Gapped Boundary Phases of Topological Insulators via Weak Coupling

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Phases of Theories

- Gapless (= massless)
  - Free theory
  - Interacting (nontrivial fixed point)
- Gapped
  - Trivial bulk theory
    - Trivial boundary
    - Gapless boundary modes
    - Gapped TQFT on the boundary

    } Bulk is not completely trivial. Symmetry Protected Topological (SPT) phase
  - Nontrivial bulk topological quantum theory
    - Same as above

Topological Insulators [Kane, Mele; …]

- Insulator
  - unbroken global $U(1)_A$. The electromagnetic gauge field A can be viewed as a classical background field.
  - Gapped and trivial bulk
- Assume it is time-reversal (T) invariant
- Nontrivial boundary
  - Can be understood in examples, more generally follows from anomalies
  - Typically, massless fermions (gapless)
  - Can also lift the fermions and have gapped boundary states. (Examples by [Metlitski, Kane, Fisher; Wang, Potter, Senthil…].)

Topological Insulator: Simple Example

Massive electron with positive real mass m outside the material and mass $-m$ inside the material.

- Since m is real, it preserves $U(1)_A$ and T (and also charge conjugation C, but this is not a symmetry in CM physics).
- As the mass varies in space, it leads to massless (gapless) fermions on the boundary [Jackiw, Rebbi].
- Perform a chiral rotation inside the material and have positive mass everywhere. Now $\theta = \pi$ inside and $\theta = 0$ outside [Qi, Hughes, Zhang; Essin, Moore, Vanderbilt]:

$$\frac{1}{8\pi}\int\left(F\wedge F+\frac{1}{24}\mathrm{Tr}\,R\wedge R\right)$$

In most of the talk we will neglect the gravitational term.

Topological Insulator: simple example

Could start with $\frac{1}{8\pi}\int F\wedge F$ inside the material, but not outside.

Another perspective on the boundary modes:

2+1-dimensional complex massless fermions have "parity anomaly."

We would like to preserve $U(1)_A$ and T. But we can preserve

- either $U(1)_A$ and violate T
- or T and violate $U(1)_A$
- or $U(1)_A$ and T, but the theory is not truly $2+1$-dimensional. It needs a bulk interaction. This is an example of anomaly inflow [Callan, Harvey].

Massless boundary modes are associated with $U(1)_A$ and T. They are robust.

Digression: The Parity Anomaly

Consider a single complex fermion coupled to a gauge field A.

Its partition function is [Alvarez-Gaumé, Della Pietra Moore]

$$Z = |\mathrm{Det}\,\mathcal{D}|\exp\left(\pm\frac{i\pi\eta}{2}\right)$$

$$\eta(A) = \lim_{\epsilon\to 0^+}\sum_i \exp(-\epsilon|\lambda_i|)\,\mathrm{sign}\,(\lambda_i)$$

The sign ambiguity reflects the T-breaking.

Cannot write the phase as $\exp\left(\pm\frac{i}{8\pi}\int AdA\right)$ because this Chern-Simons term is not well defined

$$\exp\left(\pm\frac{i\pi\eta}{2}\right) = \pm\exp\left(\pm\frac{i}{8\pi}\int AdA + \text{gravitational term}\right)$$

Return to the Topological Insulator

Recall the APS theorem

$$\text{Index}(\mathcal{D}) = \frac{1}{8\pi^2}\int_{\text{Bulk}}\left(F\wedge F + \frac{1}{24}\,\text{tr}\,R\wedge R\right) - \frac{\eta}{2}$$

Therefore, we can define [Witten]

$$Z \equiv (-1)^{\text{Index}(\mathcal{D})}|\text{Det}\,\mathcal{D}| =$$

$$|\text{Det}\,\mathcal{D}|\exp\left(\pm\frac{i\pi\eta}{2}\right)\exp\left(\mp\frac{i}{8\pi}\int_{\text{Bulk}}\left(F\wedge F + \frac{1}{24}\,\text{tr}\,R\wedge R\right)\right)$$

It is T-invariant (real) but depends on the bulk.

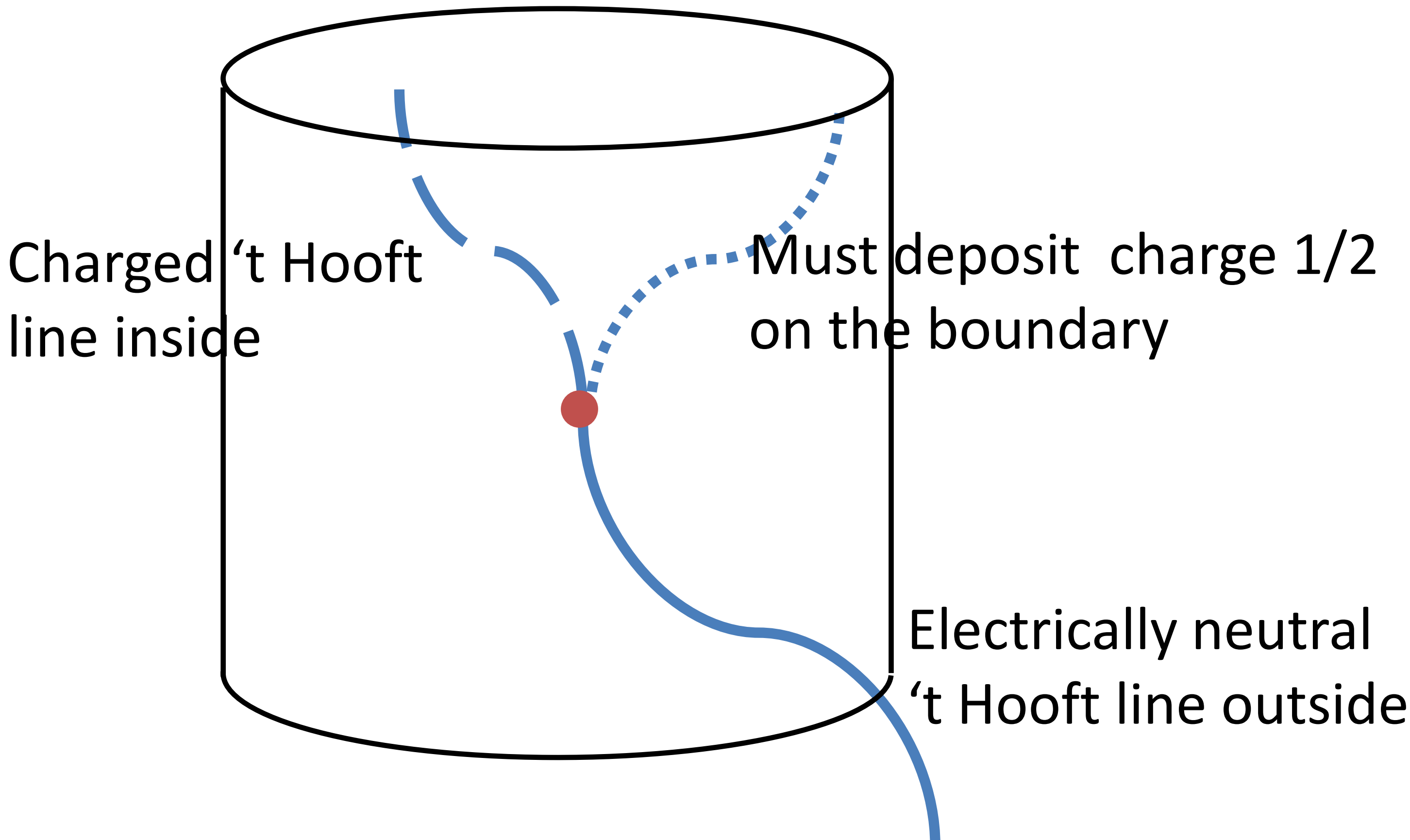
This is the partition function of the topological insulator.

Topological Insulator

Start with $\frac{1}{8\pi}\int F \wedge F$ inside the material, but not outside.

Another perspective on "nontrivial physics on the boundary. "

Bring a neutral magnetic monopole through the boundary



Topological Insulator

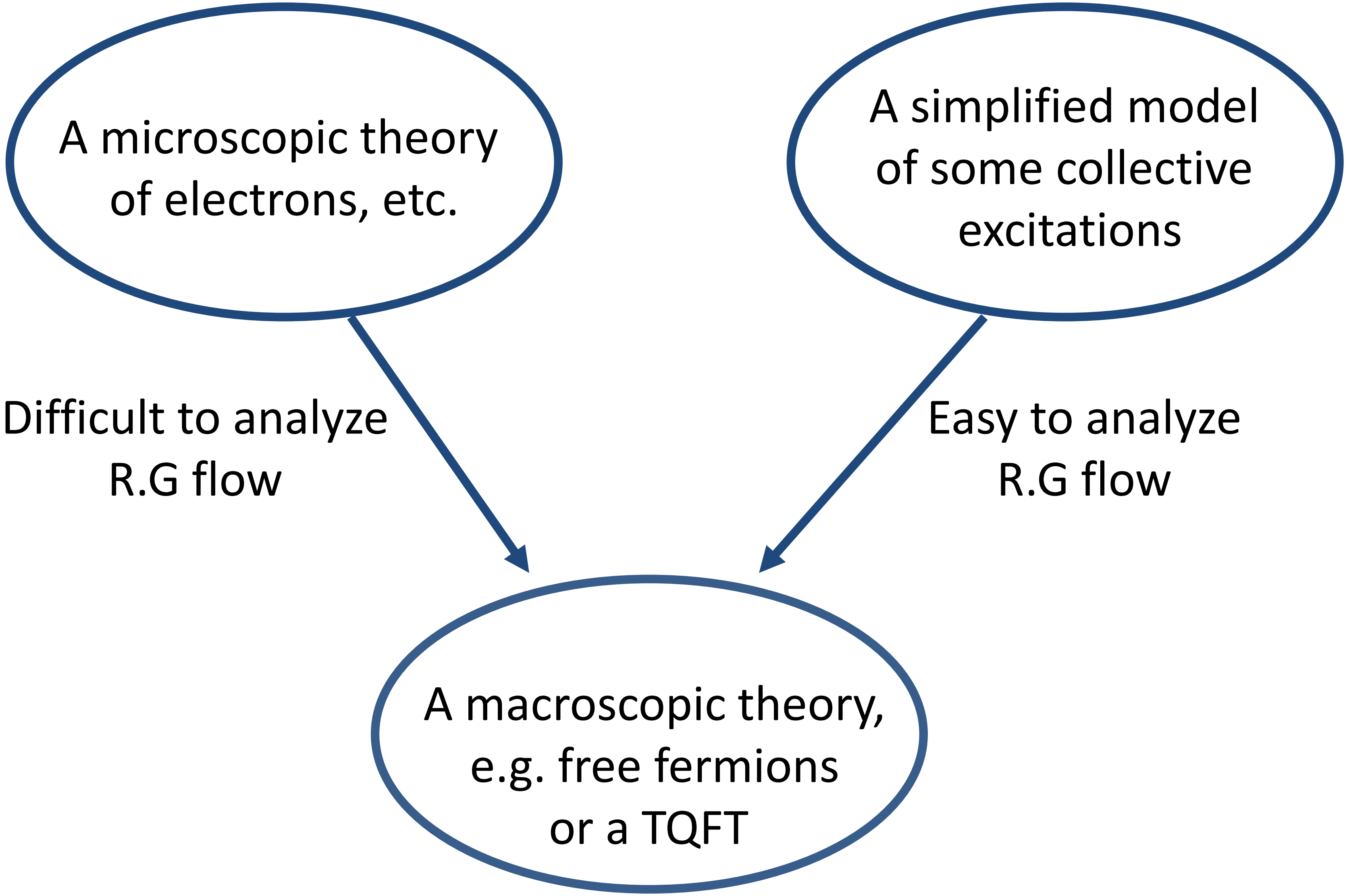
Start with $\frac{1}{8\pi}\int F \wedge F$ inside the material, but not outside.

Massless boundary modes are associated with $U(1)_A$ and T.

They are robust – cannot be lifted by small perturbations.

- Can we add a large perturbation and gap the system?
- Something must remain on the boundary to account for the anomaly inflow.
- Can there be a TQFT on the boundary with the same anomaly? (Examples by [Metlitski, Kane, Fisher; Wang, Potter, Senthil…].)
- Not obvious whether a given TQFT has the right anomaly. (Particularly challenging to control the gravitational anomaly.)

Often in CM Physics



The Three Models

- The macroscopic model gives an exact description of the long distance physics.
- The simplified model flows to the macroscopic model and captures some features of the higher energy modes (e.g. properties of some quasi-particles), but it is not exact.
- The simplified model is particularly useful when it is weakly coupled. We can vary parameters across phase transitions, while keeping $\hbar$ small. Semi-classical methods are reliable.
- Powerful consistency conditions based on anomalies can constrain the search for these models [‘t Hooft].
- We will present such a simplified model that describes the gapless and a gapped phase of a topological insulator.

The Simplified Model

Extend the model with a single massless boundary fermion.

- Emergent $U(1)_a$ gauge field on the boundary
- Scalar w of $U(1)_a$ charge 1, which can Higgs it to be trivial.
- Massless fermion χ with $U(1)_A \times U(1)_a$ charges $(1, 2s)$
  - For integer s no additional anomaly associated with a.
  - Below we will argue that s has to be even.

In a phase with $\langle w \rangle \neq 0$ the low-energy spectrum consists of a massless fermion with $U(1)_A$ charge one.

So this system contains the previous system – same anomaly.

(Even the same gravitational anomalies.)

The Simplified Model

Add:

- Scalar Φ with $U(1)_A \times U(1)_a$ charges $(2, 4s)$ such that we can have a T-invariant coupling $\chi\chi\Phi^* +$ c.c. .

In a phase with $\langle w \rangle = 0$, but $\langle \Phi \rangle \neq 0$ the theory is gapped:

- Higgs $U(1)_a \to \boldsymbol{Z}_{4s}$. No massless gauge field.
- χ acquires a mass from $\chi\chi\Phi^*$
- Unbroken T and global $U(1)_A$ symmetry (linear combination of the original global $U(1)_A$ and gauge $U(1)_a$)

- Our system has the right anomaly to be a boundary state.
- It has a gapped boundary phase with a TQFT.
- Everything can be analyzed explicitly.

The Massive Spectrum

- w quanta are $U(1)_A$ neutral bosons transforming with "charge" 1 under $\boldsymbol{Z}_{4s}$.
- χ quanta are $U(1)_A$ neutral fermions transforming with "charge" $2s$ under $\boldsymbol{Z}_{4s}$.
- Interesting spectrum of vortices from $U(1)_a \to \boldsymbol{Z}_{4s}$:
  - The elementary vortex (vorticity $v = \pm 1$) has a single χ zero mode. It exhibits non-Abelian statistics.
  - More generally, all odd v vortices have non-Abelian statistics.
  - Even v vortices have Abelian statistics.

An Aside About Emergent Gauge Fields

- There must exist quasi-particles with all allowed charges under an emergent gauge symmetry
  - Otherwise there is an unphysical one-form global symmetry – unphysical degeneracies.
  - It is associated with shifting the gauge field by a flat connection, whose holonomies are not visible by the quasi-particles [Gaiotto, Kapustin, NS].
  - Such a symmetry is not present in the microscopic theory and should not exist at long distances.
- $U(1)_a$ must be compact . Otherwise a continuum of quasi-particle charges (or infinite degeneracies).
- In this case we should consider magnetic monopoles.

Monopole Operators

- As every 2+1-dimensional $U(1)_a$ gauge theory, our system has a global $U(1)_J$ symmetry, whose current is $\frac{1}{2\pi} da$.
- The local operators charged under it are monopole operators. They correspond to vortices with $v = 4s$.
- Since the microscopic system of electrons does not have this $U(1)_J$ symmetry, we would like to add the monopole operator to the Lagrangian (Hamiltonian).
- This can be done while keeping the coefficient of the monopole operator and $\hbar$ small. Preserve weak coupling.

Monopole Operators

- Must be able to add a monopole operator to the Lagrangian (Hamiltonian).
- Quantizing the monopole operator we find that a $U(1)_A \times U(1)_a$ neutral operator has spin $\frac{s}{2}$ mod $\boldsymbol{Z}$.
- Therefore, we can add it to the Lagrangian only for s even.
- This is a symptom of a more general phenomenon...

Spin/Charge Relation

Consider a system of electrons with an arbitrary Hamiltonian such that the nuclear spins are not important.

All the states of the system (in finite volume) and all the local operators must satisfy a selection rule:

$$2\ \text{Spin} = U(1)_A\ \text{Charge mod } 2$$

This is a powerful constraint on any long distance description (and any simplified model that flows to it).

One way of thinking about the spin/charge constraint is similar to ‘t Hooft anomaly constraints…

Spin/Charge Relation

‘t Hooft coupled a system with a global symmetry to a background gauge field and tracked the anomalies to long distances.

Similarly, we couple the microscopic system to a background metric and a background A.

Naively, since we have spinors, we need the background manifold to have a spin structure.

But, we can also place the system on a non-spin manifold by letting A be a $Spin_c$ connection...

Spin/Charge Relation

We can also place the system on a non-spin manifold by letting A be a $Spin_c$ connection.

(This means that the obstruction to spin structure is corrected by A not being a $U(1)$ gauge field; i.e. $\int \frac{F}{2\pi} = \frac{1}{2} \int w_2 \bmod \mathbf{Z}$.)

Note, we are not really interested in the behavior of our system of electrons in this more general background. This is merely a device to constrain the long distance behavior .

Therefore, the macroscopic theory should also satisfy this relation. And if we study it using a simplified model that flows to it, this model should also satisfy it.

Spin/Charge Relation

Back to our model with an emergent $U(1)_a$ gauge field and

- scalar w with $U(1)_A \times U(1)_a$ charges $(0,1)$
- fermion χ with $U(1)_A \times U(1)_a$ charges $(1,2s)$
- scalar Φ with $U(1)_A \times U(1)_a$ charges $(2,4s)$

All the perturbative $U(1)_a$ invariant operators satisfy the spin/charge relation (fermions have odd $U(1)_A$ charges and bosons have even $U(1)_A$ charge).

For even s this is also true for the monopole operators.

For odd s the monopole operators do not satisfy it.

Spin/Charge Relation

For odd s the perturbative spectrum satisfies the relation. But monopole operators do not satisfy it.

This is a new anomaly.

It allows us to exclude models. In our case it forces s to be even.

The Low Energy TQFT

First, we describe the $\boldsymbol{Z}_{4s}$ gauge theory as a $U(1)_a \times U(1)_c$ Chern-Simons theory [Maldacena, Moore, NS]

$$\frac{1}{2\pi}\, c\, d(4sa + 2A)\,.$$

c is dual to the phase of the Higgs field Φ. Its equation of motion constrains a to be a $\boldsymbol{Z}_{4s}$ gauge field. The coupling to A follows from the coupling of Φ.

Second, we integrate out χ to find a Chern-Simons term

$$\frac{1}{8\pi}(2sa + A)d(2sa + A)$$

Actually, it is an η-invariant [Alvarez-Gaumé, Della Pietra, Moore]

The Low Energy TQFT

$$\frac{1}{2\pi}\, c\, d(4sa + 2A) + \frac{1}{8\pi}(2sa + A)d(2sa + A)$$

The term $\frac{2s^2}{4\pi} ada$ corresponds to a Dijkgraaf-Witten term in the $\boldsymbol{Z}_{4s}$ gauge theory.

The term $\frac{1}{8\pi} AdA$ is the only term that is not properly normalized. It reflects the anomaly. It comes form the bulk of the system.

This $U(1)_a \times U(1)_c$ gauge theory is free and easy to analyze.

For example, the world lines of the w quanta are represented by the Wilson lines $\exp(i \oint a)$.

But this cannot be the whole story…

The Low Energy TQFT

$$\frac{1}{2\pi}\, c\, d(4sa + 2A) + \frac{1}{8\pi}(2sa + A)d(2sa + A)$$

But this cannot be the whole story.

- Miss a line operator for the χ quasi-particle.
- Not T invariant
- The Wilson lines $\exp(i \oint c)$ should represent the vortices. But this misses the fact that they have non-Abelian statistics.
- Integrating out χ does not quite lead to this Chern-Simons term, but to an η-invariant.

These problems are related and can be fixed…

The Low Energy TQFT

The low energy theory must include $-\frac{\pi}{2}\eta(x)$ with a constraint that sets x to a $\boldsymbol{Z}_2$ gauge field:

$$\frac{2}{2\pi}\int y\,dx - \frac{\pi}{2}\eta(x)$$

(Without the constraint $-\frac{\pi}{2}\eta(x)$ is not a good action because it is not continuous mod 2π.)

This theory is the $2+1$-dimensional TQFT that corresponds to the $1+1$-dimensional Ising model. It has three line observables:

- 1 is the identity
- W_ψ has spin $1/2$
- W_σ has spin $1/16$ and it has non-Abelian statistics.

The Low Energy TQFT

$$\frac{2}{2\pi}\int y\, dx - \frac{\pi}{2}\eta(x)$$

We need to couple it to the Abelian sector and to ensure that $x = 2sa + A$.

We do that by a $\boldsymbol{Z}_2$quotient that acts on y and c. This means they are not good $U(1)$ gauge fields, but $\tilde{c} = c + y$ and $2c$ are. In terms of them our action is

$$-\frac{\pi}{2}\eta(2sa + A) + \frac{1}{8\pi}\int (8\,\tilde{c} + 2sa + A)d(2sa + A)$$

We can think of the first term as coming from integrating out χ and the second term as coming form the bulk (only the AdA term depends on the bulk).

This action is T-invariant.

The Low Energy TQFT

Another way to think about this $\boldsymbol{Z}_2$ quotient is to correlate v odd with W_σ and v even with 1 and W_ψ; i.e. the line observables (the quasi-particles) are

- $\exp\left(i\, n \oint a \ + i\, v \oint c\right)$ with even v
- $W_\psi \exp\left(i\, n \oint a \ + i\, v \oint c\right)$ with even v
- $W_\sigma \exp\left(i\, n \oint a \ + i\, v \oint c\right)$ with odd v

Here

- n is the number of w quanta
- W_ψ represents the fundamental χ quanta
- v is the vorticity

The Low Energy TQFT

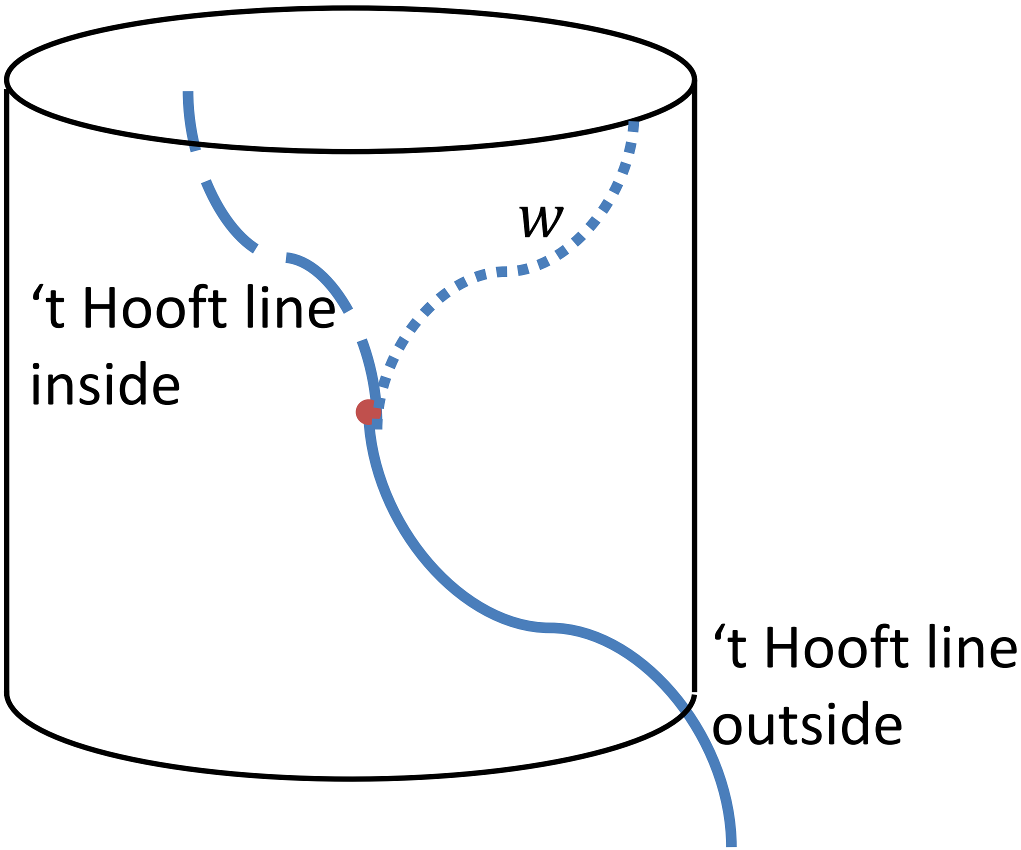
The spins, statistics, and charges of these lines reproduce the semi-classical results of the quasi-particles.

This TQFT satisfies the spin/charge relation only for even s. This is a consistency condition on our analysis and a confirmation of the anomaly in spin/charge for odd s.

This TQFT with $s = 2$ in [Metlitski, Kane, Fisher].

More about the TQFT

Recall the process of bringing a neutral magnetic monopole through the boundary. In the gapped phase it emits a w quantum from the point of entry. This manages to balance the charge.



More about the TQFT

One might not like to use $\frac{2}{2\pi}\int y\, dx - \frac{\pi}{2}\eta(x)$ for the Ising sector.

A more explicit expression arises from recalling

$$\text{Ising} = \frac{SU(2)_2}{U(1)_4} = U(2)_{2,-4}$$

2d GKO coset

Corresponding Chern-Simons description [Moore, NS]

(Not the same framing anomaly as $\frac{2}{2\pi}\int y\, dx - \frac{\pi}{2}\eta(x)$.)

$$\frac{2}{4\pi}\mathrm{Tr}\left(udu + \frac{2}{3}u^3\right) - \frac{2}{4\pi}(\mathrm{Tr}\, u)d(\mathrm{Tr}\, u)$$

More about the TQFT

$$\frac{2}{4\pi}\mathrm{Tr}\left(udu+\frac{2}{3}u^3\right)-\frac{2}{4\pi}(\mathrm{Tr}\,u)d(\mathrm{Tr}\,u)$$

Now we add the Abelian sector and perform the quotient identifying $\boldsymbol{Z}_2 \subset U(2)$ with $\boldsymbol{Z}_2 \subset U(1)_c$. This can be done by using the $\boldsymbol{Z}_2$ invariant variables

$$\tilde{u}=u\ -c\,\mathbf{1}$$

$$e=(\mathrm{Tr}\,u)-2sa$$

(the shift by $2sa$ is for convenience) to find the Lagrangian

$$\frac{2}{4\pi}\mathrm{Tr}\left(\tilde{u}d\tilde{u}+\frac{2}{3}\tilde{u}^3\right)-\frac{1}{4\pi}(\mathrm{Tr}\,\tilde{u})d(\mathrm{Tr}\,\tilde{u})\ -\frac{1}{4\pi}(e-A)d(e-A)$$
$$+\frac{1}{8\pi}(-4\,(\mathrm{Tr}\,\tilde{u})\ +6sa+3A)\,d(2sa+A)$$

e decouples. $\tilde{u}$ is effectively an $SU(2)_2$ field.

Conclusions

- Topological phases of matter are interesting.
  - They exhibit rich phenomena. Some of them have already been encountered by high energy physicists, but most of them have not.
  - Mathematics, quantum field theory, condensed matter physics…
- We have presented a weakly coupled T-invariant theory with a global $U(1)_A$ symmetry. It has two interesting phases:
  - Massless charged fermions. Hence, the correct anomaly to be the boundary of a topological insulator.
  - Gapped phase with a TQFT.
  - Explicit, calculable.

Conclusions

- The analysis of this system, despite being weakly coupled, has many interesting subtleties.
- New consistency conditions
- New anomalies
- More models
- Topological superconductors ($U(1)_A$ is broken to $\boldsymbol{Z}_2$)
- Many interesting questions