Anomalies, Conformal Manifolds, and Spheres

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Jaume Gomis, Po-Shen Hsin, Zohar Komargodski, Adam Schwimmer, NS, Stefan Theisen, arXiv:1509.08511

CFT Sphere partition function $\log Z$

- Power divergent terms are not universal. Can be removed by counterterms like $\Lambda^d \int \sqrt{\gamma} + \Lambda^{d-2} \int \sqrt{\gamma}\, R + \cdots$
- In addition
  - for odd *d* $\log Z = -F$ is universal (ambiguity in quantized imaginary part due to Chern-Simons terms – depends on framing)
  - for even *d* $\log Z = C \log(r\Lambda) - F$.
    - C is universal (in *2d* it is $c/3$ and in *4d* it is – a)
    - F is not universal; it can be absorbed in a local counterterm, $\int \sqrt{\gamma}\, E_d\, F$ with E_d the Euler density
- Used in c-theorem and its generalizations, entanglement entropy, …

Conformal manifolds

- S$= S_0 + \int \lambda^i O_i(x)$ is an exactly marginal deformation
- Family of CFTs labeled by coordinates λ^i
- Metric on the conformal manifold – the Zamolodchikov metric

$$\langle O_i(x)\, O_j(0)\rangle = \frac{g_{ij}(\lambda)}{|x|^{2d}}$$

- Focus on $d = 2$ (later $d = 4$)

Conformal manifolds of *2d* (2,2) SCFT

- Typical example: sigma models with Calabi-Yau target space.
- In string theory the coordinates on the conformal manifold are the moduli – massless fields in *4d*
  - *2d* chiral fields λ
  - *2d* twisted chiral fields $\tilde{\lambda}$
- The conformal manifold is Kahler with

$$K = K_c(\lambda, \bar{\lambda}) + K_{tc}\left(\tilde{\lambda}, \bar{\tilde{\lambda}}\right)$$

2d (2,2) curved superspace

- For rigid SUSY in curved spacetime use supergravity [Festuccia, NS]
- Simplification in *2d*
  - (locally) pick conformal gauge $\gamma_{\mu\nu} = e^{2\sigma}\delta_{\mu\nu}$
  - for SUSY (locally) pick superconformal gauge
  - use flat space expressions with explicit σ's
- Two kinds of $(2,2)$ supergravities. In the superconformal gauge they depend on
  - A chiral $\Sigma = \sigma + i\, a + \cdots$ with $A_\mu = \epsilon_{\mu\nu}\partial^\nu a$ an axial R-gauge field (Lorentz gauge). The curvature is in a chiral superfield $\mathcal{R} = \overline{D}^2\overline{\Sigma}$. We will focus on this.
  - A twisted chiral $\tilde{\Sigma} = \sigma + i\, \tilde{a} + \cdots$ with $\widetilde{A_\mu} = \epsilon_{\mu\nu}\partial^\nu \tilde{a}$ a vector R-gauge field (Lorentz gauge).

2d (2,2) curved superspace

- In this language the curvature is in a chiral superfield $\mathcal{R} = \bar{D}^2\bar{\Sigma}$. It contains $\partial^2\sigma$.
- To preserve rigid SUSY in a non-conformal theory we need to add terms to the flat superspace Lagrangian.
- Various backgrounds in the literature are easily described, e.g.
  - Topological twist is $\Sigma = 0$, $\bar{\Sigma} = 2\sigma \neq 0$
  - Omega background

$$\Sigma = 0$$

$$\bar{\Sigma} = 2\sigma + 2\, i\, \epsilon \bar{z}\;\; \partial_{\bar{z}}(2\,\sigma + \log \bar{z}\,)\, \bar{\theta}^2$$

2d (2,2) curved superspace

- Supersymmetry on $\mathbb{S}^2$ [Benini, Cremonesi; Doroud, Gomis, Le Floch, Lee] is achieved with (suppress r dependence)

$$\Sigma = -\log(1+|z|^2) + \theta^2 \frac{i}{1+|z|^2}$$

$$\bar{\Sigma} = -\log(1+|z|^2) + \bar{\theta}^2 \frac{i}{1+|z|^2}$$

- $\bar{\Sigma}$ is not necessarily the complex conjugate of Σ.

(2,2) sphere partition functions

- [Benini, Cremonesi; Doroud, Gomis, Le Floch, Lee] placed a nonconformal gauged linear sigma model on a sphere with

$$\Sigma = -\log(1+|z|^2) + \theta^2 \frac{i}{1+|z|^2}$$

$$\bar{\Sigma} = -\log(1+|z|^2) + \bar{\theta}^2 \frac{i}{1+|z|^2}$$

- This theory flows in the IR to a nonlinear sigma model with Calabi-Yau target space.

- They computed Z of the IR theory using localization in the UV theory.

(2,2) sphere partition functions

- Amazing conjecture [Jockers, Kumar, Lapan, Morrison, Romo]: this $\mathbb{S}^2$ partition function is

$$Z = \ r^{c/3} e^{-K_c(\lambda,\bar{\lambda})}$$

(restoring the radius r, whose power reflects the ordinary conformal anomaly).

Similarly, using $\tilde{\Sigma}$ it is $\mathrm{Z} = r^{c/3} e^{-K_{tc}(\tilde{\lambda},\bar{\tilde{\lambda}})}$.

- Proofs based on localization, squashed sphere, *tt**, twisting, counterterms and properties of the background [Gomis, Lee; Gerchkovitz, Gomis, Komargodski; ...] .

Questions/confusions

- Given that the one point function of a marginal operator vanishes, how can the sphere partition function depend on λ?
- Why is it meaningful?
  - Can add a local counterterm $\int \sqrt{\gamma}\, R\, f\left(\lambda, \bar{\lambda}\right)$, making the answer non-universal
- In an SCFT on the sphere there is no need to add terms to the Lagrangian to preserve SUSY
  - Why does it depend on the background Σ?
  - If it does not, what determines whether we used Σ or $\tilde{\Sigma}$ to find e^{-K_c} or $e^{-K_{tc}}$?
  - Where is the freedom in Kahler transformations?
- What's the conceptual reason for it? Is it UV or IR?

Conformal manifolds and anomalies (w/o SUSY)

Zamolodchikov metric $\langle O_i(x)\, O_j(0)\rangle = \frac{g_{ij}(\lambda)}{|x|^4}$

In momentum space $\int e^{ipx}\langle O_i(x)\, O_j(0)\rangle \sim g_{ij}(\lambda) p^2 \log(\mu^2/p^2)$

Dependence on the scale μ leads to a conformal anomaly: with position dependent λ (suppressed coefficients) [Osborn; Friedan, Konechny]

$$T_\mu{}^\mu = c\, R + g_{ij}(\lambda)\, \partial_\mu \lambda^i\, \partial^\mu \lambda^j + \ \ldots$$

Ordinary conformal anomaly

A more subtle anomaly (actually, less subtle)

More about anomalies

- The partition function Z is a nonlocal functional of the background fields (the metric $\gamma_{\mu\nu}$, exactly marginal couplings λ^i, background gauge fields, etc.).
- Its variation under changing the conformal factor $\delta_\sigma \log Z$ is an integral of a local functional of the background fields and $\delta\sigma$.
- $\delta_\sigma \log Z$
  – Must be coordinate invariant in spacetime (assume that the regularization preserves it, i.e. $T_{\mu\nu}$ is conserved also at coincident points)
  – Must be coordinate invariant in the conformal manifold
  – Must obey the Wess-Zumino consistency conditions
  – In SUSY theories it must be supersymmetric

More about anomalies

- A term in $\delta_\sigma \log Z$ that is a Weyl variation of a local term is considered trivial.
- An anomaly is a "cohomologically nontrivial" term.
  - It cannot be removed by changing a local counterterm.
  - It cannot change by changing the renormalization scheme.
- Therefore, even though it arises due to a short distance regulator, it is universal – it does not depend on the choice of regulator.

Returning to conformal manifolds and anomalies (w/o SUSY)

$$\langle O_i(x)\, O_j(0)\rangle = \frac{g_{ij}(\lambda)}{|x|^4}$$

$$\int e^{ipx}\langle O_i(x)\, O_j(0)\rangle \sim g_{ij}(\lambda) p^2 \log(\mu^2/p^2)$$

leads to [Osborn; Friedan, Konechny]

$$\delta_\sigma \log Z \sim \int \sqrt{\gamma}\, \delta\sigma \left(c\, R + g_{ij}(\lambda)\, \partial_\mu \lambda^i\, \partial^\mu \lambda^j + \ \dots \right)$$

Ordinary conformal anomaly

A more subtle anomaly

Conformal manifolds and anomalies (w/o SUSY)

Another possible anomaly in *2d*

$$\delta_\sigma \log Z \ \sim \int \sqrt{\gamma}\, \delta\sigma \epsilon^{\mu\nu} B_{ij}(\lambda) \partial_\mu \lambda^i \, \partial_\nu \lambda^j$$

can be ruled out as follows.

$$\langle O_{i_1}(p_1) O_{i_2}(p_2) \cdots \rangle = \log \mu^2 \, \delta(\textstyle\sum p_i) A_{i_1 i_2 \cdots} + \cdots$$

where $\cdots$ are finite.

$A_{i_1 i_2 \cdots}$ is a second order polynomial in the momenta.

Set e.g. $p_3 = p_4 = \cdots = 0$ and find

$$A_{i_1 i_2 \cdots} = p_1^2 \partial_{i_3} \partial_{i_4} \cdots g_{i_1 i_2}.$$

Therefore, all these anomalies are generated from the metric and there is no "B-field" anomaly.

Warmup: $2d\ \mathcal{N} = 1$

- The conformal manifold is parameterized by real superfields λ^i.
- The anomaly should be expressed in superspace – need to use curved superspace.
- Use the superconformal gauge – the conformal factor σ is in a real superfield Σ and the curvature is in $\mathcal{R} = D^2\Sigma$.
- The anomaly $\delta_\sigma \log Z \sim \int \sqrt{\gamma}\, \delta\sigma \left(c\, R + g_{ij}(\lambda)\, \partial_\mu\lambda^i\, \partial^\mu\lambda^j + \cdots\right)$ is supersymmetrized in the superconformal gauge as

$$\delta_\Sigma \log Z \sim \int d^2\theta\, \delta\Sigma \left(c\, D^2\Sigma + g_{ij}(\lambda)\, D_+\lambda^i\, D_-\lambda^j + \cdots\right)$$

Ordinary conformal anomaly

A more subtle anomaly

- Ambiguity due to a local counterterm $\int d^2\theta\, \mathcal{R}\, f(\lambda)$ prevents us from making any statement about Z.

(2,2) SCFT

- The conformal manifold is parameterized by chiral superfields λ^i and twisted chiral superfields $\tilde{\lambda}^{\tilde{a}}$.
- The anomaly should be expressed in superspace – need to use curved superspace.
- Focus on the supergravity, where the conformal factor σ is in a chiral superfield Σ and the curvature is in a chiral superfield $\mathcal{R} = \bar{D}^2\bar{\Sigma}$.

(2,2) SCFT

- The supersymmetrization of the anomaly
  $$\delta_\sigma \log Z \sim \int \sqrt{\gamma}\, \delta\sigma \left(c\, R + g_{i\bar{\imath}} \partial_\mu \lambda^i \partial^\mu \bar{\lambda}^{\bar{i}} + \tilde{g}_{a\bar{a}}\, \partial_\mu \tilde{\lambda}^a \partial^\mu \bar{\tilde{\lambda}}^{\bar{a}} \right)$$
  after gauge fixing is
  $$\delta_\Sigma \log Z \sim$$
  $$\int d^4\theta\, (\delta\Sigma + \delta\bar{\Sigma}) \left(c\, (\Sigma + \bar{\Sigma}) + K_c(\lambda, \bar{\lambda}) - K_{tc}\left(\tilde{\lambda}, \bar{\tilde{\lambda}}\right)\right)$$
  The first term can also be written as c $\int d^2\theta\, \delta\Sigma\, \mathcal{R} + c.c.$
- Lack of "B-anomaly" proves that K is such a sum of two terms.
- Invariance under Kahler transformations of $K_{tc}\left(\tilde{\lambda}, \bar{\tilde{\lambda}}\right)$ by $\tilde{f}\left(\tilde{\lambda}\right)$ and of $K_c(\lambda, \bar{\lambda})$ by $f(\lambda)$ up to a variation of a local counter term $\int d^4\theta \delta\Sigma\, \bar{f}\left(\bar{\lambda}\right) = \int d^2\theta\, \delta\mathcal{R}\, f(\lambda)$.
- Alternatively, Kahler invariance, involves a shift $\Sigma \to \Sigma + \frac{1}{c} f(\lambda)$.

(2,2) SCFT

With an axial R-symmetry the supersymmetry current multiplet $J_{\pm\pm}$ is a real superfield satisfying (slightly simplified)

$$\overline{D}_{\mp} J_{\pm\pm} = \mp D_{\pm} W$$

with chiral W.

In an SCFT $W = 0$ at separated points, but the anomaly sets a contact term

$$W = \overline{D}^2 \left(c\, \overline{\Sigma} + K_c(\lambda, \bar{\lambda}) - K_{tc}\left(\tilde{\lambda}, \bar{\tilde{\lambda}}\right)\right) =$$

$$= \mathrm{c}\, \mathcal{R} + \overline{D}^2 \left(K_c(\lambda, \bar{\lambda}) - K_{tc}\left(\tilde{\lambda}, \bar{\tilde{\lambda}}\right)\right)$$

- It is invariant under Kahler transformations provided we also shift $\Sigma \to \Sigma + \frac{1}{c} f(\lambda)$.

Ambiguities

- In the superconformal gauge the local terms are expressed in terms of the chiral curvature superfield $\mathcal{R} = \bar{D}^2\bar{\Sigma}$.
- Terms that depend on Σ not through $\mathcal{R}$ are non-local.
- Therefore, the anomaly is not a variation of a local term.
- Freedom in the local term

$$\int d^2\theta\, \mathcal{R}\, f(\lambda) + c.c. = \int d^4\theta\, \bar{\Sigma} f(\lambda) + c.c.$$

with holomorphic $f(\lambda)$ leads to freedom in Kahler transformations of $K_c(\lambda, \bar{\lambda})$. (It can be absorbed in a shift of Σ.)

- Other than that, the anomaly is unambiguous.

The anomaly in components

For a purely conformal variation $\delta\Sigma = \delta\sigma$ the anomaly is

$$\delta_\Sigma \log Z \sim \int d^4\theta \, (\delta\Sigma + \delta\bar{\Sigma}) \left(c\,(\Sigma + \bar{\Sigma}) + K_c(\lambda\,, \bar{\lambda}) - K_{tc}\left(\tilde{\lambda}, \bar{\tilde{\lambda}}\right) \right)$$

$$= \int \Big[\delta\sigma \, \left(c \,\square\sigma + g_{i\bar{\imath}} \partial_\mu \lambda^i \partial^\mu \bar{\lambda}^{\bar{\imath}} + \tilde{g}_{a\bar{a}} \, \partial_\mu \tilde{\lambda}^a \partial^\mu \bar{\tilde{\lambda}}^{\bar{a}} \right)$$

$$- \square\delta\sigma \, K_c(\lambda\,, \bar{\lambda}) \Big]$$

The last term leads to

$$\log Z \sim \int \sqrt{\gamma} \, R \, K_c(\lambda, \bar{\lambda}) + \ \ldots$$

and hence on $\mathbb{S}^2$

$$Z = \ r^{c/3} e^{-K_c}$$

Q.E.D.

The anomaly in components

For a purely conformal variation $\delta\Sigma = \delta\sigma$ the anomaly is

$$\delta_\Sigma \log Z \sim \int d^4\theta\, (\delta\Sigma + \delta\bar{\Sigma})\Big(c\,(\Sigma + \bar{\Sigma}) + K_c(\lambda\,,\bar{\lambda}) - K_{tc}\left(\tilde{\lambda},\bar{\tilde{\lambda}}\right)\Big)$$

$$= \int \Big[\, \delta\sigma\, \Big(c\,\Box\sigma + g_{i\bar{\imath}}\partial_\mu\lambda^i\partial^\mu\bar{\lambda}^{\bar{\imath}} + \tilde{g}_{a\bar{a}}\,\partial_\mu\tilde{\lambda}^a\partial^\mu\bar{\tilde{\lambda}}^{\bar{a}}\Big)$$

$$- \Box\delta\sigma\, K_c(\lambda\,,\bar{\lambda})\Big]$$

The last term leads to

$$\log Z \sim \int \sqrt{\gamma}\, R\, K_c(\lambda,\bar{\lambda}) + \ \ldots$$

Without SUSY this term is not universal. It is the variation of the local counterterm . With SUSY it is related to the universal term and hence it is meaningful.

Kahler invariance

$$Z = \ r^{c/3} e^{-K_c}$$

- Kahler transformations of K_c can be absorbed in a local counter term.
- Alternatively, full Kahler invariance when Σ is also shifted. With this interpretation Z is invariant under the combined transformation

$$K_c \to K_c + f(\lambda) + \bar{f}(\bar{\lambda})$$

$$r \to r\, e^{\frac{3}{c}\left(f(\lambda)+\bar{f}(\bar{\lambda})\right)}$$

Extensions

- Trivial to repeat with $\tilde{\Sigma}$ and to find K_{tc}.
  - Here we choose the contact terms to preserve the vector R-symmetry – use the other $(2,2)$ supergravity.
  - Equivalently, we use a different regulator that preserves the supergravity of $\tilde{\Sigma}$.
- *2d* $\mathcal{N} = (0,2)$
- *4d*
  - $\mathcal{N} = 1$
  - $\mathcal{N} = 2$

$2d\ \mathcal{N} = (0,2)$

- The conformal manifold is parameterized by chiral superfields λ^i.
- The conformal factor σ is in a chiral superfield Σ and the curvature is in a chiral superfield $\mathcal{R}_- = \partial_{--}\bar{D}_+\bar{\Sigma}$.
- Several possible anomalies including:
  - The ordinary anomaly
    $i\int d\theta^+ c\ \delta\Sigma\, \mathcal{R}_- + c.c. = i\int d^2\theta^+\ c\ \delta\Sigma\, \partial_{--}\bar{\Sigma} + c.c.$
  - The anomaly associated with the metric
    $$i\int d^2\theta^+ \left(\delta\Sigma + \delta\bar{\Sigma}\right)\left(K_i\, \partial_{--}\lambda^i - K_{\bar{\imath}}\partial_{--}\bar{\lambda}^{\bar{\imath}}\right)$$
  - …

$2d\ \mathcal{N} = (0,2)$

- The anomaly associated with the metric
  $$i\int d^2\theta^+ \left(\delta\Sigma + \delta\bar{\Sigma}\right)\left(K_i\, \partial_{--}\lambda^i - K_{\bar{\imath}}\partial_{--}\bar{\lambda}^{\bar{\imath}}\right)$$
- Using the lack of "B−anomaly"
  $$i\int d^2\theta^+ \left(\delta\Sigma + \delta\bar{\Sigma}\right)\left(K_i\, \partial_{--}\lambda^i - K_{\bar{\imath}}\partial_{--}\bar{\lambda}^{\bar{\imath}}\right)$$
  $$= i\int d^2\theta^+ \left(\delta\Sigma + \delta\bar{\Sigma}\right)\left(\partial_i\, K\partial_{--}\lambda^i - \partial_{\bar{\imath}}\, K\partial_{--}\bar{\lambda}^{\bar{\imath}}\right)$$
  with real K.
- Hence, the metric on the conformal manifold must be Kahler.
- The last term includes in its component expansion $\Box\delta\sigma K$ and could lead to an interesting Z. But…

Ambiguities in *2d* $\mathcal{N} = (0,2)$

- As in all our examples, a local counterterm
$$i\int d\,\theta^+ \mathcal{R}\_f(\lambda) + c.c. = i\int d^2\theta^+\, \partial_{--}\bar{\Sigma}\, f(\lambda) + c.c.$$
with holomorphic $f(\lambda)$ – Kahler transformations of K.
- Can redefine the *2d* metric by a function of the moduli.
  – Locally a chiral superfield is the same as a real superfield (not in (2,2)). Hence, in the conformal gauge we can shift (more carefully, use SUGRA)
$$\Sigma \to \Sigma + \frac{i}{\partial_{++}} \bar{D}_+ D_+ H\left(\lambda, \bar{\lambda}\right).$$
  – This shifts the ordinary anomaly term
$$i\int d^2\theta^+\, \delta\Sigma\, \partial\_{--}\bar{\Sigma} \to i\int d^2\theta^+\, \delta\Sigma\, \partial\_{--}\bar{\Sigma} + i\int d^2\theta^+\, \delta\Sigma\, \partial\_{--} H\,,$$
  which includes in components $\Box\delta\sigma H$.
  – Hence, Z is ambiguous (note, there is no local counterterm).

4d

- Without SUSY
  - Anomaly $\int \sqrt{\gamma}\, \delta\sigma(a\, E_4 - \ c\, W^2 + g_{ij}\Box\lambda^{\,i}\Box\lambda^{\,j} + \cdots)$
  - Ambiguous counterterms $\int \sqrt{\gamma}\, \left(R^2 F_1(\lambda\,) + R^2_{\mu\nu} F_2(\lambda\,) + R^2_{\mu\nu\rho\sigma} F_3(\lambda\,) + \cdots\right)$
- With SUSY need to
  - Supersymmetrize – in $\mathcal{N} = 1, 2$ σ and λ in chiral superfields.
  - Covariantize in spacetime
  - Covariantize in the conformal manifold

$4d\ \mathcal{N} = 1$

- Anomaly

$$\int \sqrt{\gamma}\, \delta\sigma\left(a\, E_4 - \; c\, W^2 + g_{i\,\bar{\imath}} \Box\lambda^{\,i} \Box\bar{\lambda}^{\,\bar{\imath}} + \cdots\right)$$

- Ambiguous counterterms

$$\int \sqrt{\gamma}\left(R^2 F_1(\lambda\,,\bar{\lambda}\,) + R_{\mu\nu}^2 F_2(\lambda\,,\bar{\lambda}\,) + R_{\mu\nu\rho\sigma}^2 F_3(\lambda\,,\bar{\lambda}\,) + \cdots\right)$$

- Can be supersymmetrized and then the local counterterm makes the sphere partition function ambiguous [Gerchkovitz, Gomis, Komargodski] .

$4d\ \mathcal{N}=2$

- Example: theories of class S with moduli λ

- After a lot of algebra (using relevant formulas in the literature) the expressions without SUSY are supersymmetrized and covariantized to

$$\int \sqrt{\gamma}\,\delta\sigma\left(a\,E_4 + g_{i\bar{\imath}}\square\lambda^{i}\square\bar{\lambda}^{\bar{\imath}} + \cdots\right) + \sqrt{\gamma}\,K(\square^2\delta\sigma + \cdots)$$

Universal

Without SUSY this is not universal, but SUSY relates it to this and hence it is universal.

$4d\ \mathcal{N} = 2$

- Key fact,
  - In $\mathcal{N}$=1 the counterterm that is proportional to the curvature square is an arbitrary function of λ and $\bar{\lambda}$.
  - In $\mathcal{N}$=2 the counterterm that is proportional to the curvature square must be holomorphic in λ.
  - Therefore, here the ambiguity is only in a holomorphic function of λ.
  - Only Kahler transformations of K.

$4d\ \mathcal{N} = 2$

- Collecting all the terms $\log Z \sim \int \sqrt{\gamma}\, E_4\, K + \cdots$ and

$$Z = r^{-a}\, e^{K(\lambda,\bar{\lambda})/12}$$

Ordinary conformal anomaly

The more subtle anomaly

- This reproduces a result of [Gerchkovitz, Gomis, Komargodski; Gomis, Ishtiaque]

Conclusions

- Anomaly under conformal transformations when the coupling constants depend on position
  - Unrelated to supersymmetry
  - This is a UV phenomenon
    - Visible on flat $\mathbb{R}^d$
    - Independent of the background
- Supersymmetry restricts
  - the form of the anomaly
  - the ambiguity due to local counterterms

Conclusions

- In *2d* $\mathcal{N} = (2,2)$ the $\mathbb{S}^2$ partition function depends on the anomaly.
  - New derivation of

$$Z = r^{c/3} e^{-K_c}$$

    - The usual conformal anomaly
    - The more subtle conformal anomaly
  - Addresses the questions/confusions we raised…

Conclusions

- Addresses the questions/confusions we raised
  - One point function nonzero because of a term in the operator proportional to the curvature (like in the dilaton)
  - Z can be unambiguous when the counterterm proportional to the curvature depends holomorphically on the couplings.
  - Dependence on Σ due to an anomaly. Different choices lead to K_c or K_{tc}.
  - The anomaly is set in the UV and is detected by the sphere (IR).

Conclusions

- Three step process
  - the anomaly
  - the ambiguity (freedom in counterterms)
  - the sphere
- Other cases
  - $2d\ \mathcal{N}=1$ ambiguous
  - $2d\ \mathcal{N}=(0,2)$ ambiguous
  - $4d\ \mathcal{N}=1$ ambiguous
  - $4d\ \mathcal{N}=2$ $Z=r^{-a}\, e^{K/12}$