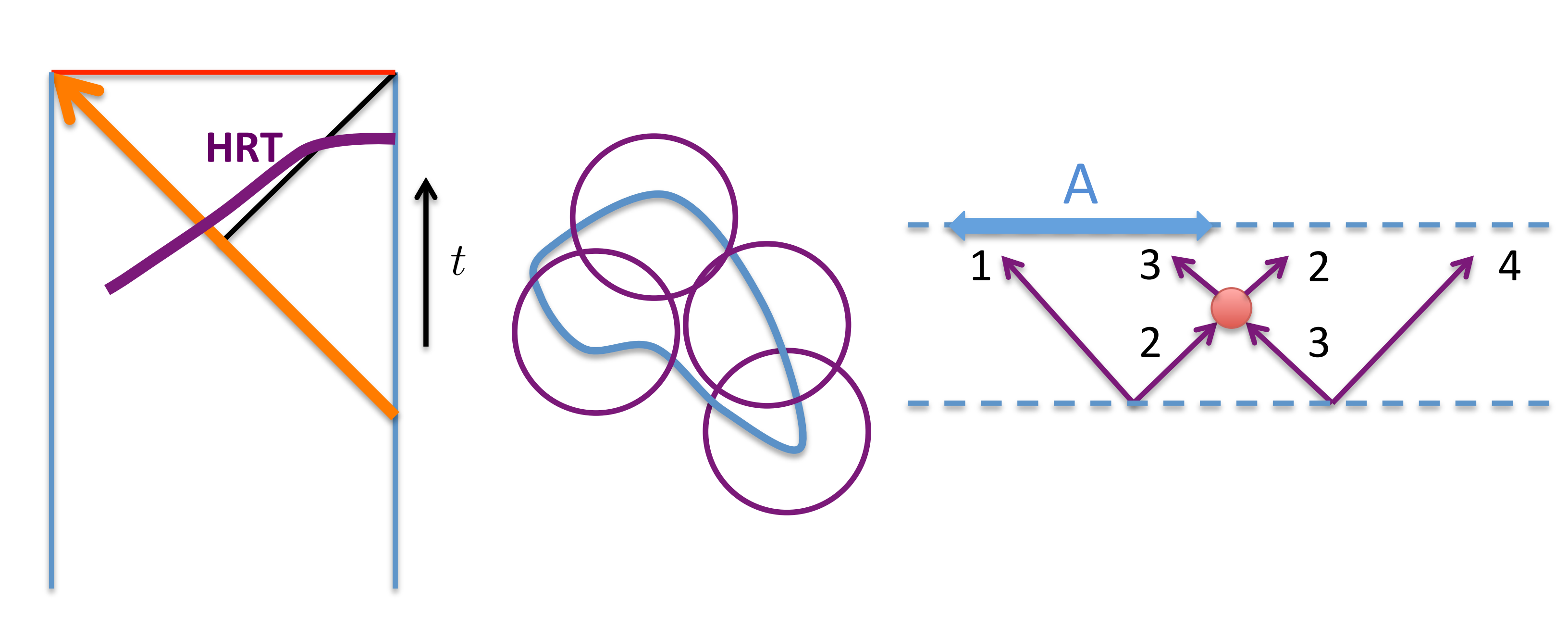
Spread of entanglement and chaos



Márk Mezei (Princeton)

MM, Stanford [to appear]; Casini, Liu, MM [1509.05044]

String Theory Seminar
IPMU, 6/6/2016

Outline

•Introduction
• Entanglement entropy
• Thermalization in subsystems
• Relation to chaos

Relations between the velocities
• General considerations
• Holographic results
• Spin chain results

Benchmarking and interpretation
• Free streaming
• Free scalar theory
• Operator and tensor network models

Summary and open questions

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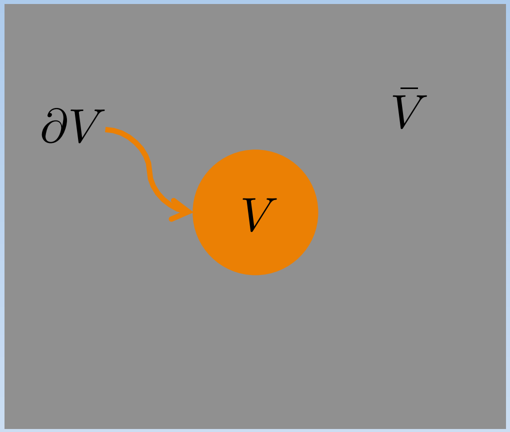
Entanglement entropy and thermalization



Entanglement entropy is an essential quantity in the study of subsystems.

- In a local theory: $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_{\bar{V}}$
- Reduced density matrix: $\rho_r = \mathrm{Tr}_{\bar{V}} \, |0\rangle \langle 0|$
- Entanglement entropy: $S_V \equiv -\mathrm{Tr}_V \, \rho_r \log \rho_r$
- Ground states of local Hamiltonians have area law entanglement
  $$S_V = \# \, \frac{A_{\partial V}}{\delta^{d-2}} + \ldots$$
- Most states in the Hilbert obey volume scaling

In chaotic systems eigenstate thermalization is believed to hold [Deutsch, Srednicki]

$$\langle E_m | \mathcal{O} | E_n \rangle = \delta_{mn} \, \mathrm{Tr} \left(\mathcal{O} \, e^{-\beta(E_m)\, H} \right) + \ldots$$

- It follows for small subsystems that $\rho_r \left(|E\rangle \right) \propto \mathrm{Tr}_{\bar{V}} e^{-\beta(E)\, H}$

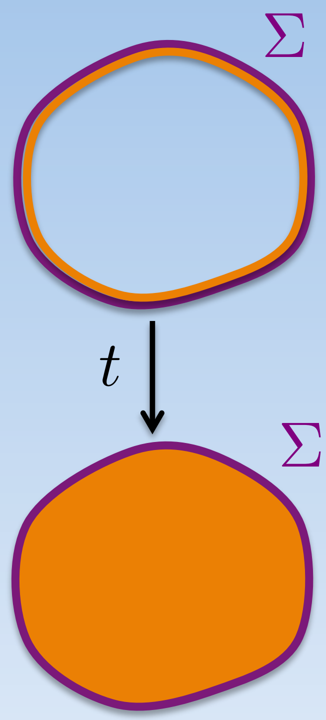
Thermalization in a closed system

- Start with short-range entangled state $|\psi(t)\rangle$
- Its decomposition into energy eigenstates dephases
- Good diagnostic of thermalization is how thermal $\rho_r \left(|\psi(t)\rangle \right)$ is

Entanglement generation in global quenches

Global quench:

- Start with QFT in a short-range entangled state at t=0. (E.g. inject uniform energy density or change the Hamiltonian)
- One-point functions reach thermal value $t_{\rm loc} \sim 1/T$
- EE (similarly to $\langle \phi(R)\, \phi(0) \rangle$) take $t_s \sim R$ to saturate to thermal value



$$S_0 = \frac{A_\Sigma}{\delta^{d-2}} + \ldots$$

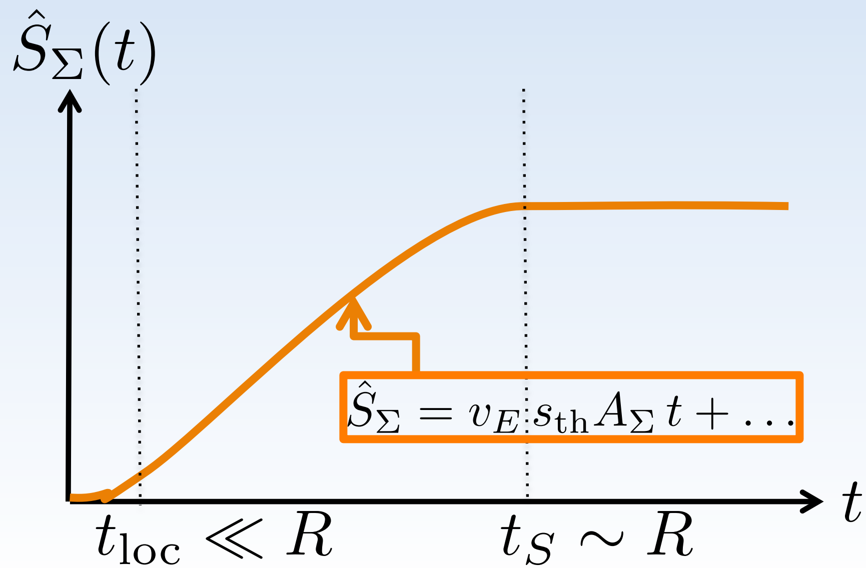
Typical point inside is unentangled with outside

$$S_{\rm eq} = s_{\rm th}\, V_\Sigma + \ldots$$

Typical point inside is entangled with outside

What is the time evolution of EE?

- 2d: numerics, CFT techniques [Huse, Kim; MM, Stanford; Calabrese, Cardy]
- d>2: holography, free field theory [Hartman, Maldacena; Liu, Suh; Cotler, Hertzberg, MM, Mueller]

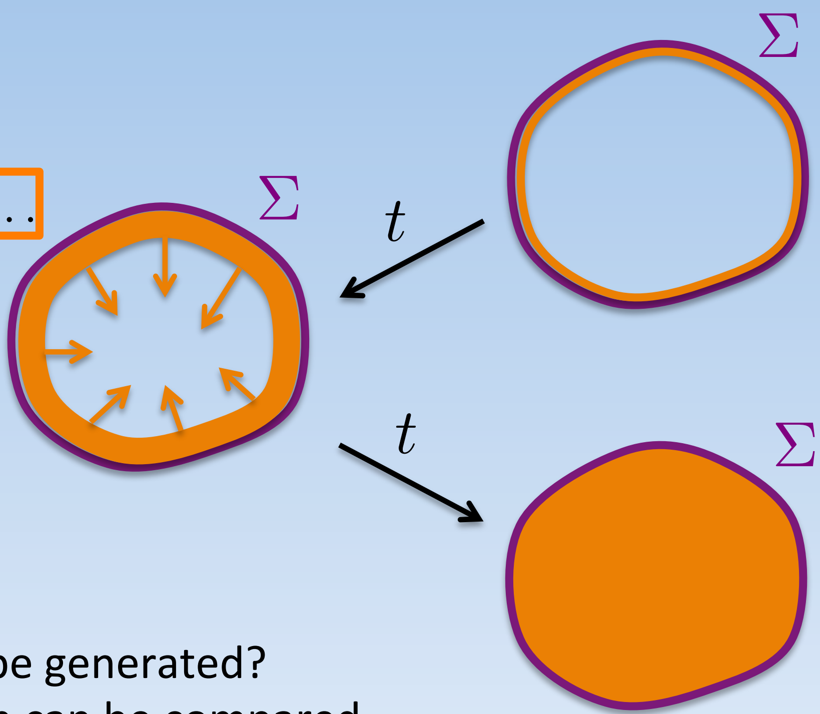


Entanglement generation in global quenches

$$S_0 = \frac{A_\Sigma}{\delta^{d-2}} + \ldots$$

Typical point inside is unentangled with outside

$$\hat{S}_\Sigma = v_E\, s_{\text{th}} A_\Sigma\, t + \ldots$$



Suggests a entanglement wave moving in with v_E . Natural picture for a local Hamiltonian.

$$S_{\text{eq}} = s_{\text{th}}\, V_\Sigma + \ldots$$

Typical point inside is entangled with outside

How fast can entanglement be generated?

- Normalized rate of growth can be compared across systems and regions:

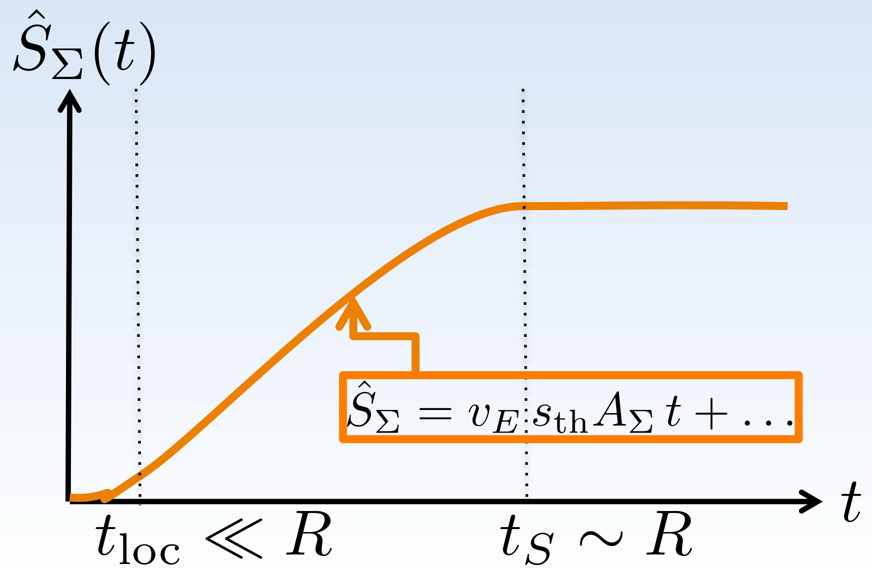
$$v_E = \frac{1}{s_{\text{th}} A_\Sigma} \frac{dS_\Sigma}{dt} \qquad (t_{\text{loc}} \ll t \ll R)$$

- Entanglement saturates with speed

$$c_E \equiv \frac{R}{t_S} \qquad (t_{\text{loc}} \ll R, t_S)$$

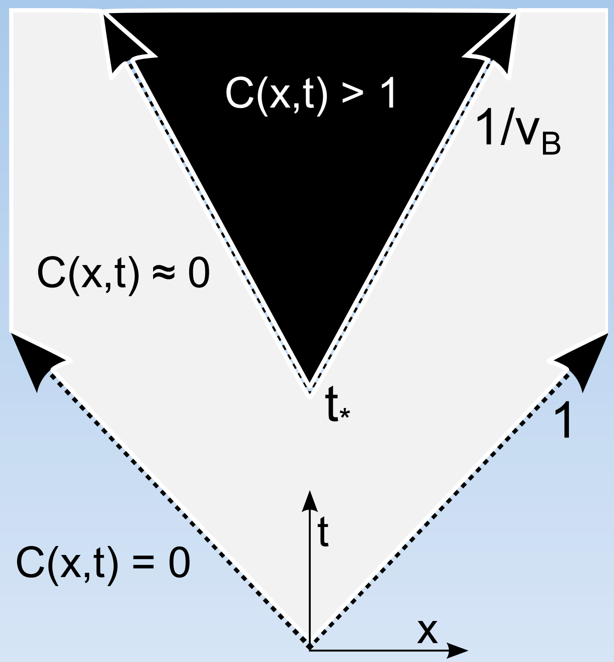
Should be constrained by causality.

- What is the relation to data characterizing chaos?



Relating entanglement growth and chaos

- Lieb-Robinson bound: even in nonrelativistic systems quantum information propagates with a finite speed
- Field theory generalization gives emergent light cone (with v_B<1) at finite energy density [Shenker, Stanford; Roberts, Stanford, Susskind]



We have defined the three velocities

v_E c_E v_B

Summary

- Studied EE spread in a global quench
- Bound from chaos and thermal relative entropy: $v_E,\, c_E \leq v_B$
- In holography: $c_E = v_B$
- Can solve for the entire $\hat{S}(t)$ curve analytically
- In chaotic spin chain: $v_E = c_E < v_B$
- Free streaming is slower than holography
- Operator and tensor network models

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Chaos and the emergent light cone at finite temperature

- Lieb-Robinson bound: even in nonrelativistic systems quantum information propagates with a finite speed

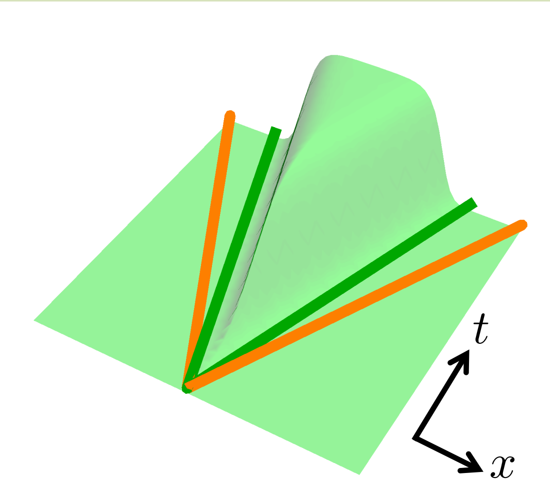
$$\frac{\|[W(x,t),V(0,0)]\|}{\|W\|\,\|V\|} \leq c_0 \exp\left[\lambda_L\left(t-\frac{x}{v_B}\right)\right]$$

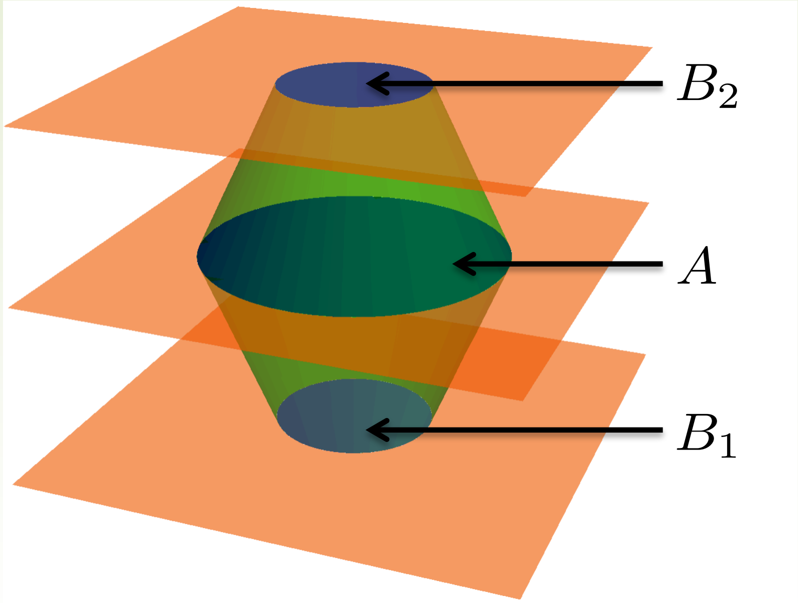
- Field theory generalization OTO thermal 4-point function [Shenker, Stanford; Roberts, Stanford, Susskind]

$$\|[W(x,t),V(0,0)]\|_2 \leq c_0 \exp\left[\lambda_L\left(t-\frac{x}{v_B}\right)\right]$$

$$\|\mathcal{O}\|_2 \equiv \mathrm{Tr}\left(\rho\,\mathcal{O}^\dagger\mathcal{O}\right)$$

- Emergent light cone (with v_B<1) at finite energy density
- $B_{1,2}$ are subsystems of A





Relations between the three velocities

Using the fact that EE is the property of causal diamonds one can show that [Casini, Liu, MM; Afkhami-Jeddi, Hartman]

$v_E,\, c_E \leq 1$

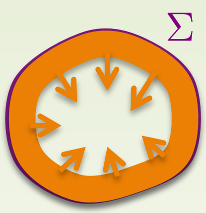
Using the emergent light cones with speed v_B we can strengthen this bound to

$v_E,\, c_E \leq v_B$

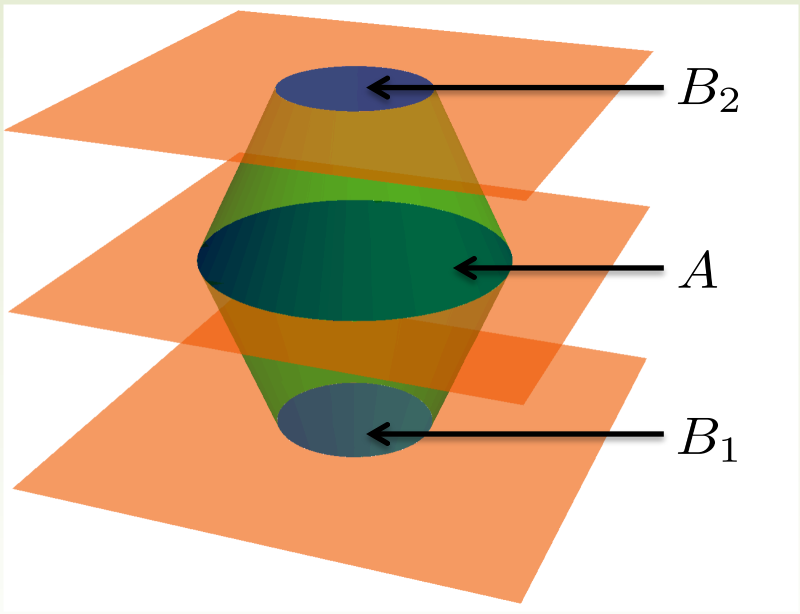
- Define thermal relative entropy $S_{\rm rel}(A) \equiv S\left(\rho_A \middle| \rho_A^{\rm th}\right) = s_{\rm th}(\beta)\, V_A - \hat{S}(\rho_A)$

- Use monotonicity of relative entropy for subregion B_1 of A to obtain

$$\hat{S}_A(t) \leq s_{\rm th}(\beta)\, V_{\rm tsunami}(t)$$



The tsunami wave front propagates with v_B.
- Analogous lower bound exists.
- The inequality between the speeds follows.



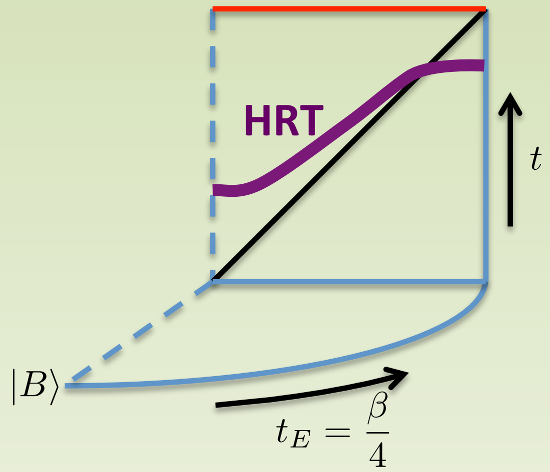
Holographic results on entanglement

Holographic models of quenches

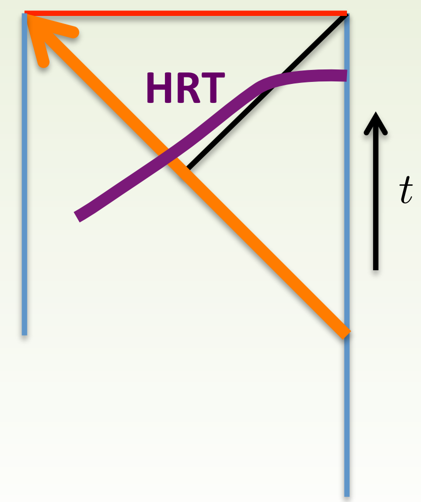
- Dual of Cardy-Calabrese boundary state is eternal BH with end of world brane [Hartman, Maldacena]
- Injecting energy density is dual to a collapsing shell. Saturation happens when the HRT surface touches the shell [Liu, Suh]
- The two setups are equivalent for large R
- v_E is determined by behind the horizon physics
- c_E is determined by near horizon physics
- Using the NEC, we can show that there are non-trivial constraints on these velocities:

$$v_E \leq v_E^{(\mathrm{SBH})}\,, \quad c_E \leq c_E^{(\mathrm{SBH})}\,,$$
$$v_E \leq c_E$$

End of the world brane quench



Vaidya quench

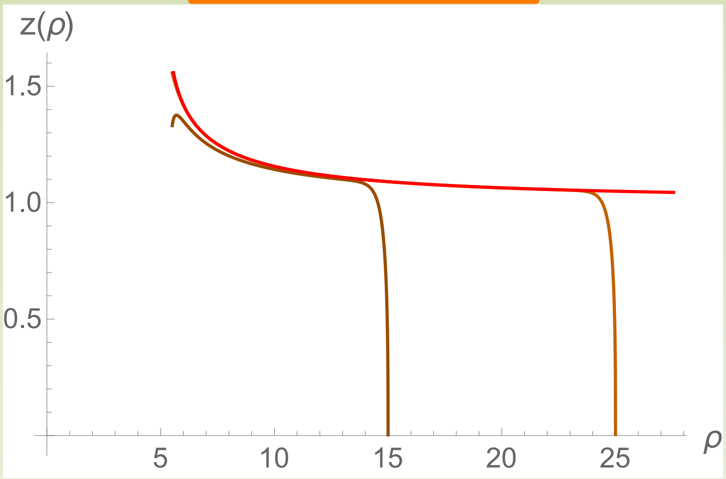


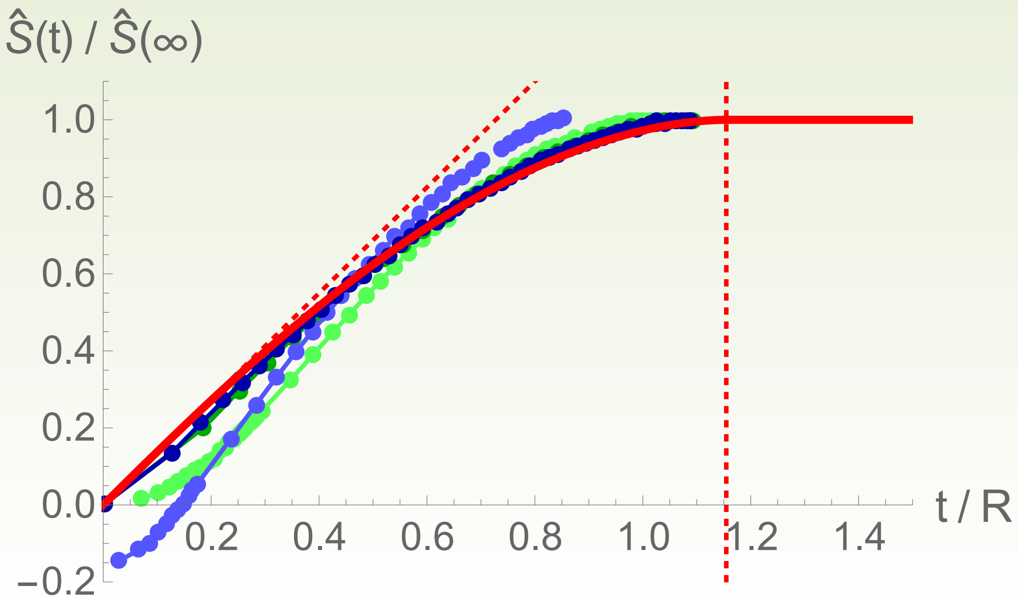
Holographic results on entanglement

Detailed understanding of how HRT surfaces are behaving

- For large R, we can understand the entropy analytically
- In both setups the minimal surfaces are close to a critical surface determined by an **algebraic equation**.
- They shoot out to the boundary exponentially fast.
- Entropy and time are given by the critical surface

Critical surface

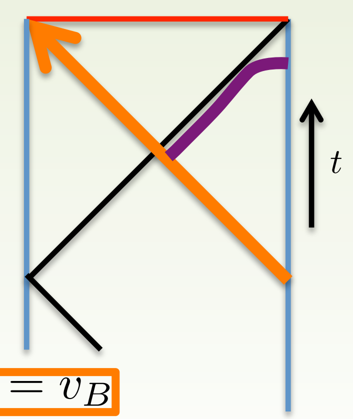




Holographic results on chaos

In holography we find that $c_E = v_B\,, \quad v_E \leq c_E$

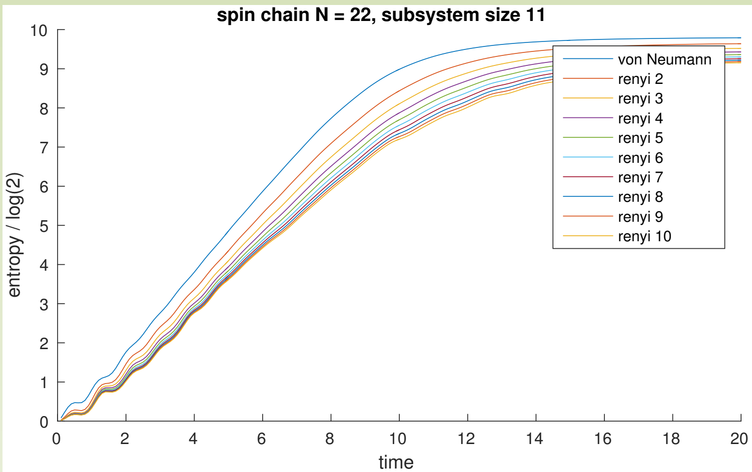
- That $c_E = v_B$ in Einstein gravity can be shown using explicit computation. Instead more insightful derivation based on entanglement wedge reconstruction.
- The size of an operator can be measured by studying its commutator with other operators: $\|[W(x,t), V(0,0)]\|_2$
- It can also be defined by the smallest ball from which it can be reconstructed.
- Holographic setup:
  - Thermal state is represented by BH
  - Acting with V(0,0) creates a particle near the boundary that falls into the BH
  - Want to find bulk subregion that contains the particle
  - By subregion-subregion duality this corresponds to a boundary subregion
  - **Summary:** find RT surface anchored on a boundary ball that contains the infalling particle
- Infalling particle's trajectory: $z(t) \approx 1 - f_1/f_2\, e^{-f_1 t}$
- Near horizon RT surface [Liu, MM]: $z(x) \approx 1 - \epsilon I_0\left(\sqrt{f_1}\,x\right)^2 \quad (d=3)$
  Reaches boundary at: $R \approx -\dfrac{\log\epsilon}{2\sqrt{f_1}}$
- Size of operator: $R = v_B\, t = \dfrac{\sqrt{f_1}}{2}\, t$
- This is the same RT surface that we needed to determine c_E → $c_E = v_B$
- Survives Gauss-Bonnet correction

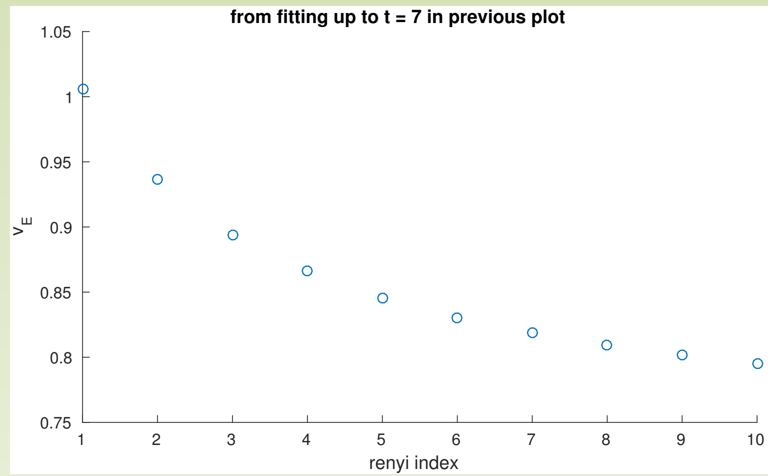


Spin chain results on entanglement and chaos

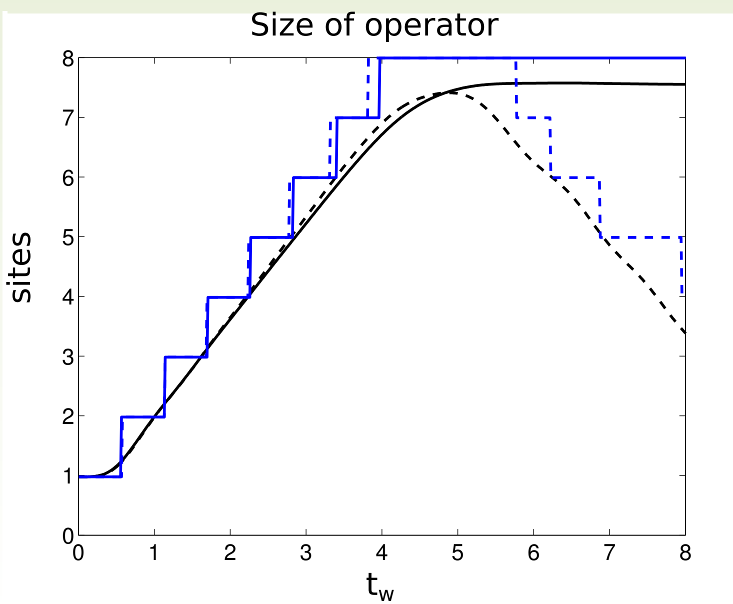
Chaotic spin chain Hamiltonian: $H = -\sum_i (Z_i Z_{i+1} - 1.05 X_i + 0.5 Z_i)$

- Entropy growth and v_E:





- Operator growth [Roberts, Susskind, Stanford]



$v_B = 2.0 > v_E, c_E$

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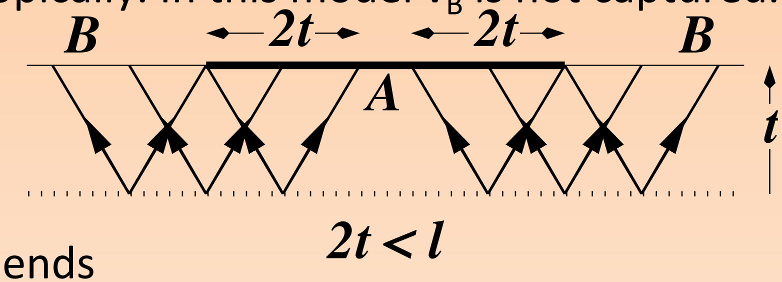
Free streaming model for entanglement spread

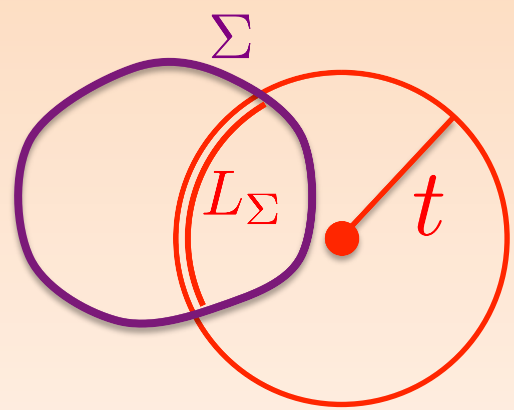
Calabrese-Cardy model: energy injection from quench creates a finite density of EPR pairs, subsequently travel freely at the speed of light isotropically. In this model v_B is not captured.

- Leads to linear growth with $v_E = 1$ in 2d



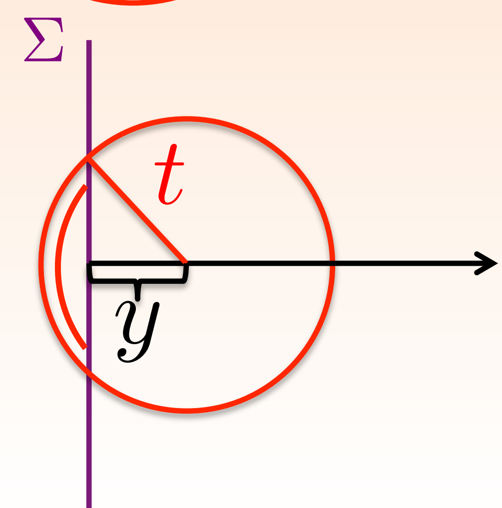
- Higher dimensions: entanglement spreading depends on entanglement pattern on the light cone $\mu[L_\Sigma]$ Contribution from each light cone has to be added:

$$\hat{S}_\Sigma(t) = \int d^{d-1}x\, \mu[L_\Sigma(\vec{x};t)]$$



- Properties of the measure:
  - Purity: $\mu[A] = \mu[\bar{A}]$
  - SSA: $\mu[A] + \mu[B] \geq \mu[A \cap B] + \mu[A \cup B]$
  - Thermalization: $\lim_{\Delta\theta \to 0} \mu[A]/\xi_A = s_{\text{th}}$
  - Upper bound: $\mu[A] \leq s_{\text{th}} \min(\xi_A, \xi_{\bar{A}})$
- Linear growth for $t_{\text{loc}} \ll t \ll R$

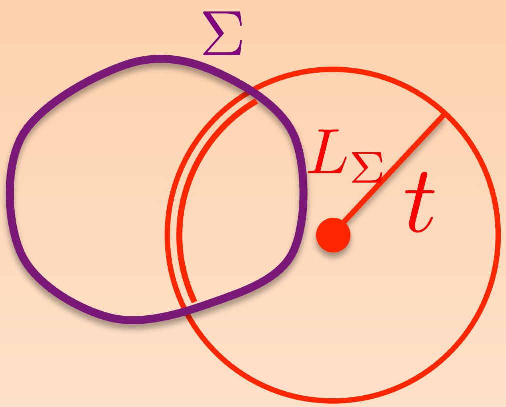
$$S_\Sigma(t) = 2A_\Sigma \int_0^t dy\, \mu_{\text{cap}}(\xi(y/t)) = \underbrace{\frac{2}{s_{\text{th}}} \int_0^1 dx\, \mu_{\text{cap}}(\xi(x))}_{v_E}\, s_{\text{th}} A_\Sigma t$$



Free streaming model for entanglement spread

Examples for measures

- EPR pairs (for small A): $\mu_{\text{EPR}}[A] = s\,\xi_A$
- 2m particle GHZ block:
  $\mu_{\text{GHZ}}[A;m] = \frac{s_{\text{th}}}{2m}\left[1-(1-2\xi_A)^m\right]$
- Random Pure State (RPS) measure inspired by [Page]
  $\mu_{\text{RPS}}[A] = s_{\text{th}}\min(\xi_A, \xi_{\bar{A}})$
  Saturates the bound $\mu[A] \leq s_{\text{th}}\min(\xi_A, \xi_{\bar{A}})$



Bound on the entanglement velocity

$$v_E = \frac{2}{s_{\text{th}}}\int_0^1 dx\,\mu_{\text{cap}}(\xi(x)) \leq 2\int_0^1 dx\,\xi(x) \equiv v_E^{\text{free}}$$

- Slower than holography: $v_E^{\text{free}} = \frac{\Gamma(\frac{d-1}{2})}{\sqrt{\pi}\,\Gamma(\frac{d}{2})} < v_E^{(\text{SBH})}$
- In strongly coupled systems, entanglement propagates faster than what's possible for free particles streaming at the speed of light!
- $c_E = 1 > c_E^{(\text{SBH})}$ is achievable, makes free streaming look even less effective
- Consider the effect of interactions: tensor network picture emerging from scattering particles is natural [Hartman, Maldacena; Casini, Liu, MM]

Entanglement spread in free scalar theory

In a free theory for Gaussian states we can use the correlation matrix to compute EE

- Time evolution of a Gaussian initial state is Gaussian (with time dependent complex kernel)
- Correlation matrix determines all correlation functions due to Wick's theorem

$$\chi_I = \begin{pmatrix} \phi_i \\ \pi_i \end{pmatrix}, \qquad [\chi_I, \chi_J] = i\, J_{IJ}$$

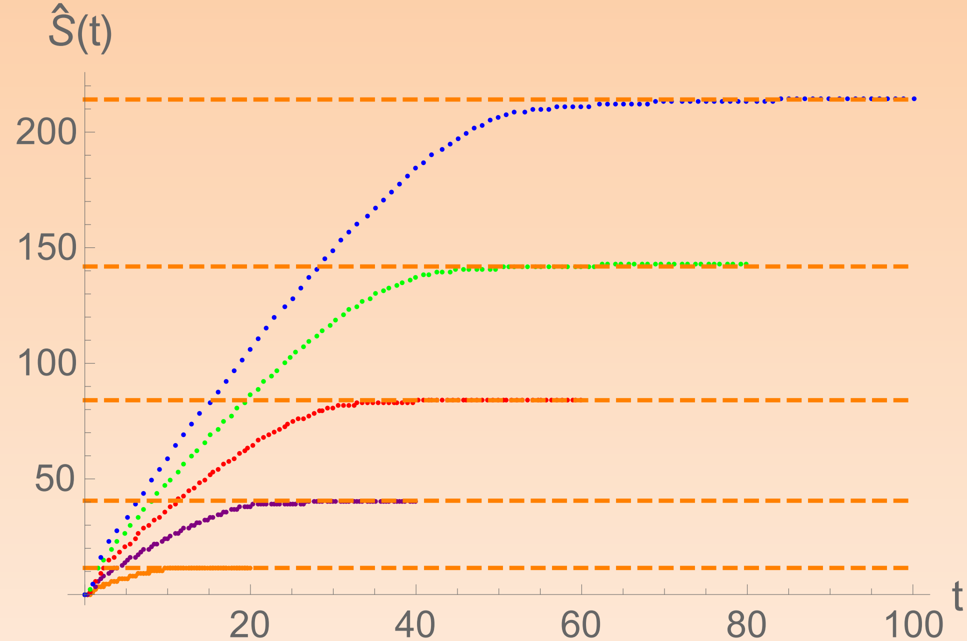
$$\Gamma_{IJ} = \frac{1}{2}\langle\psi|\{\chi_I, \chi_J\}|\psi\rangle$$

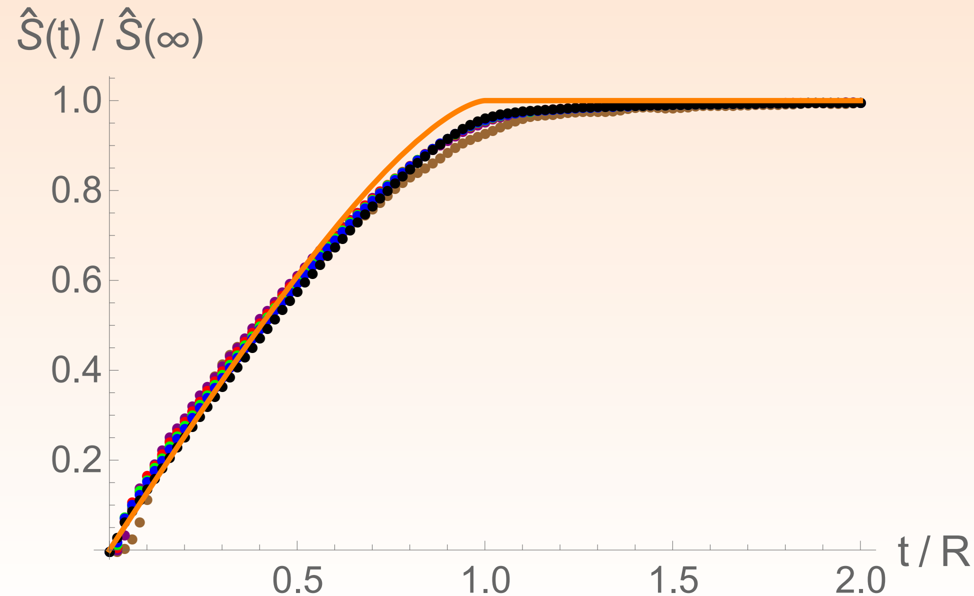
- The symplectic eigenvalues of the correlation matrix give the eigenvalues of the reduced density matrix

$$\tilde{\chi} = S\chi\,, \qquad SJS^T = J\,,$$

$$\tilde{\Gamma} = S\Gamma S^T = \begin{pmatrix} \mathrm{diag}\,(\gamma_k) & 0 \\ 0 & \mathrm{diag}\,(\gamma_k) \end{pmatrix}$$

- Numerical results for 3d mass quench for scalar field [Cotler, Hertzberg, MM, Mueller]





Operator and tensor network models

Operator counting model [Abanin, Ho]

- Closer in spirit to spin chains, infinite temperature
- The reduced density matrix is an operator, so it also spreads

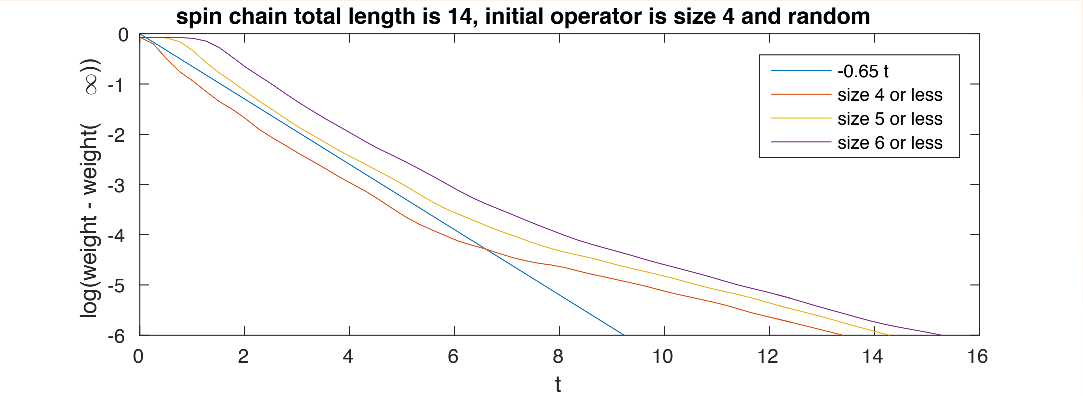
$$\rho(0) = \prod_i \frac{\mathbb{I}_i + \sigma_i^3}{2} = \frac{1}{2^{V/2}} \sum_{\mathcal{O}(0)} \mathcal{O}(0) \longrightarrow \rho_\Sigma(t) = \frac{1}{2^{V_\Sigma/2}} \sum_{\mathcal{O}(0)} \mathcal{O}(t)_\Sigma$$

- Second Rényi entropy:

$$\mathrm{Tr}_\Sigma \rho_\Sigma(t)^2 = \frac{1}{2^{V_\Sigma}} \sum_{\mathcal{O}(0),\mathcal{P}(0)} \mathrm{Tr}_\Sigma \left(\mathcal{O}(t)_\Sigma \mathcal{P}(t)_\Sigma\right)$$

$$= \frac{1}{2^{V_\Sigma}} \sum_{\mathcal{O}(0)} \mathrm{Tr}_\Sigma \left(\mathcal{O}(t)_\Sigma^2\right)$$

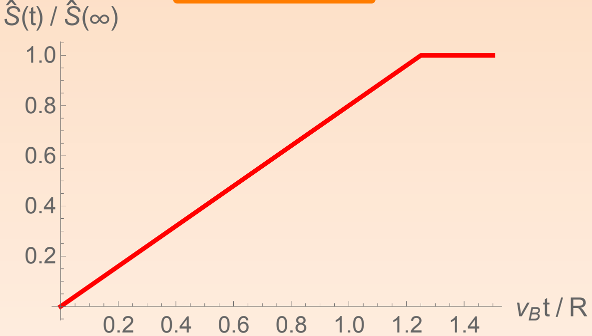
- Small operators contribution: 1
  Big operators contribution: probability of staying inside $\mathrm{Tr}_\Sigma \left(\mathcal{O}(t)_\Sigma^2\right) = 2^{-\alpha\, A[\mathcal{O}(0)](t-t_{\mathrm{delay}})}$
- Predictions:
  - d=2: linear growth until saturation with $v_E = \min(v_B, \alpha)$
    Can measure α independently $\alpha = \frac{0.65}{\log 2} = 0.94$
    Precise agreement with $S_2(t)$



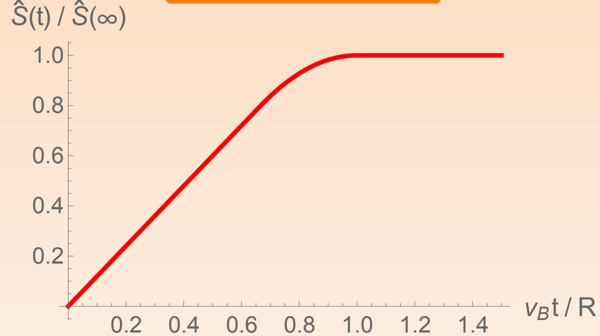
Operator and tensor network models

- Predictions:
  - d=2: linear growth until saturation with
    $v_E = \min(v_B, \alpha)$
    Precise agreement with spin chain numerics
  - d>2: three regimes

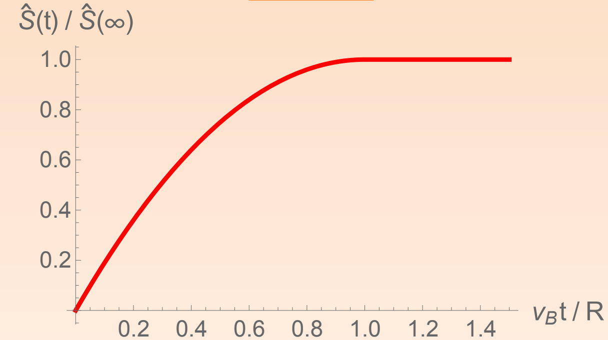
$$\frac{\alpha}{v_B} < \frac{1}{d-1}$$



$$\frac{1}{d-1} \leq \frac{\alpha}{v_B} \leq 1$$



$$1 < \frac{\alpha}{v_B}$$



Middle regime in good agreement with holographic theories

Tensor network model gives identical predictions

- Combines insight from thermal relative entropy and cuts in tensor networks
- New way of bounding the entropy introduced by a cut in a tensor network

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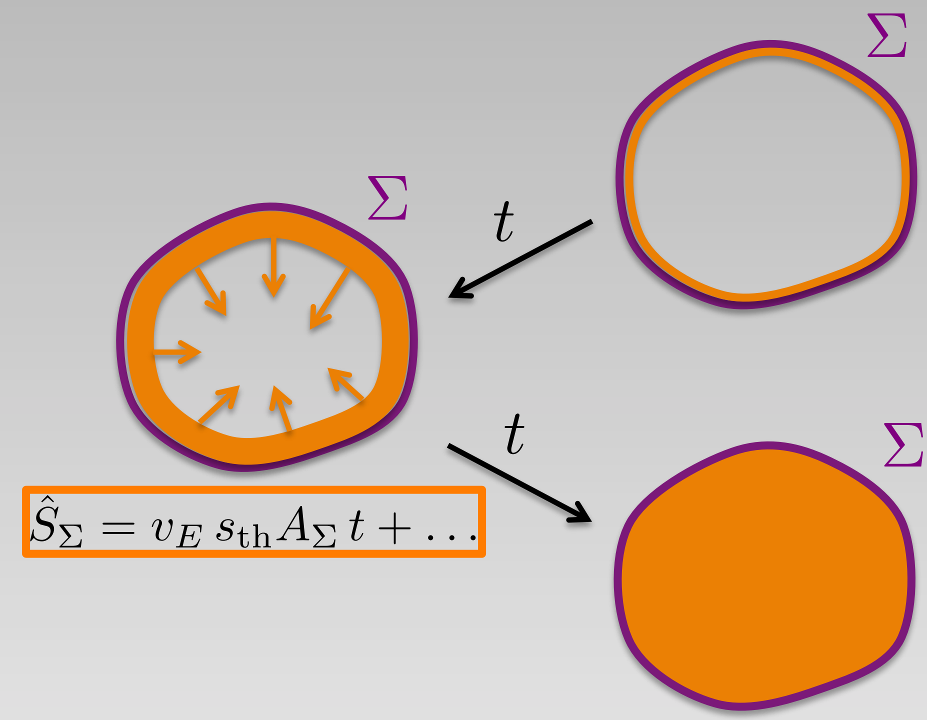


Open questions

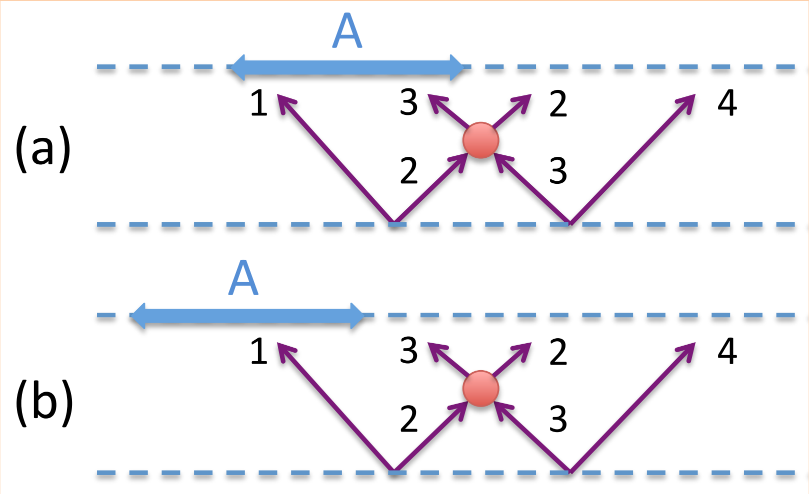
- **What is the relation between v_E and v_B?**
- Can the bound from relative entropy be realized in a CFT? Are the holographic bounds $v_E \leq v_E^{(\mathrm{SBH})}\,, \quad c_E \leq c_E^{(\mathrm{SBH})}$ universal?
- The three velocities are new observables in a QFT. Are they calculable?
  - ➢ What are they in weakly coupled theories? [v_B: Stanford]
  - ➢ What are they for perturbed 2d CFTs? [v_E: Cardy]
  - ➢ What is v_E and c_E for free theories? [Cotler, Hertzberg, MM, Mueller]

Effect of interactions

How does EE of A change in a scattering event between particles 2,3?

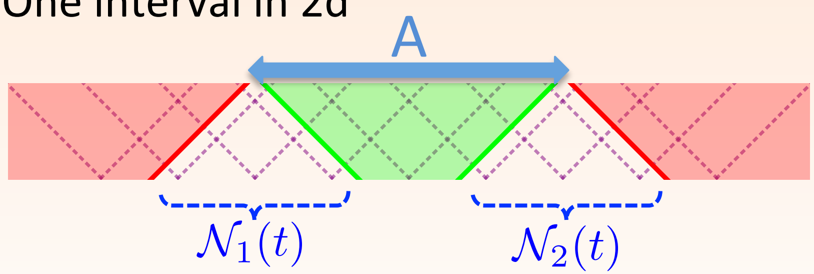
a) (1,2) and (3,4) are entangled initially. Scattering is taken into account as a unitary transformation on $\mathcal{H}_{23}$ EE can change. YES

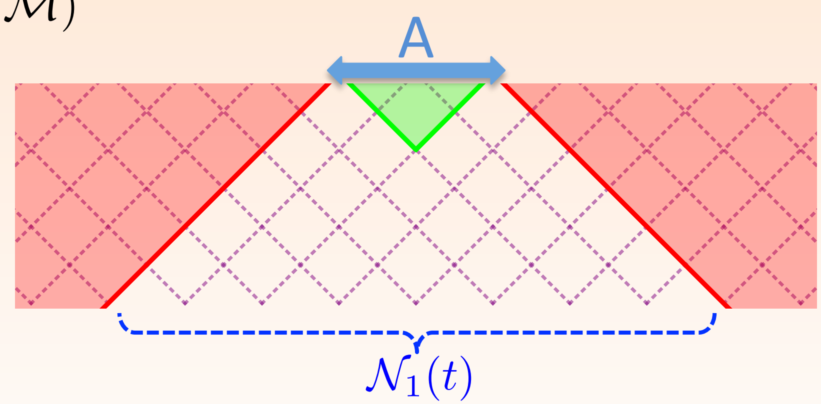
b) 1 is not affected, scattering is a change of basis in $\mathcal{H}_{\bar{A}} = \mathcal{H}_{234}$. NO



Assume that the mean free path $\ell \ll R, t$

- Scatterings that happen in $\mathcal{D}_-(\bar{A})$ or $\mathcal{D}_-(A)$ don't matter for EE
- Scatterings like (a) are effective scatterings, only particles that originate in $\mathcal{N}(t) \equiv \mathcal{M} - (\mathcal{D}_-(A) \cap \mathcal{M}) - (\mathcal{D}_-(\bar{A}) \cap \mathcal{M})$ participate in them.
- One interval in 2d



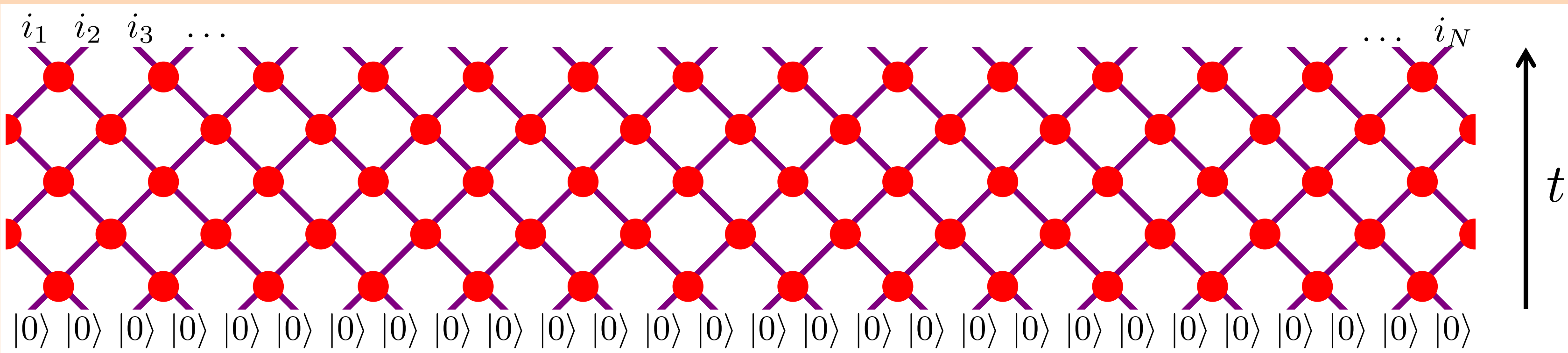


- Infinitely many scattering leads to loss of memory of the initial state, postulate RPS
- Leads to tsunami, caveat for concave regions. [Casini, Liu, MM; Leichenauer, Moosa]

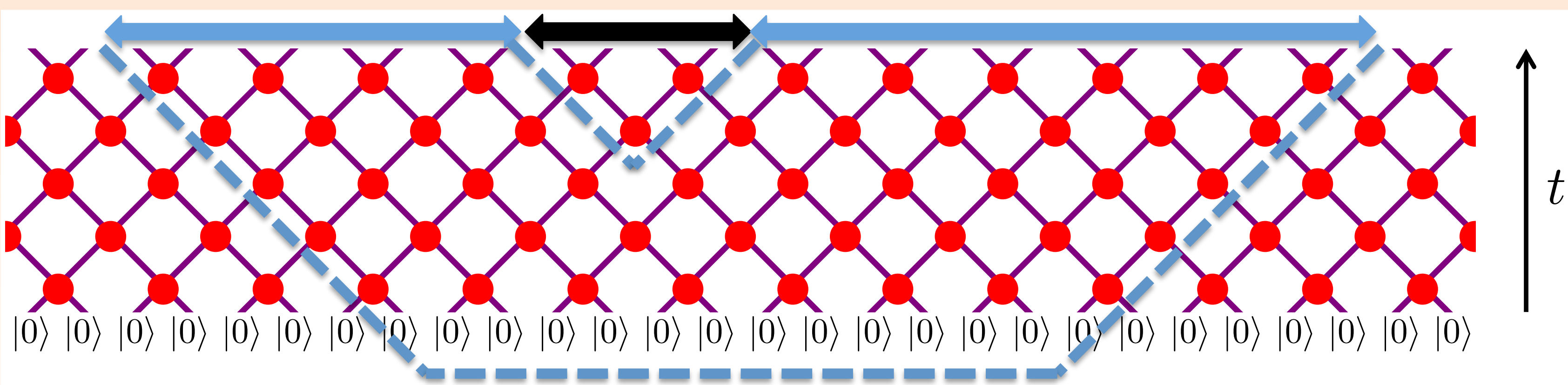
Reinterpreting the infinite scattering model

The scattering picture can be reinterpreted as a tensor network:

$$|\psi(t)\rangle = \sum_{\{i_j=1\}}^{\chi} T_{i_1 i_2 i_3 \ldots i_N}(t)\, |i_1 i_2 i_3 \ldots i_N\rangle$$ [also: Hartman, Maldacena]



- Bound on entropy from minimal cut: $S_A \leq \ell_{\text{cut}} \log \chi$



- Reproduces holography for 2d for multiple intervals. Higher dimensions?

Geometric model inspired by tensor networks

Want an entropy function that satisfies all known criteria for a (relativistic) CFT:

- Rotational invariance and scaling property
- SSA even for boosted regions
- Linear regime and saturation

The model is defined by

- Spacetime ends at t=0
- Entropy is given by a minimal surface area:
  - ➢ Area is measured using the metric $ds^2 = d\vec{x}^2$ Many degenerate surfaces
  - ➢ Slope is bounded $n = (n^0, \vec{n})$

$$\frac{|\vec{n}|}{|n^0|} \leq \frac{1}{\alpha}, \quad ~~0 < \alpha < \infty~~ \rightarrow \boxed{0 < \alpha \leq 1}$$

  - ➢ Proof of SSA is similar to [Headrick, Takayanagi]
- Tsunami velocity $v_E = \alpha$

Choosing $\alpha = v_B$ saturates the bound obtained from thermal relative entropy $\hat{S}_A(t) = s_{\rm th}(\beta)\, V_{\rm tsunami}(t)$

Reinterpretation of this model is the next task.