A Consistent Calculation of Sommerfeld Effect

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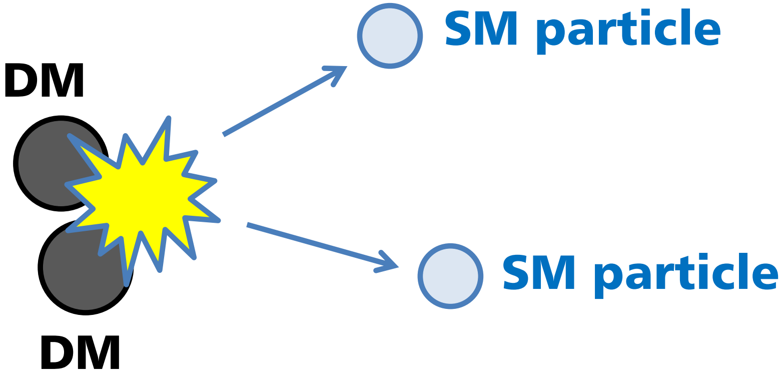
[arXiv:1603.01383]

2016. 4. 6 @ Kavli IPMU

Dark matter and its annihilation cross section



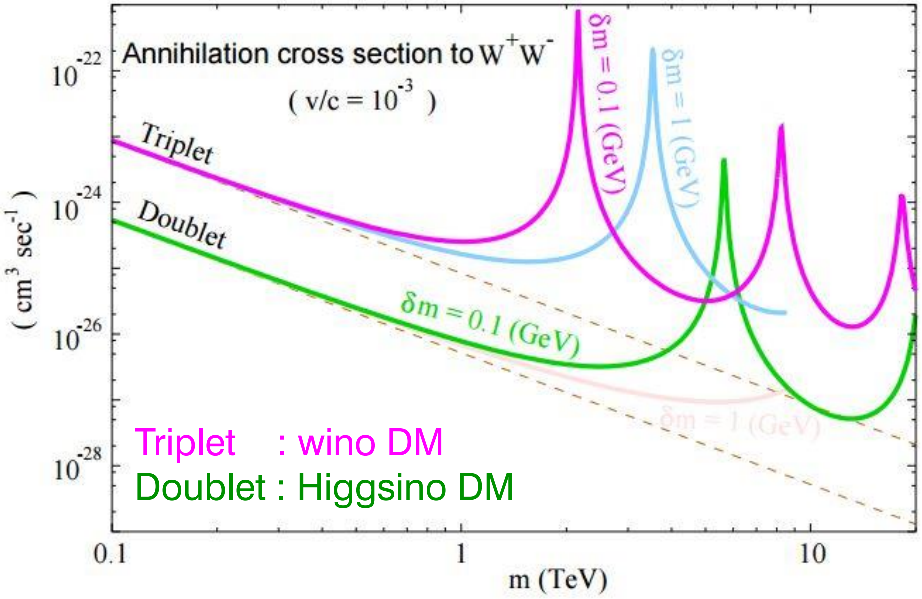
DM annihilation cross section is very important to discuss DM phenomenology.

- Thermal relic abundance
- Effect on CMB
- Observation of cosmic ray (anti-proton, positron, …)

Sommerfeld enhancement

[Sommerfeld (1931)]
[Hisano, Matsumoto, Nojiri (2003)]

If the dark matter couples with a light force carrier ($m_{force} \ll m_{DM}$),
There is large enhancement of the cross section in non-relativistic regime.



Triplet : wino DM
Doublet : Higgsino DM

Dotted lines : LO results
Solid lines : Resummed results

[Hisano, Matsumoto, Nojiri (2003)]

Non-perturbative resummation is required to calculate this effect.

“Usual” way to calculate

See, e.g., [Hisano, Matsumoto, Nojiri, Saito (2004)], [Cirelli, Strumia, Tamburini (2007)]

Long range force gives distortion from plane-wave.

1. Solve Schrodinger Eq. with long-range force

$$\left(-\frac{1}{2\mu}\nabla^2 + V(\boldsymbol{x}) - \frac{p^2}{2\mu}\right)\psi(\boldsymbol{x}) = 0 \qquad \text{with} \qquad \psi \to e^{ipz} + f(\theta)\frac{e^{ipr}}{r}$$

2. Compare the result with $V(r) = 0$ $\longrightarrow$ $\boldsymbol{\sigma} = \boldsymbol{\sigma_0} \times \boldsymbol{S(v)}$

LO cross section : σ_0

Enhancement factor : $S(v) = |\psi(\mathbf{0})|^2/|\psi_0(\mathbf{0})|^2$ with $\psi_0(\boldsymbol{x}) = e^{ipz}$

A problem of the “usual” way

$\boldsymbol{\sigma} = \boldsymbol{\sigma_0} \times \boldsymbol{S(v)}$

LO cross section : σ_0

Enhancement factor : $S(v) = |\psi(\mathbf{0})|^2 / |\psi_0(\mathbf{0})|^2$ with $\psi_0(\boldsymbol{x}) = e^{ipz}$

σ_0 and $S(v)$ are irrelevant each other.

⇕

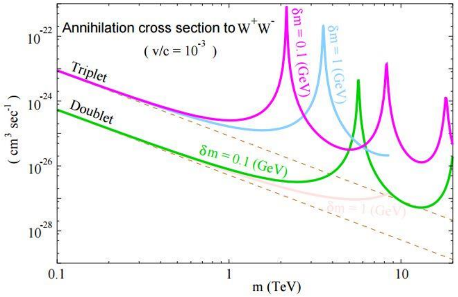
Unitarity bound on s-wave : $\sigma \leq \dfrac{\pi}{p^2}$

[Griest, Kamionkowski (1992)]
[Landau-Lifshits’s textbook]

Problematic situations

1. Large $\sigma_0 v$
2. At zero energy resonance $(S(v) \propto v^{-2} \rightarrow \sigma \propto v^{-3})$

e.g., peaks in right figure



Outline of this talk

- **Motivation**

  To obtain a formula which satisfies the **unitarity bound**.

- **Strategy**

  We treat DM annihilation as scattering problem in **non-relativistic QM**.
  An effective description is given by

$$\left(-\frac{1}{2\mu}\nabla^2 + V(|\boldsymbol{x}|) + u\delta^3(\boldsymbol{x}) - \frac{p^2}{2\mu}\right)\psi(\boldsymbol{x}) = 0$$

  What we have to do is **to solve this equation!**

- **Outline**

  1. Introduction
  2. Formalism
  3. Examples

2. Formalism

Basics: Scattering problem in QM

1. Solve the Schrodinger equation:

$$\left(-\frac{1}{2\mu}\nabla^2 + V(\boldsymbol{x}) - \frac{p^2}{2\mu}\right)\psi(\boldsymbol{x}) = 0$$

2. Determine asymptotic form ($r \to \infty$) of s-wave solution :

$$\psi(r) \ \rightarrow \ S_0 \frac{e^{ipr}}{r} - \frac{e^{-ipr}}{r}$$

3. Calculate the cross section from the above S_0.

$$\sigma_{sc} = \frac{\pi}{p^2} |S_0 - 1\,|^2, \quad \sigma_{ann} = \frac{\pi}{p^2}(1 - |S_0|^2)$$

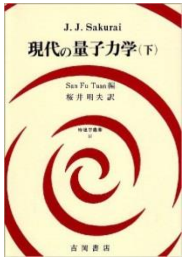
DM-DM elastic scattering

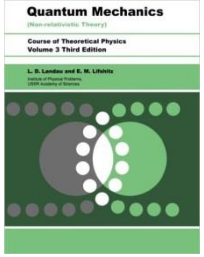
DM-DM annihilation

That's it.

What we need is s-wave solution.

See also





etc.

1. Solve the Schrodinger equation c.f. [Jackiw (1992)]

We take short range-effect as delta function potential:

$$\left(-\frac{1}{2\mu}\nabla^2 + V(|\boldsymbol{x}|) + u\delta^3(\boldsymbol{x}) - \frac{p^2}{2\mu}\right)\psi(\boldsymbol{x}) = 0$$

(u is a complex parameter.)

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(u is a complex parameter.)

⇩

$$\psi(x) = \psi_0(x) - 2\mu u G_p(x,0)\psi(0)$$

We have to solve a consistency condition at the origin.

Wave function w/o short range effect :

$$\left(-\frac{1}{2\mu}\nabla^2 + V(|\boldsymbol{x}|) - \frac{p^2}{2\mu}\right)\psi_0(\boldsymbol{x}) = 0$$

Green's function :

$$\left(-\frac{1}{2\mu}\nabla^2 + V(|\boldsymbol{x}|) - \frac{p^2}{2\mu}\right)G_p(\boldsymbol{x},0) = \frac{1}{2\mu}\delta^3(\boldsymbol{x}), \qquad G_p(\infty,0) \propto \frac{e^{ipr}}{4\pi r}$$

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(u is a complex parameter.)

$$\psi(x) = \psi_0(x) - G_p(x,0)\psi_0(0)\left(\frac{1}{2\mu u} + G_p(0,0)\right)^{-1}$$

We solved a consistency condition at the origin.

c.f.) Perturbative expansion of u:

$$\psi(x) = \underbrace{\overset{O(u^0)}{\psi_0(x)} - \overset{O(u^1)}{2\mu u G_p(x,0)\psi_0(0)}}_{\text{This part corresponds to the “usual” calculation.}} + \overset{O(u^2)}{4\mu^2 u^2 G_p(x,0)G_p(0,0)\psi_0(0)} + \overset{O(u^3)}{\cdots}$$

Wave function w/o short range effect :

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1. Solve the Schrodinger equation c.f. [Jackiw (1992)]

$G_p(0,0)$ is divergent. ex) $G_p(r,0) = \frac{e^{ipr}}{4\pi r}$ for $V(r) = 0$

It is relevant to short-range physics (UV divergence). **We need renormalization.**

$G_p(0,0) - \mathrm{Re}G_{p_0}(0,0)$ is finite

For $V(r) = \frac{V_{-1}}{r} + V_0 + V_1 r + \cdots$

$$\frac{k_{p_0}}{4\pi} \equiv \frac{1}{2\mu u} + \mathrm{Re}G_{p_0}(0,0)$$

$$\psi(x) = \psi_0(x) - G_p(x,0)\psi_0(0)\left(\frac{1}{2\mu u} + G_p(0,0)\right)^{-1}$$

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2. Determine asymptotic form of Ψ

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$$\boxed{\psi(x) = \psi_0(x) - G_p(x,0)\psi_0(0)\left(\frac{k_{p_0}}{4\pi} - \mathrm{Re}G_{p_0}(0,0) + G_p(0,0)\right)^{-1}}$$

$$G_p(x,0) = \frac{g_p(x)}{4\pi x}$$ with $g_p(0) = 1,\ g_p(x) \to d_p e^{ipx}$

We can take $\psi_0(x) = c\,\mathrm{Im}G_p(x,0)$, and $\mathrm{Re}G_p(0,0) = \frac{1}{4\pi} g_p'(0)$

$$\boxed{\psi(\infty) \propto \frac{1}{r}\left(\underbrace{\frac{d_p}{d_p^*}\frac{k_{p_0} - \mathrm{Re}g_{p_0}'(0) + g_p'^{\,*}(0)}{k_{p_0} - \mathrm{Re}g_{p_0}'(0) + g_p'(0)}}_{\text{amplitude of DM-DM s-wave scattering}}\ e^{ipr} - e^{-ipr}\right)}$$

3. Calculate the cross sections

$$S_0 = \frac{d_p}{d_p^*}\,\frac{k_{p_0} - \mathrm{Re}g'_{p_0}(0) + {g'_p}^*(0)}{k_{p_0} - \mathrm{Re}g'_{p_0}(0) + g'_p(0)}$$

$$\left(-\frac{1}{2\mu}\nabla^2 + V(|\boldsymbol{x}|) + u\delta^3(\boldsymbol{x}) - \frac{p^2}{2\mu}\right)\psi(\boldsymbol{x}) = 0$$

$$\psi(r) \rightarrow S_0\frac{e^{ipr}}{r} - \frac{e^{-ipr}}{r}$$

$$\left(-\frac{1}{2\mu}\frac{d^2}{dr^2} + V(r) - \frac{p^2}{2\mu}\right)g_p(r) = 0$$

$$g_p(0) = 1, \qquad \lim_{r\to\infty} e^{-ipr}g_p(r) = d_p$$

$$\Downarrow$$

$$\sigma_{sc} = \frac{4\pi}{p^2}\left|\frac{1-S_0}{2i}\right|^2, \qquad \sigma_{ann} = \frac{4\pi}{p^2}\frac{1-|S_0|^2}{4}$$

DM-DM elastic scattering

DM-DM annihilation

$\sigma_{sc}, \sigma_{ann}$ (S_0) are determined from k_{p_0}, $\underbrace{d_p \text{ and } g'_p(0) - Reg'_{p_0}(0)}_{\text{Long-range effect}}$.

Short-range effect

Large momentum limit (determination of k)

$$S_0 = \frac{d_p}{d_p^*}\frac{k_{p_0} - \mathrm{Re}g'_{p_0}(0) + {g'_p}^*(0)}{k_{p_0} - \mathrm{Re}g'_{p_0}(0) + g'_p(0)}$$

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DM-DM elastic scattering

DM-DM annihilation

Large momentum limit $(p^2 \gg |2\mu V|)$

$$d_p \to 1, \qquad g'_p \to ip$$

and for small coupling $(|k_{p_0}^{-1}| \ll p^{-1}, p_0^{-1})$,

$$\sigma_{sc} \to 4\pi\left|k_{p_0}^{-1}\right|^2, \qquad \sigma_{ann} \to \frac{4\pi}{p}\frac{\mathrm{Im}k_{p_0}}{|k_{p_0}|^2}$$

$$\mathrm{Re}k_{p_0}^{-1} \simeq \pm\sqrt{\frac{\sigma_{sc}}{4\pi} - \left(\frac{\mu\sigma_{ann}v}{4\pi}\right)^2}$$

$$\mathrm{Im}k_{p_0}^{-1} \simeq -\frac{\mu\sigma_{ann}v}{4\pi}$$

Relation with conventional formulae

$$S_0 = \frac{d_p}{d_p^*}\frac{k_{p_0} - \mathrm{Re}g'_{p_0}(0) + {g'_p}^*(0)}{k_{p_0} - \mathrm{Re}g'_{p_0}(0) + g'_p(0)}$$

$$\Downarrow$$

$$\sigma_{ann} = \frac{4\pi}{p^2}\frac{1-|S_0|^2}{4} = \frac{4\pi}{p^2}\frac{\mathrm{Im}k_{p_0}\mathrm{Im}g'_p(0)}{\left|k_{p_0} - \mathrm{Re}g'_{p_0} + g'_p(0)\right|^2}$$

$$\left(-\frac{1}{2\mu}\nabla^2 + V(|\boldsymbol{x}|) + u\delta^3(\boldsymbol{x}) - \frac{p^2}{2\mu}\right)\psi(\boldsymbol{x}) = 0$$

$$\psi(r) \;\rightarrow\; S_0\frac{e^{ipr}}{r} - \frac{e^{-ipr}}{r}$$

$$\left(-\frac{1}{2\mu}\frac{d^2}{dr^2} + V(r) - \frac{p^2}{2\mu}\right)g_p(r) = 0$$

$$g_p(0) = 1, \qquad \lim_{r\to\infty} e^{-ipr}g_p(r) = d_p$$

At the leading order of k^{-1} (or u),

$$\sigma v \simeq \frac{4\pi}{\mu}\frac{\mathrm{Im}k_{p_0}}{\left|k_{p_0}\right|^2} \times \frac{\mathrm{Im}g'_p(0)}{p}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\sigma v \simeq \quad (\sigma v)_0 \quad \times \quad S(v)$$

$$W_p(r) \equiv \frac{1}{2i}\left(g_p^*(r)g'_p(r) - g_p(r){g_p^*}'(r)\right)$$

$$\left\{\begin{array}{l} W'_p(r) = 0 \\ W_p(0) = \mathrm{Im}g'_p(0) \\ W_p(\infty) = p\left|d_p\right|^2 \end{array}\right. \longrightarrow \frac{1}{p}\mathrm{Im}g'_p(0) = \left|d_p\right|^2 = S(v)$$

A formula of annihilation cross section

$$\sigma v \simeq \frac{\sigma v_0\, S(v)}{\left| 1 + \left(\eta \sqrt{\frac{\mu^2 \sigma_{sc,0}}{4\pi} - \left(\frac{\mu^2 \sigma v_0}{4\pi}\right)^2} - i \frac{\mu^2 \sigma v_0}{4\pi} \right) (T(v) + iS(v))\, v \right|^2}$$

Short-range force

σv_0 : LO annihilation cross section

$\sigma_{sc,0}$: LO DM-DM elastic scattering cross section

$\eta = \mathrm{sgn} k_{p_0}^{-1}$ It is determined by matching the scattering amplitude
It is relevant that short-range force is attractive / replusive.

long-range force

$S(v) = \frac{1}{p} \mathrm{Im} g_p'(0)$ ("usual" Sommerfeld factor)

$T(v) = \frac{1}{p} [\mathrm{Re} g_p'(0) - \mathrm{Re} g_{p_0}'(0)]$

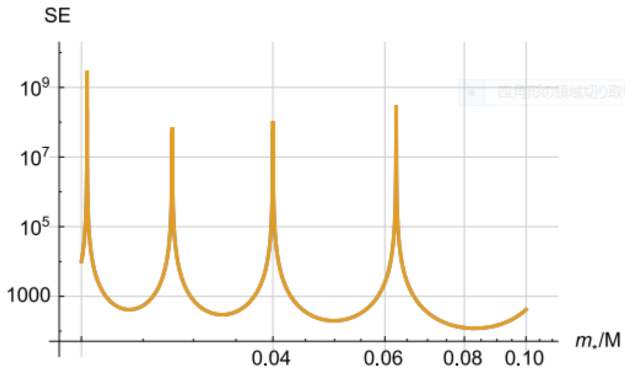
3. Examples

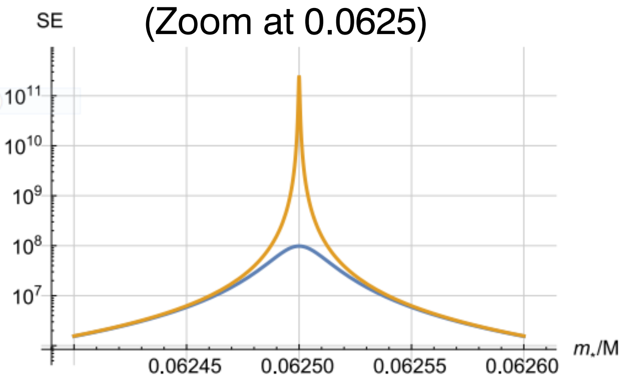
Example 1 (small bare cross section)

Hulthen potential : $V(r) = -\dfrac{\alpha m_* e^{-m_* r}}{1-e^{-m_* r}}$ (Good approximation of $V(r) = -\frac{\alpha e^{-mr}}{r}, m_* = \frac{\pi^2}{6} m$)

$$\alpha = 1, \qquad \sigma_{ann} v = \frac{1}{32\pi M^2}, \qquad \sigma_{sc} = \frac{\mu^2}{4\pi}(\sigma_{ann} v)^2$$

$v = 10^{-6}$ ⇒




(Zoom at 0.0625)

Yellow : usual formula

Blue : our formula

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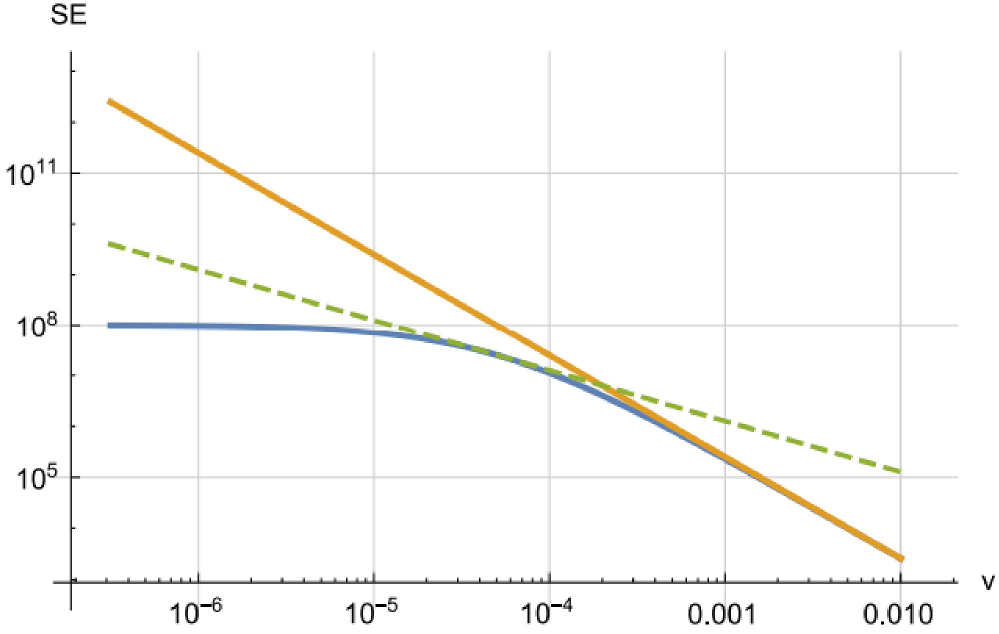
$m_* = 0.0625M$



Yellow : usual formula

Blue : our formula

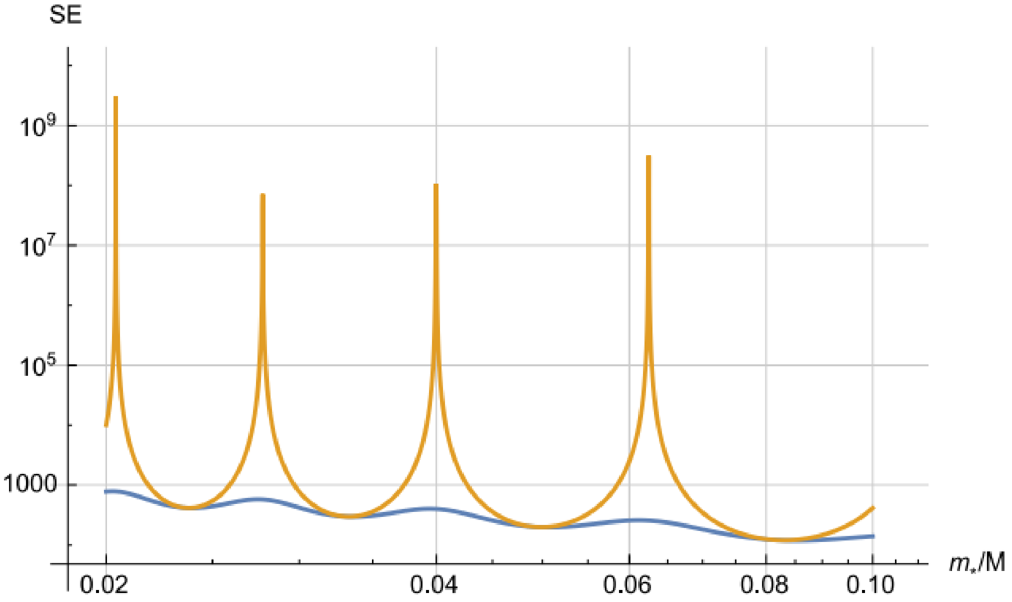
Green dotted : Unitarity bound

Example 2 (large bare cross section)

Hulthen potential : $V(r) = -\dfrac{\alpha m_* e^{-m_* r}}{1 - e^{-m_* r}}$ (Good approximation of $V(r) = -\frac{\alpha e^{-mr}}{r}, m_* = \frac{\pi^2}{6} m$)

$$\alpha = 1, \qquad \sigma v = \frac{2\pi}{M^2}, \qquad \sigma_{sc} = \frac{\mu^2}{4\pi}(\sigma v)^2$$

$v = 10^{-6}$



Yellow : usual formula

Blue : our formula

Summary

- Solved Schrodinger Eq with long-range potential and delta function potential.
- Constructed DM annihilation cross section from non-relativistic QM.
- Our formula satisfies the unitarity bound.

$$S_0 = \frac{d_p}{d_p^*}\,\frac{k_{p_0} - \mathrm{Re}\,g'_{p_0}(0) + {g'_p}^*(0)}{k_{p_0} - \mathrm{Re}\,g'_{p_0}(0) + g'_p(0)}$$

$$\Downarrow$$

$$\sigma_{sc} = \frac{4\pi}{p^2}\left|\frac{1 - S_0}{2i}\right|^2, \qquad \sigma_{ann} = \frac{4\pi}{p^2}\,\frac{1 - |S_0|^2}{4}$$

Backup

An examples of zero energy resonance

- S-wave bound-state in square-well potential

$$\left(-\frac{1}{2\mu}\nabla^2 + V(r)\right)\psi(\boldsymbol{r}) = E_{bin}\psi(\boldsymbol{x}), \qquad V(r) = -\frac{k_V^2}{2\mu}\theta(r_0 - r), \quad E_{bin.} = -\frac{\kappa^2}{2\mu}$$

$$\psi(\boldsymbol{x}) = \frac{\chi(|\boldsymbol{x}|)}{|\boldsymbol{x}|} \quad \Rightarrow \quad \left(-\frac{1}{2\mu}\frac{d^2}{dr^2} + V(r) + \frac{\kappa^2}{2\mu}\right)\chi(r) = 0, \qquad \chi(0) = 0$$

- Condition for a bound state

For $r < r_0$, $\chi(r) = A\sin(\tilde{k}r)$

For $r > r_0$, $\chi(r) = e^{-\kappa r}$

$$\Rightarrow \quad \kappa = -\tilde{k}\cot\tilde{k}r_0 \qquad \left(\tilde{k} = \sqrt{k_V^2 - \kappa^2}\right)$$

- Zero energy resonance

Tuning of the potential

$$k_V r_0 = \frac{2n-1}{2}\pi + \epsilon$$

Binding energy of the most “shallow” state

$$\kappa \simeq k_V\epsilon \to E_{bin.} \simeq -\frac{k_V^2\epsilon^2}{2\mu}$$

Another way to see

$$\left(-\frac{1}{2\mu}\nabla^2 + V(|\boldsymbol{x}|) + u\delta^3(\boldsymbol{x}) - \frac{p^2}{2\mu}\right)\psi(\boldsymbol{x}) = 0$$

(u is a complex parameter.)

Probability flux

$$j = \frac{1}{\mu}\mathrm{Im}\psi^*\nabla\psi$$

⇨

"Lost" of probability

$$\nabla \cdot j = 2\mathrm{Im}u|\psi(0)|^2\delta^3(x)$$

Annihilation cross section

$$\sigma v \ = \ -\int d^3x(\nabla \cdot j) \ = \ -2\mathrm{Im}u|\psi(0)|^2$$

Short-range	Long-range	
LO	Neglect	$-2\mathrm{Im}u$
LO	full	$-2\mathrm{Im}u|\psi_0(0)|^2$
full	full	$-2\mathrm{Im}u|\psi(0)|^2$

Finite piece of Green function

$$G_p(r,0) = \frac{g_p(r)}{4\pi r} \quad \begin{cases} \left(-\frac{d^2}{dr^2} + 2\mu V(r) - p^2\right) g_p(r) = 0 \\ g_p(0) = 0, \qquad \lim_{r\to\infty} g_p(r) e^{-ipr} = d_p \end{cases}$$

$$2\mu V(r) = \frac{V_{-1}}{r} + V_0 + V_1 r + \cdots$$

$$\Rightarrow \begin{cases} g_p(r) = (1 + g_1 r + \cdots) + V_{-1}(r + h_2 r^2 + \cdots)\log r \\ G_p(r,0) = \frac{1}{4\pi r} + \frac{V_{-1}}{4\pi}\log r + \frac{g_1}{4\pi} + \cdots \end{cases}$$

Divergent parts Finite parts

Finite part of $G_p(0,0)$ is only in real part and it is independent on p.

$G_p(0,0) - \mathrm{Re}G_{p_0}(0,0)$ is finite for any p and p_0

Regularization velocity

Full Sommerfeld factor is approximated by shifting velocity.

$$\frac{\sigma v}{\sigma v_0} \simeq S(v + v_c),$$

$$\begin{aligned} v_c &= \frac{\alpha\, m\, M\, \sigma v_0}{4\pi} \\ &\approx 2 \times 10^{-3}\, \alpha \left(\frac{M}{10\ \mathrm{TeV}}\right)^2 \left(\frac{m/M}{0.1}\right) \left(\frac{\sigma v_0}{3 \times 10^{-26}\ \mathrm{cm}^3/\mathrm{sec}}\right) \end{aligned}$$

Scattering problem in quantum mechanics

Boundary condtion of wave function ψ at large $r \equiv |\mathbf{x}|$:

$$\psi(\mathbf{x}) \to e^{ipz} + f(\theta)\frac{e^{ipr}}{r}$$

Initial-wave Scattered-wave

$$e^{ipz} \to \sum_{\ell} (2\ell+1)P_\ell(\cos\theta)\left(\frac{e^{ikr}}{2ikr} - \frac{(-1)^\ell e^{-ikr}}{2ikr}\right), \qquad f(\theta) = \sum_{\ell} (2\ell+1) f_\ell P_\ell(\cos\theta)$$

$$\psi(\mathbf{x}) \to \left(\underbrace{(1+2ipf_0)}_{S_0}\frac{e^{ipr}}{2ipr} - \frac{e^{-ipr}}{2ipr}\right) + \quad (\ell \geq 1 \text{ wave})$$

Asymptotic behavior of ψ tells us the (s-wave) cross sections.

$$\sigma_{sc} = 4\pi|f_0|^2 = \frac{\pi}{p^2}|S_0 - 1\,|^2$$ (amplitude of scattered wave)

$$\sigma_{ann} = \frac{\pi}{p^2}(1 - |S_0|^2)$$ (difference between e^{ipr} and e^{-ipr})