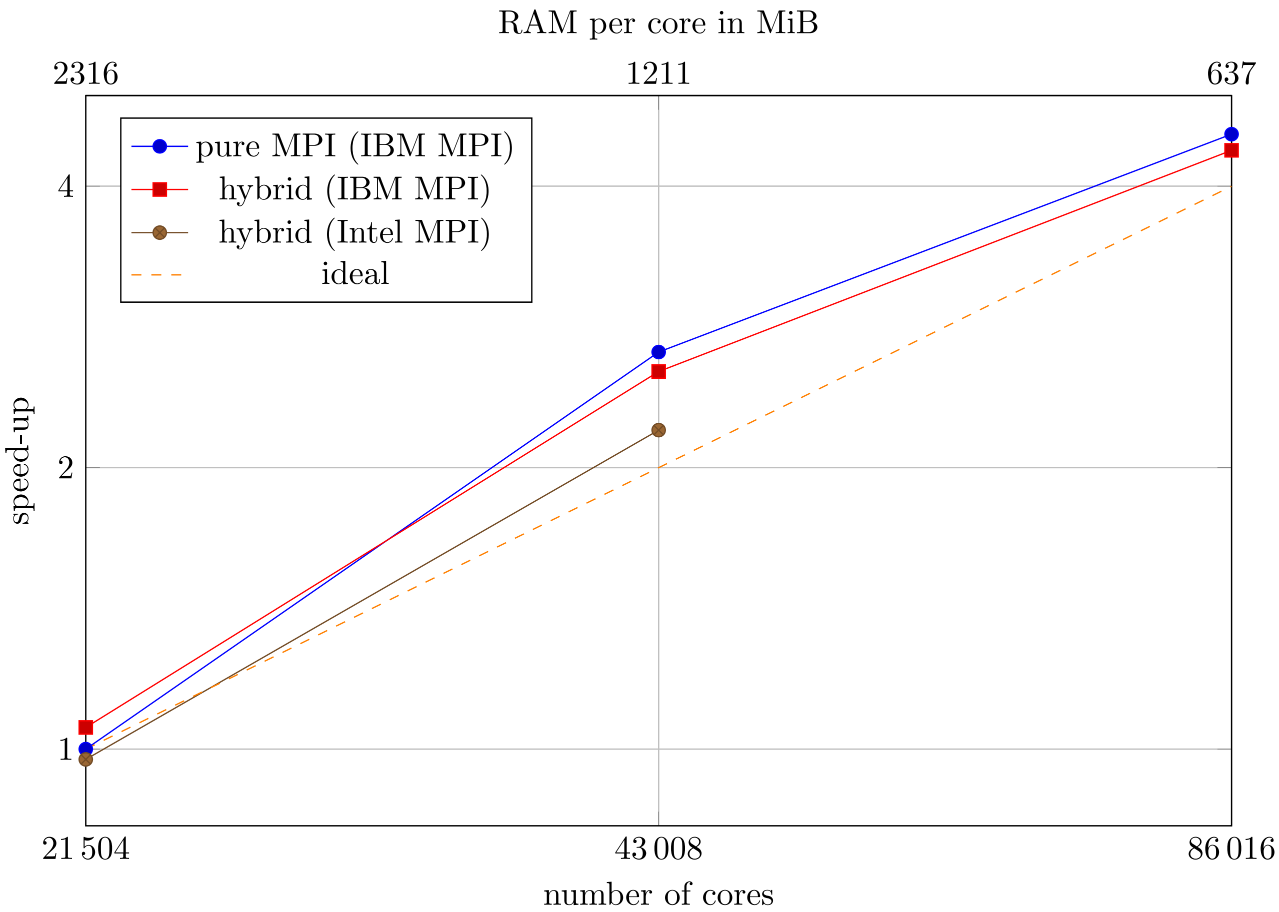


SCALING ON LARGE HPC SYSTEMS

SuperMUC Phase 2

Leibniz Computing Center, Garching, Germany

86 016 cores Intel Haswell architecture



DYNAMICAL SHEAR

collaborators:
Raphael Hirschi (Keele) and Cyril Georgy (Geneva)
Friedrich Röpke (HITS), Leonhard Horst (Würzburg)

STARS WITH ROTATION

- rotating stars are oblate
- not 1D problem anymore
- assume isobaric shells of constant Ω and composition (shellular rotation)
- slightly changed equations of stellar structure still in 1D (1.5D simulation)

SOME SHORTCOMINGS

- shellular structure is not certain
- some latitutes could be convective, others stable
- shear is introduced as diffusion coefficient
- shear criterion is resolution dependent (finite difference of Ω)

SOME STABILITY CRITERIA

- Convective Stability (Ledoux criterion)

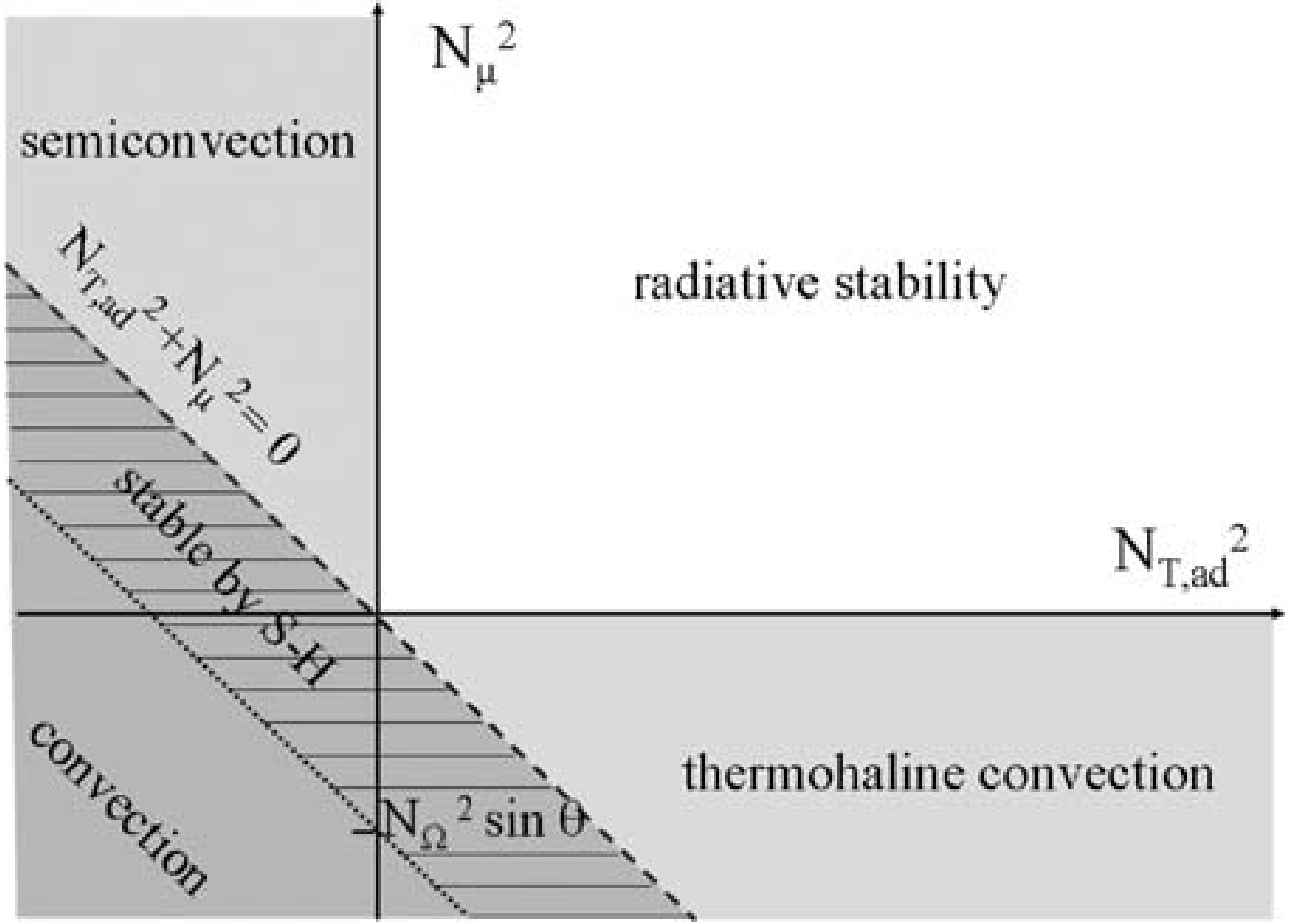
  Brunt–Väisälä frequency $N^2 = \frac{g\delta}{H_P}\left(\nabla_{\rm ad} - \nabla + \frac{\varphi}{\delta}\nabla_\mu\right)$

  unstable if $N^2 < 0$
- including rotation (Solberg–Høiland criterion)

  $$N^2 = \frac{g\delta}{H_P}\left(\nabla_{\rm ad} - \nabla + \frac{\varphi}{\delta}\nabla_\mu\right) + \sin\vartheta \frac{1}{\varpi^3}\frac{d(\Omega^2\varpi^4)}{d\varpi}$$
- Dynamical Shear

  Richardson number $Ri = \frac{N^2}{(\partial u/\partial z)^2}$

  unstable if $Ri > Ri_c = \frac{1}{4}$



$N_\mu{}^2$
semiconvection
radiative stability
$N_{T,ad}{}^2 + N_\mu{}^2 = 0$
$N_{T,ad}{}^2$
stable by S-H
convection
thermohaline convection
$-N_\Omega{}^2 \sin\theta$

Maeder (2009)

DYNAMICAL SHEAR

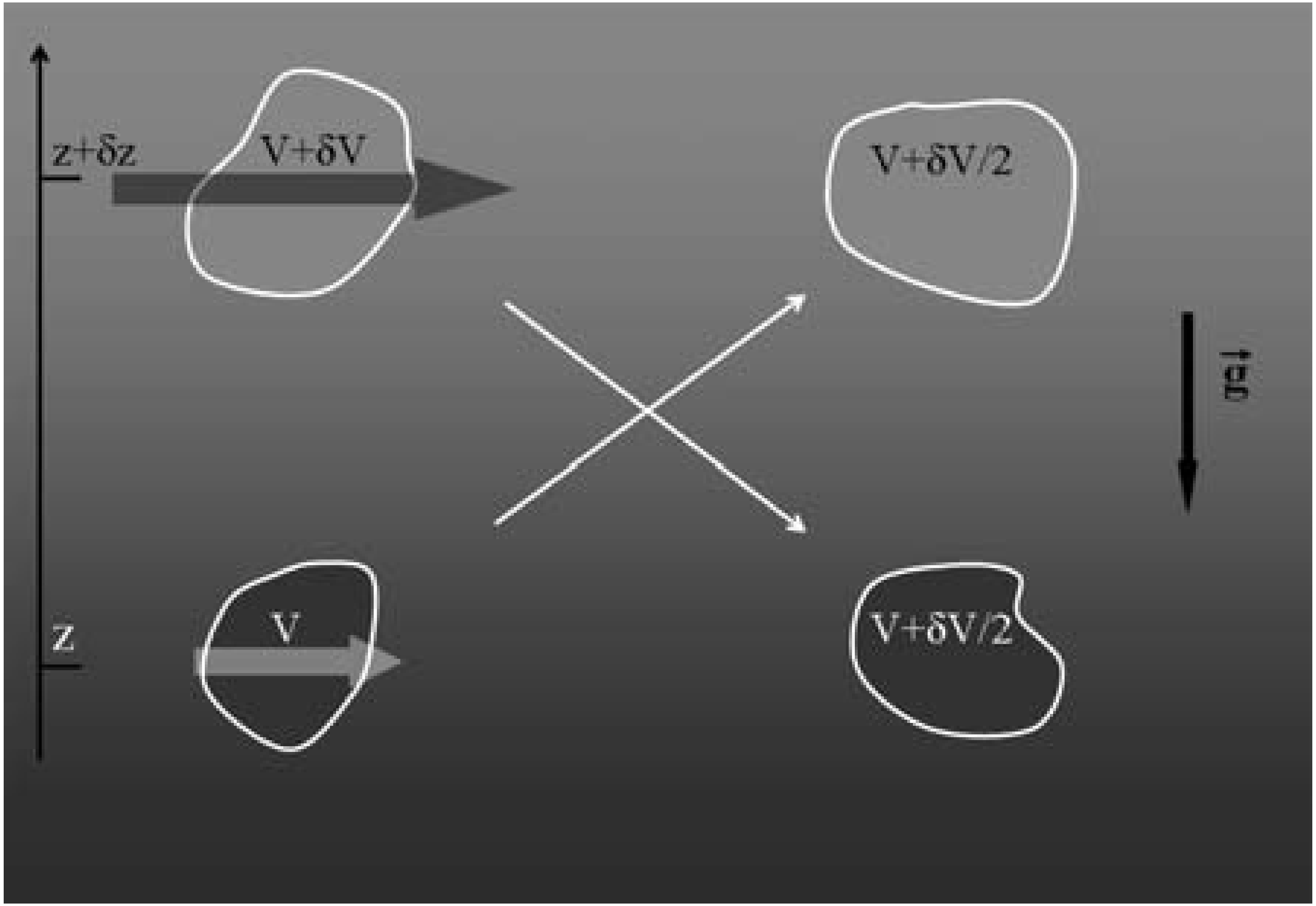


image credit: Maeder (2009), originally Talon (1997)

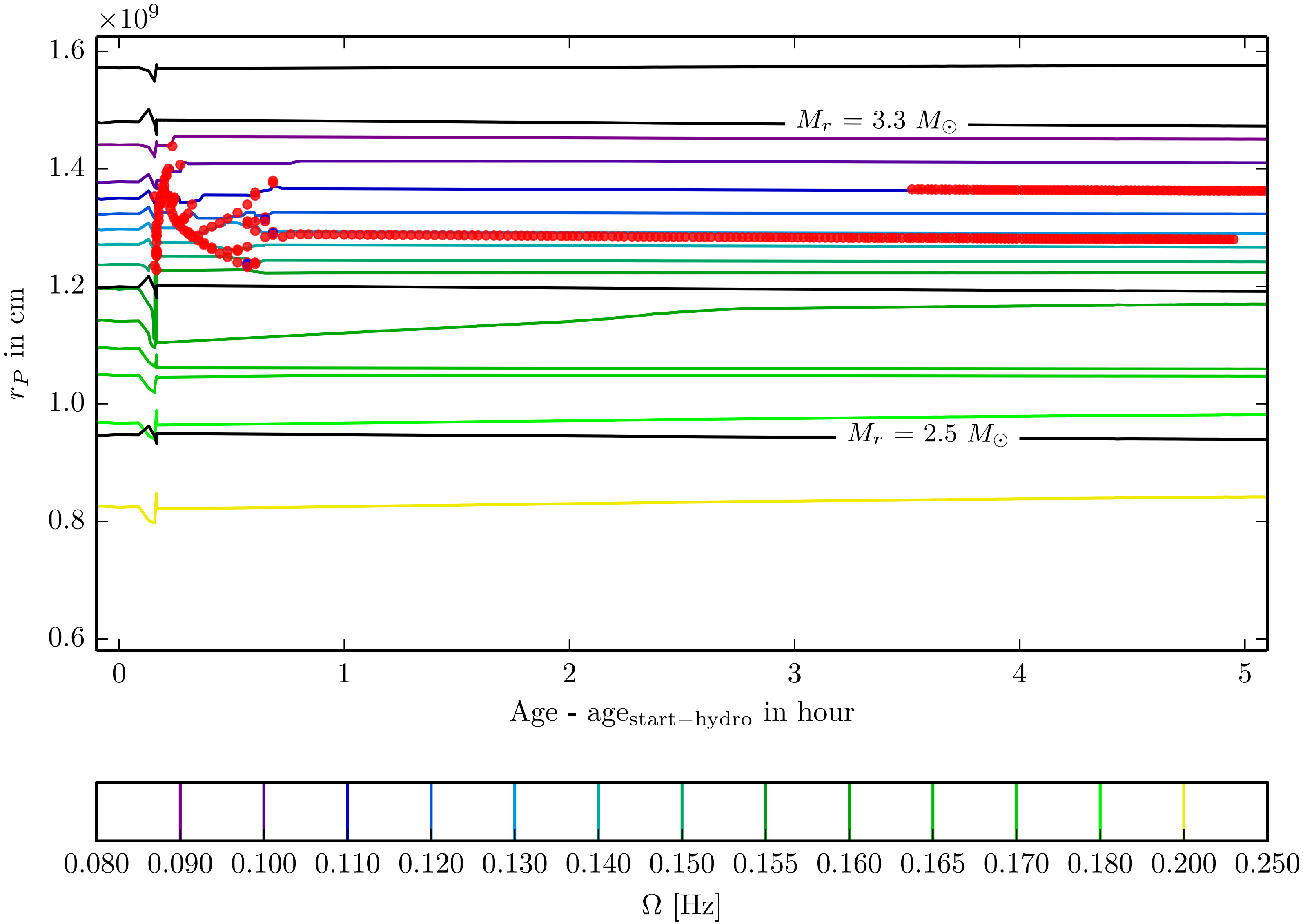
THE QUEST FOR A GOOD INITIAL MODEL

- should become shear unstable in stellar evolution code
- should not show other instabilities at the same time
- ideally similar time scale in stellar evolution and hydro code

A LOT OF WORK BY R. HIRSCHI

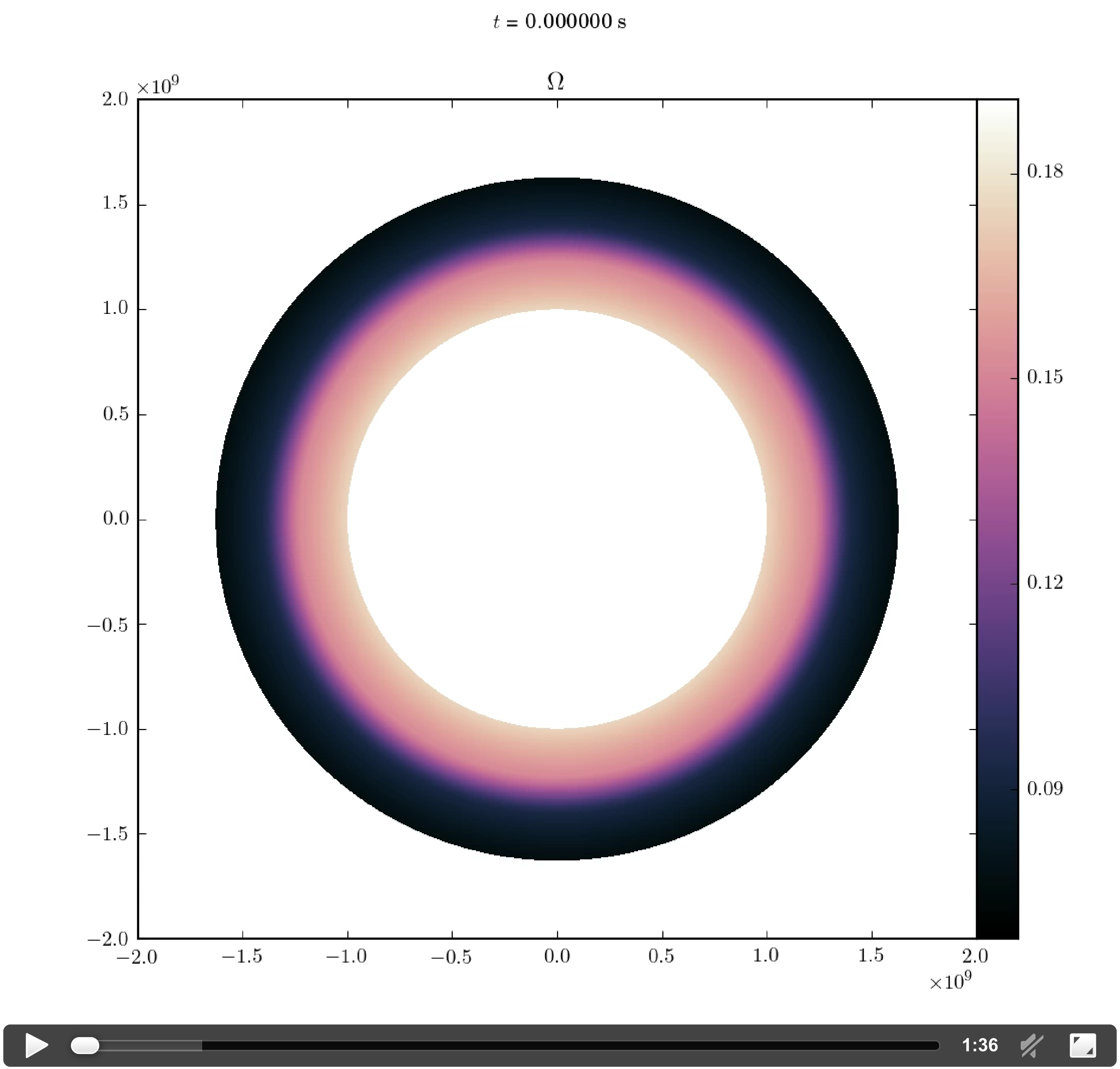
- $20\ M_\odot$ ZAMS star, 40% crit. rotation
- core O burning phase
- Ne burning shell
- convectively stable
- Ri unstable

SIMULATION WITH GENEC



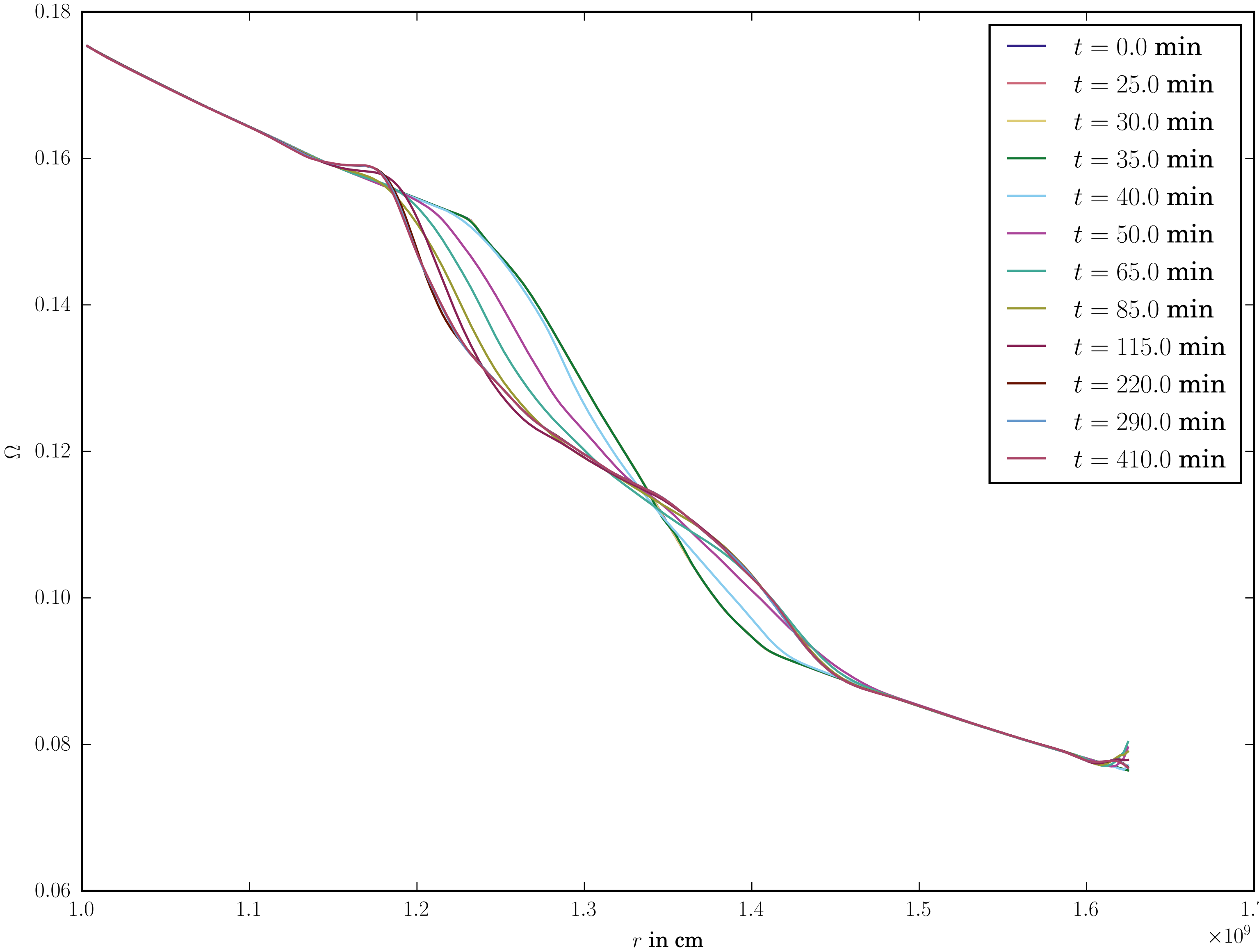
SIMULATIONS WITH SLH

- 2D equatorial plane
- more than 6 hours of physical time
- special mapping of GENEC data to keep convective stability



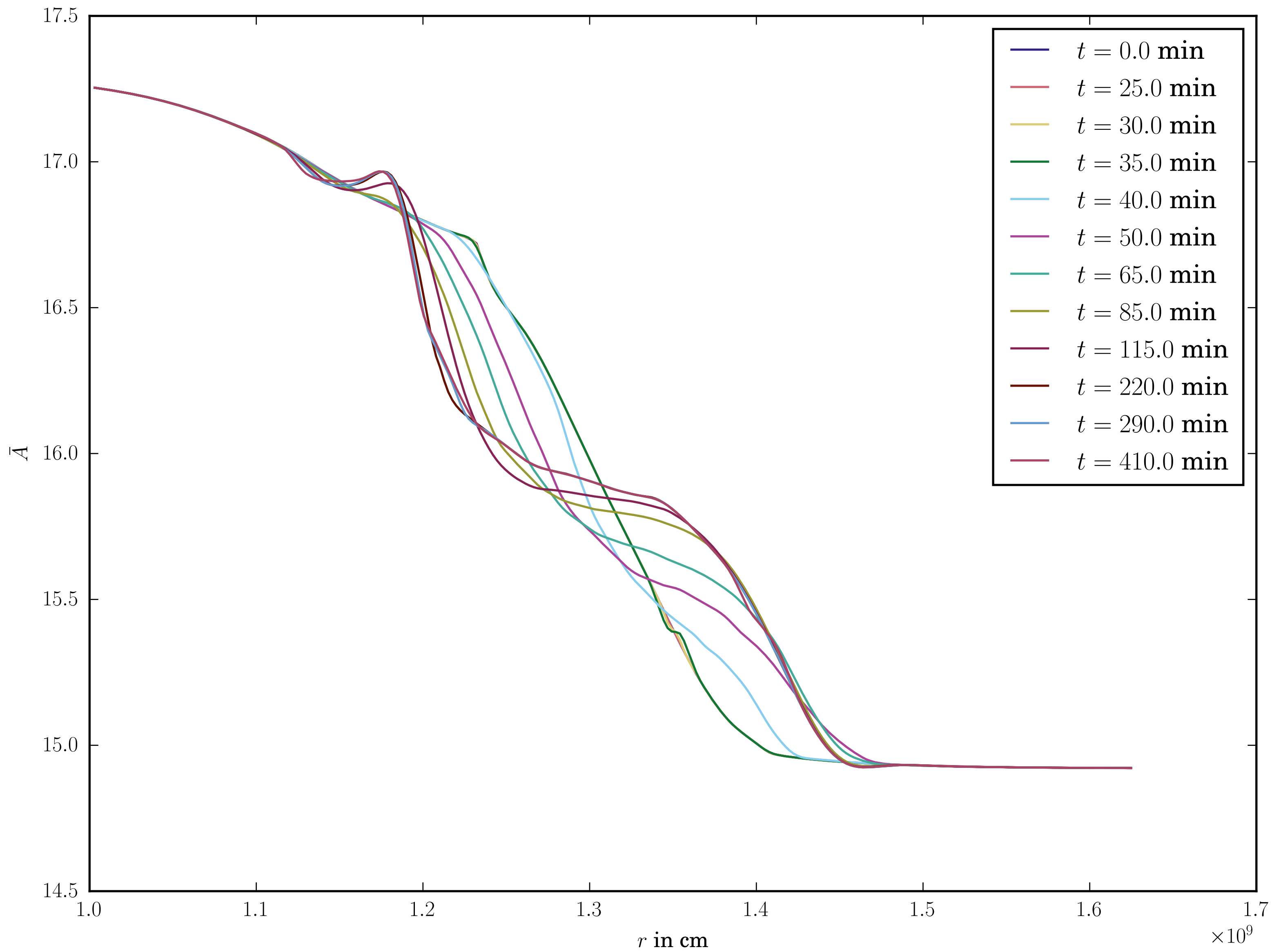
t = 0.000000 s
Ω
×10^9
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0 −1.5 −1.0 −0.5 0.0 0.5 1.0 1.5 2.0
×10^9
0.18
0.15
0.12
0.09
1:36

EVOLUTION OF ANGULAR MOMENTUM



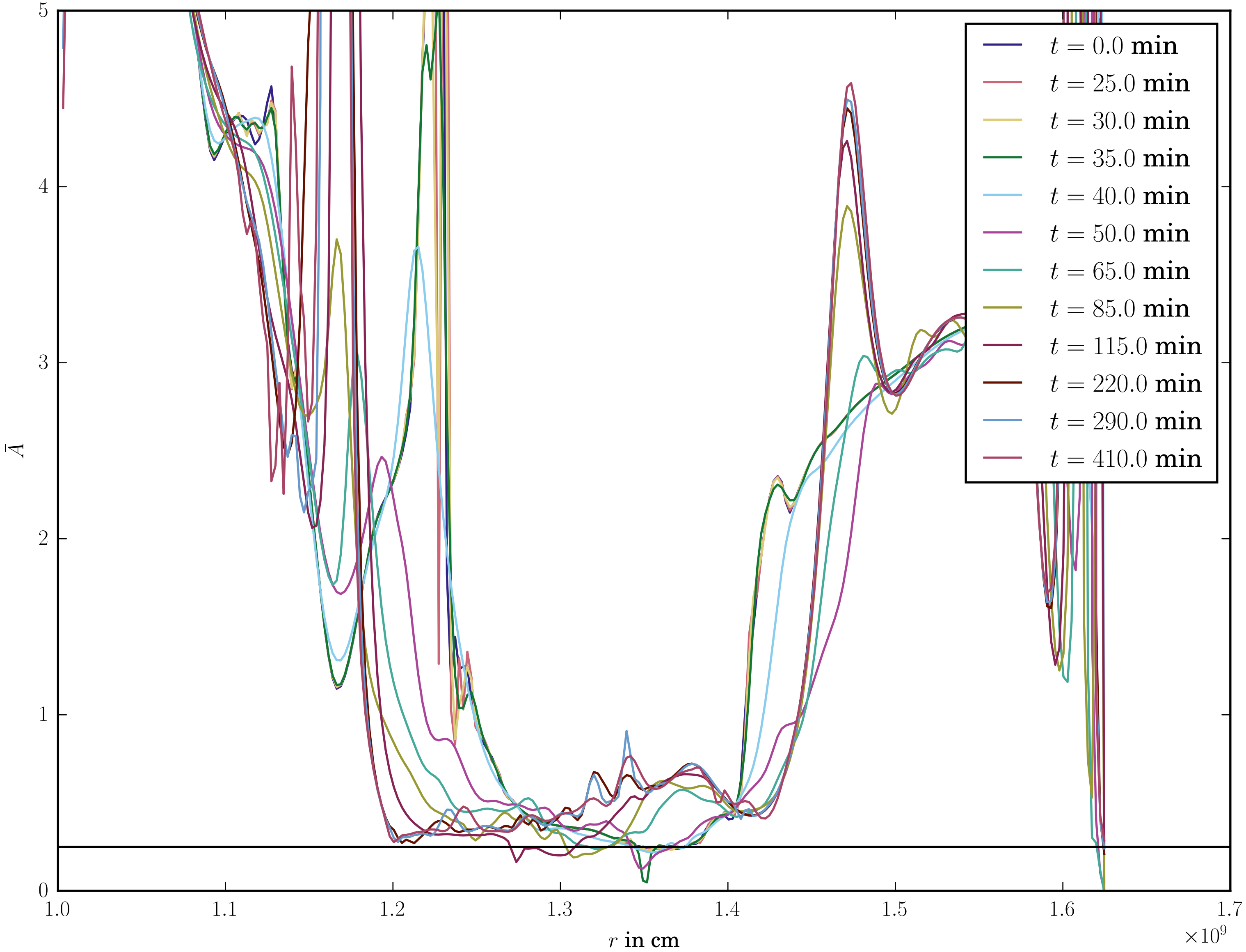
0.18
0.16
0.14
0.12
0.10
0.08
0.06
Ω
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
$\times 10^9$
r in cm
$t = 0.0$ min
$t = 25.0$ min
$t = 30.0$ min
$t = 35.0$ min
$t = 40.0$ min
$t = 50.0$ min
$t = 65.0$ min
$t = 85.0$ min
$t = 115.0$ min
$t = 220.0$ min
$t = 290.0$ min
$t = 410.0$ min

EVOLUTION OF MEAN ATOMIC MASS



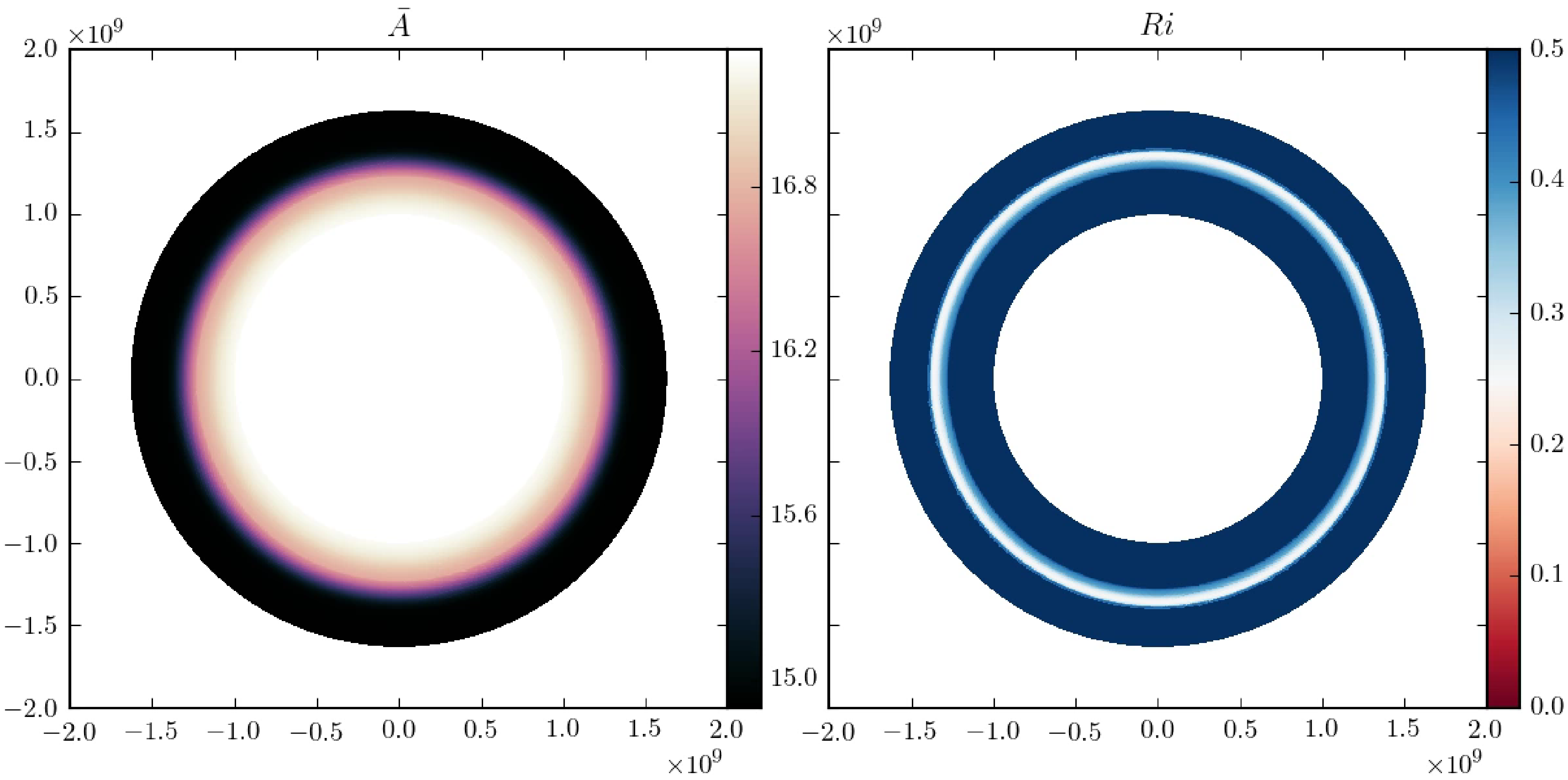
17.5
17.0
16.5
16.0
15.5
15.0
14.5
$\bar{A}$
1.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
$\times 10^9$
r in cm
$t = 0.0$ min
$t = 25.0$ min
$t = 30.0$ min
$t = 35.0$ min
$t = 40.0$ min
$t = 50.0$ min
$t = 65.0$ min
$t = 85.0$ min
$t = 115.0$ min
$t = 220.0$ min
$t = 290.0$ min
$t = 410.0$ min

EVOLUTION OF RICHARDSON NUMBER



$\bar{A}$
r in cm
$\times 10^9$
$t = 0.0$ min
$t = 25.0$ min
$t = 30.0$ min
$t = 35.0$ min
$t = 40.0$ min
$t = 50.0$ min
$t = 65.0$ min
$t = 85.0$ min
$t = 115.0$ min
$t = 220.0$ min
$t = 290.0$ min
$t = 410.0$ min

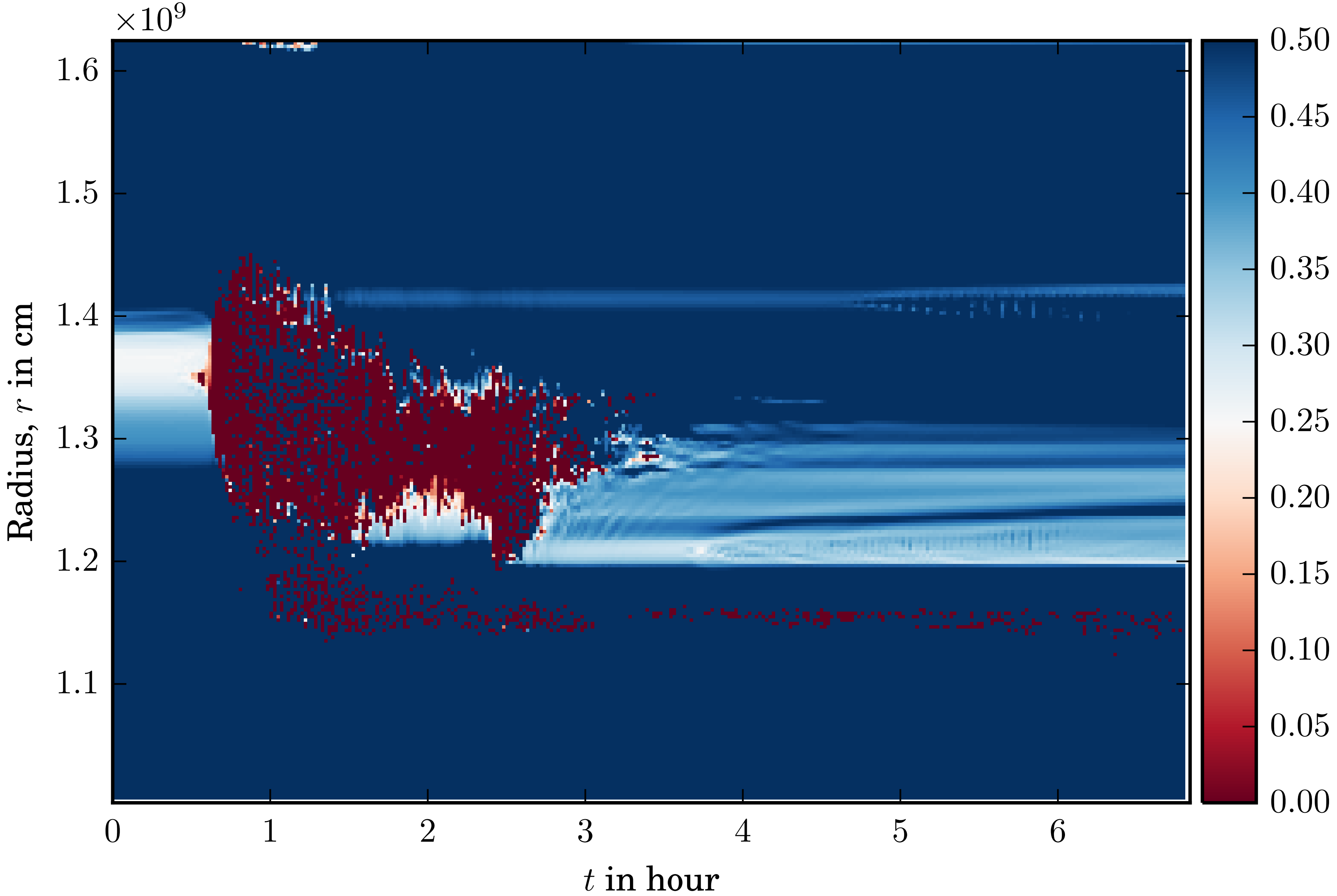
$t = 0.00$ s



$\bar{A}$
Ri
$\times 10^9$
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
16.8
16.2
15.6
15.0
0.5
0.4
0.3
0.2
0.1
0.0

3:16

EVOLUTION OF RICHARDSON NUMBER



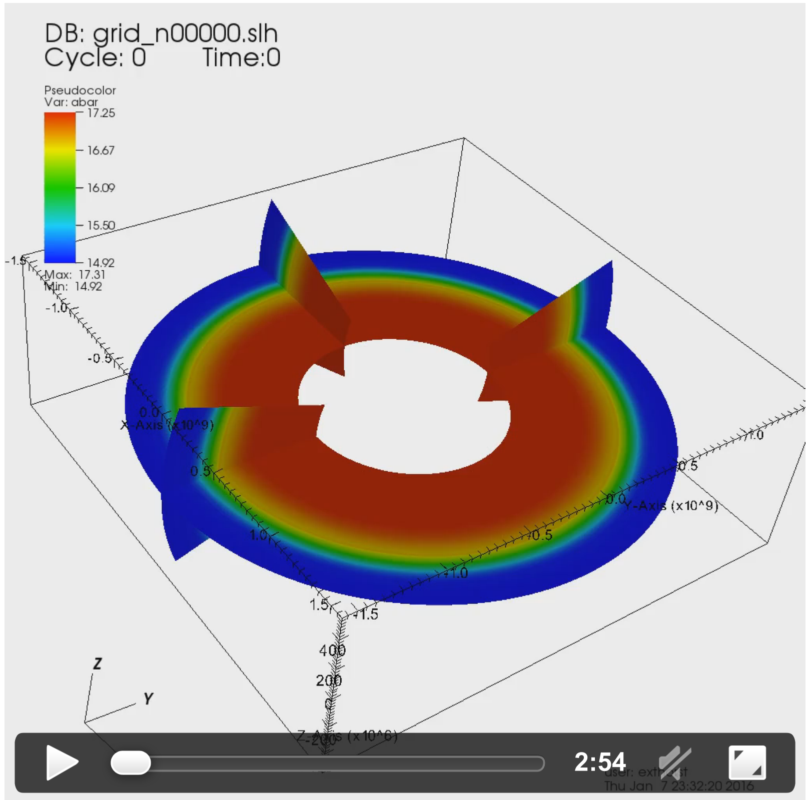
×10^9
1.6
1.5
1.4
1.3
1.2
1.1
Radius, r in cm
0
1
2
3
4
5
6
t in hour
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00

FIRST 3D WORK

by Leonhard Horst

- not straightforward to map model to 3D
- strict shellular rotation cannot always be upheld, while keeping a stable model
- some modifications to Ω profile to get a stable model in the equatorial plane

$\bar{A}$



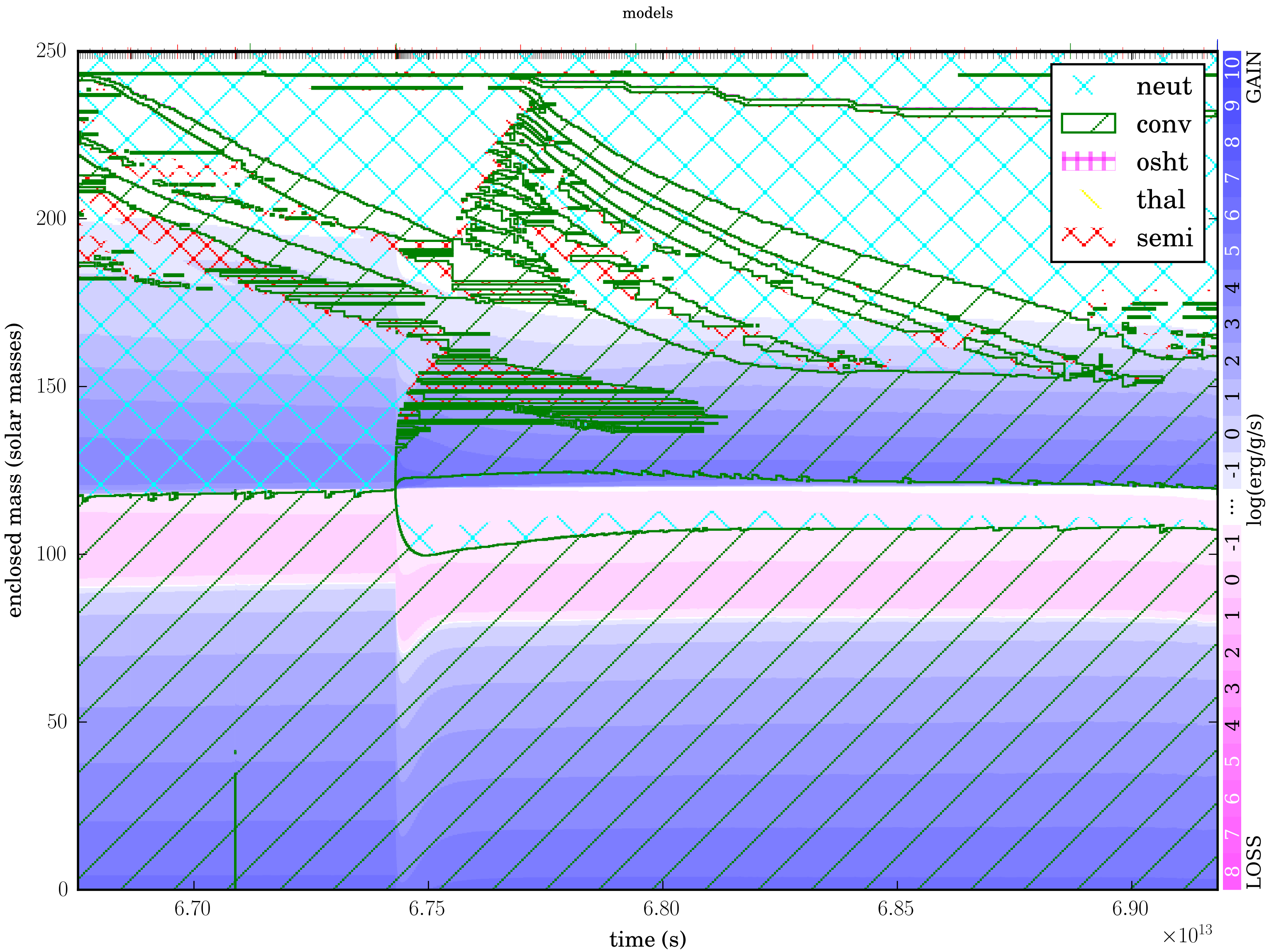
CONVECTIVE OVERSHOOTING IN POP III STARS

collaborators:
Alexander Heger (Monash), Friedrich Röpke (HITS)

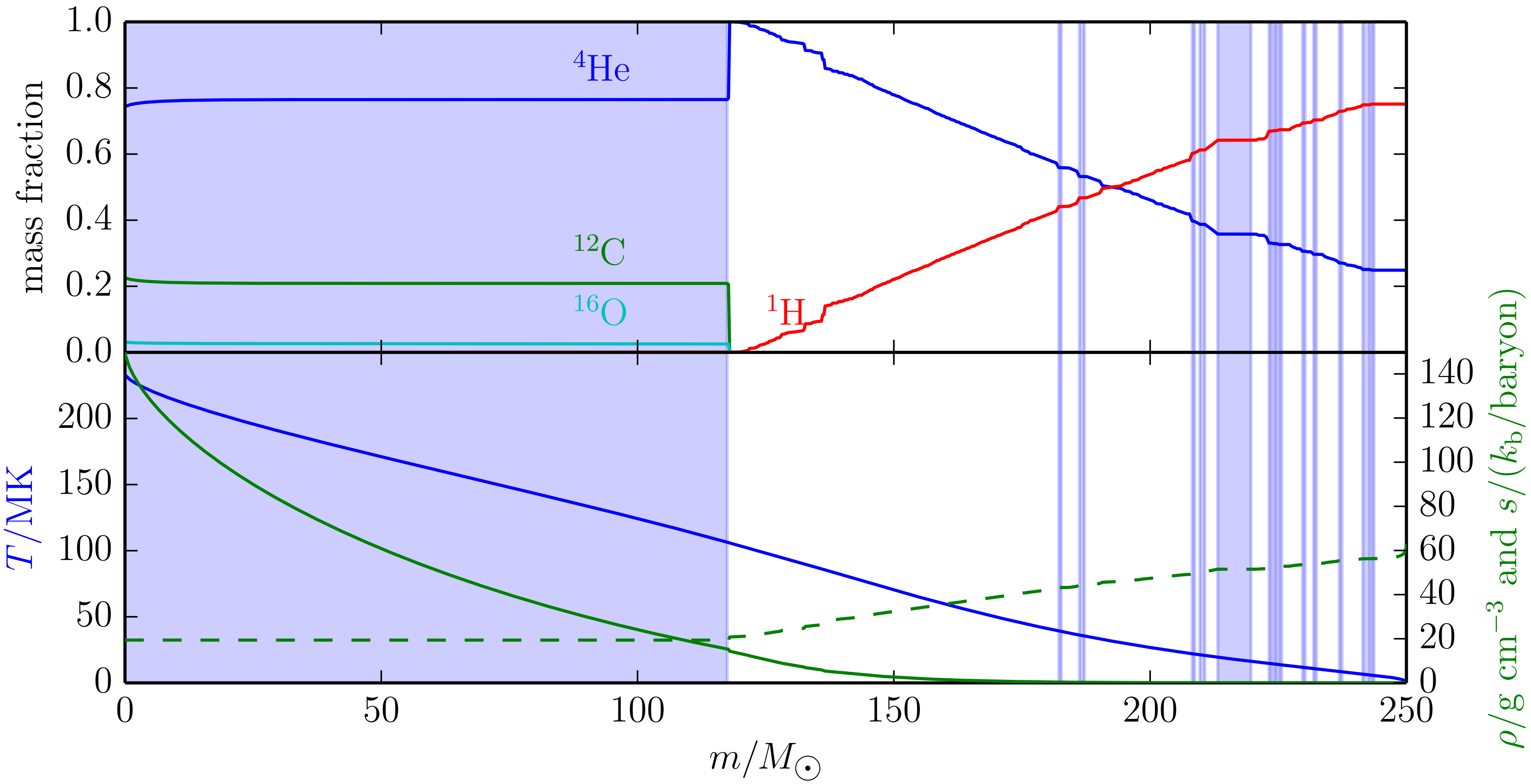
SETTING

- zero metallicity initial model
- core He burning produced already significant amount of $^{12}\mathrm{C}$
- H burning shell with $X(^{12}\mathrm{C}) = 10^{-9}$
- convective core grows, reaches H burning shell

$250\,M_\odot$ star ($Z = 0$) during core He burning

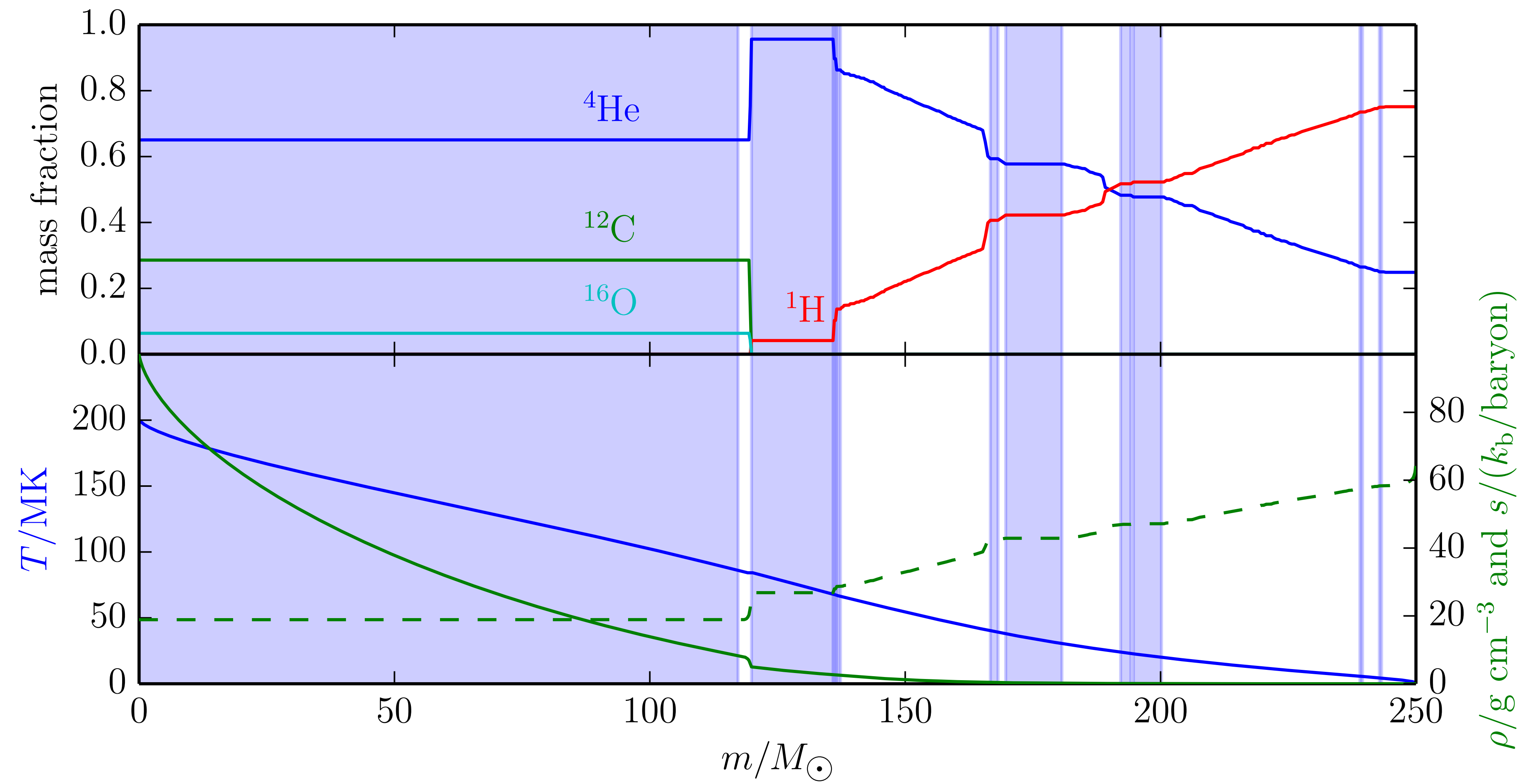


STRUCTURE



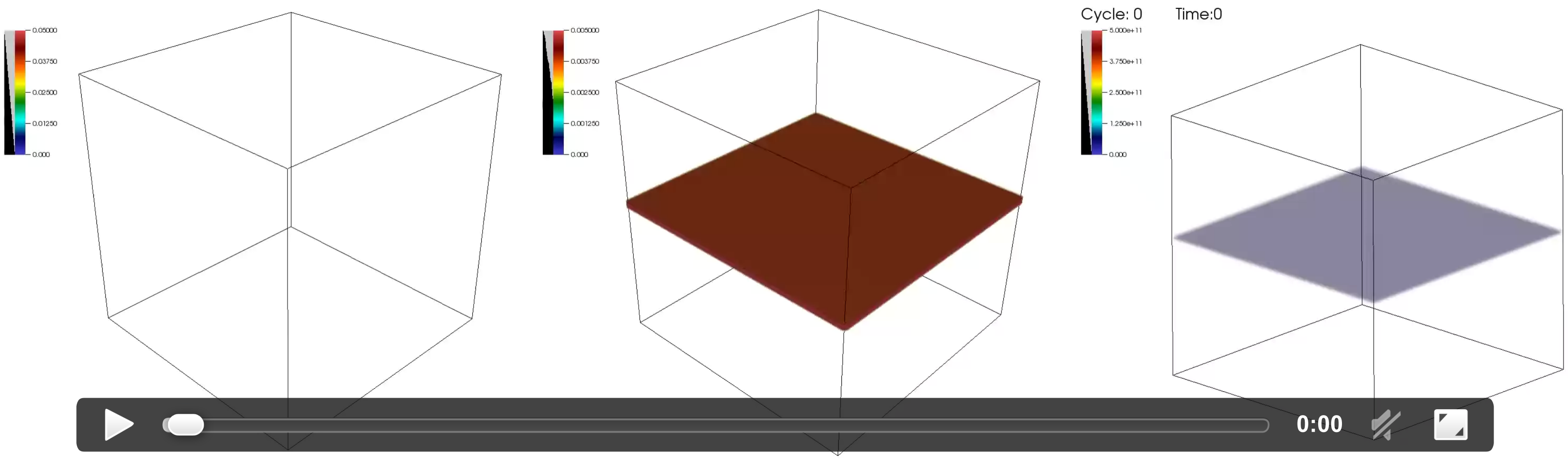
1.0
0.8
0.6
0.4
0.2
0.0
mass fraction
^{4}He
^{12}C
^{16}O
^{1}H
250
200
150
100
50
0
T/MK
140
120
100
80
60
40
20
0
ρ/g cm^{-3} and $s/(k_\mathrm{b}$/baryon)
0
50
100
150
200
250
$m/M_\odot$

STRUCTURE (20 000 YEARS LATER)



3D BOX (2013)

128^3 grid for about 4 days



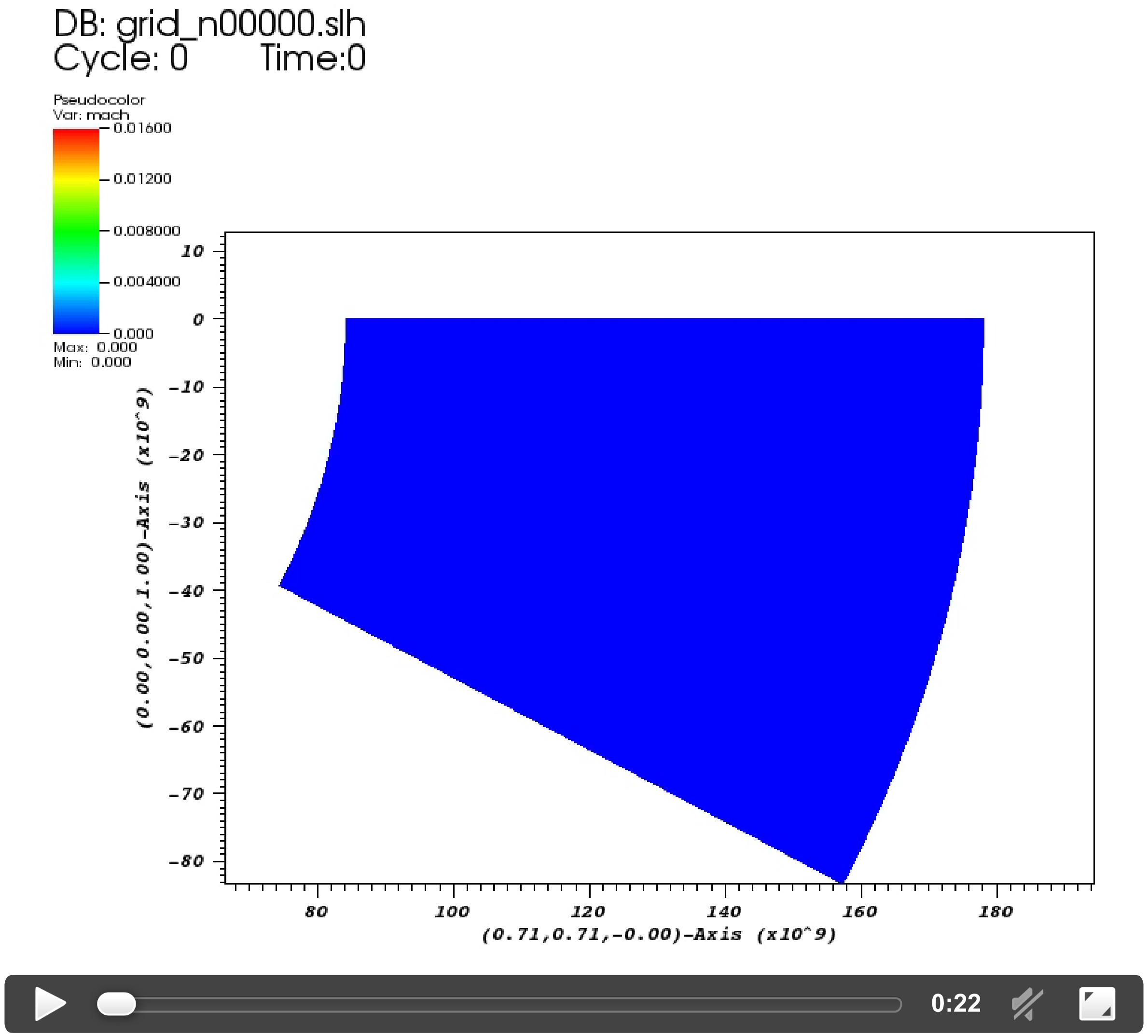
Mach number

14_{N}

energy release

3D WEDGE (2014)

512^3 grid



NEXT STEPS

- start simulation right before overshooting reaches H shell
- include core using cubed sphere

CONCLUSIONS

- In many aspects stars should be treated as 3D, dynamical objects.
- SE codes are still needed to cover evolutionary timescale.
- Low Mach numbers require special numerical methods.
- Fully implicit, 3D hydro is possible and scales well to large supercomputers.
- We can now look at many poorly understood phenomena from stellar evolution in greater detail using hydro simulations.