Boundedness results on Fano varieties. $/\mathbb{C}$.

§ Fano varieties.

X : normal projective variety with mild singularities.

$\omega_X := \bigwedge^{\dim X} \Omega_X$: canonical sheaf.
$\cong \mathcal{O}_X(K_X)$ canonical divisor.

Def. X : Fano $\iff$ $-K_X$: ample.

$$\left(\iff i : X \hookrightarrow \mathbb{P}^N \quad \mathcal{O}_X(-\overset{\exists}{m}K_X) \simeq i^*\mathcal{O}_{\mathbb{P}^N}(1) \right)$$

$\exists m \in \mathbb{Z}_{>0}$

Example: (1) $\mathbb{P}^n$: Fano

(2) $X_d \subset \mathbb{P}^n$: smooth hypersurface of deg d
Fano if $d < n+1$.

(3) X : smooth Fano curve $\iff$ $X \simeq \mathbb{P}^1$

(4) X : smooth Fano surface $\iff$ $X \simeq \begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 \text{ or} \\ S_d : \text{blowup "general" } d \text{ pts on } \mathbb{P}^2 \end{cases}$
("del Pezzo") $0 \leq d \leq 8$

(5) $\mathbb{P}(a_0, a_1, \cdots, a_n)$: weighted projective space.
is Fano with "klt" singularities.

§. Birational geometry

X, Y : algebraic varieties

Def: $X \underset{bir}{\simeq} Y$: birational equivalent

$\iff \exists\, U \subset X, V \subset Y$: open Zariski subset
s.t $U \simeq V$.

Goal of birational geometry:

Classify all algebraic varieties up to birational equivalence.

To do this, we need:

<u>Step 1</u> Find some "good model" in each birational equivalence class ← Minimal model program

<u>Step 2</u> Classify all "good models". ← Moduli problem.

<u>Conj</u> (Good Minimal Model)

$$X = \overbrace{X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow X_3 \dashrightarrow \cdots \dashrightarrow X_N}^{\text{Minimal model program}} \xrightarrow{f} Z.$$

algebraic variety
normal, proj,
w/ mild singularities

birational modifications called "divisorial contraction" or "flip"

s.t. (1) if X is not uniruled (covered by $\mathbb{P}^1$'s)

then K_{X_N}: semiample, i.e. $\exists\, X_N \xrightarrow{f} Z$

$K_{X_N} = f^*H$. H: ample div. on Z.

(2) if X is uniruled

then $X_N \xrightarrow{f} Z$: Mori fiber space
(general fiber = Fano)

<u>Example</u> (surface)

$$X \to X_1 \to X_2 \to X_3 \to \cdots \to X_N = X'$$

smooth
proj surface

contracting (-1)-curves.

s.t. either (1) $K_{X'}$: semi-ample

or (2) $X' \to Z$: MFS $\begin{cases} (2\text{-}1)\ X' \to C : \text{ruled surface} \quad (C\text{: sm. curve}) \\ (2\text{-}2)\ X' \cong \mathbb{P}^2. \end{cases}$

(~~Strat~~. bir geom of

Study of X $\leftarrow$ Study of $X_N \xrightarrow{f} Z$

$\leftarrow$ Study of general fibers & Z.
F.

blocks of algebraic varieties in birational geometry $\xleftarrow{\text{form}}$
$\begin{cases} (1)\ K_F : \text{ample} & (K_F > 0) \Leftrightarrow \text{general type} \\ (2)\ K_F \equiv 0 & (K_F = 0) \Leftrightarrow \text{Calabi-Yau} \\ (3)\ -K_F : \text{ample} & (K_F < 0) \Leftrightarrow \text{Fano} \end{cases}$

Our interest: classify Fano varieties.

Hope: $\exists$ "finitely many" Fano varieties.
= "boundedness"

Evidence: curves, surfaces.

Thm (Kollár–Miyaoka–Mori 92) Fix $n \in \mathbb{Z}_{>0}$

$\{X \mid X : \text{smooth Fano n-fold}\}$: bounded.

Def: $\{X_t\}_{t \in T}$: set of algebraic varieties

is bounded

$\overset{\text{def}}{\Leftrightarrow}$ $\exists\, \mathcal{X} \to S$: projective morphism between Noetherian schemes

s.t. $\forall t \in T$, $\exists s \in S$ s.t. $X_t \simeq \mathcal{X}_s$.
cl.pt

Note: We must consider "singularities" ~~of Fano~~ var

because, In MMP even we start from smooth X.

we get singular X_N. (with terminal singularities)

§ Singularities in MMP

X : normal projective variety

K_X : canonical div. s.t. K_X : $\mathbb{Q}$-Cartier (i.e. $\mathcal{O}_X(mK_X)$: line bundle, $\exists m \in \mathbb{Z}$

Take log resolution $Y \xrightarrow{f} X$ $\left(\begin{array}{l} = \text{a proper birational morphism s.t.} \\ Y \text{ is smooth, } \mathrm{Ex}(f) \text{ is } \underbrace{\text{divisor}}_{\text{SNC}} \end{array} \right)$

$\mathrm{Ex}(f)$ = the set where f is not biregular. $\mathrm{Ex}(f) = \sum E_i$

we may write

$$K_Y = f^* K_X + \sum_i a_i E_i$$

X is terminal if $a_i > 0$ $\forall i$. (the best sing we may expect in MMP)

canonical if $a_i \geq 0$. $\forall i$

Kawamata log terminal (klt) if $a_i > -1$ $\forall i$

log canonical if $a_i \geq -1$ (the worst sing we should have in MMP)

fix $\varepsilon \in [0,1]$ ε-klt if $a_i > -1+\varepsilon$ $\forall i$.

Note: ◦ Mori classified terminal sing in dim 3.

◦ in general, we don't have explicit description for these singularities.

two directions:
- explicit study of terminal Fano 3-folds.
- general theory of klt Fano varieties.

§. Explicit results on terminal Fano 3-folds. (= $\mathbb{Q}$-Fano 3-folds.)

Thm [Kawamata 92, Kollár-Miyaoka-Mori-Takagi 00]

{$\mathbb{Q}$-Fano 3-folds}: bounded.

Explicit boundedness results on $\mathbb{Q}$-Fano 3-folds, X.

◦ $\frac{1}{330} \leq (-K_X)^3 \leq 64$ ("$\leq$": Prokhorov, "$\geq$": J.A. Chen - M. Chen 08)

• $|-mK_X|$ gives a birational map $\forall m \geq 97$ (M. Chen - J 15)

($\forall m \geq 39$ if $\rho(X) = 1$)

$$X \overset{|-mK_X|}{\dashrightarrow} \mathbb{P}^{N = \dim|-mK_X|}$$
$$x \longmapsto [s_0(x) : \cdots : s_N(x)]$$

Tools for 3-dim ~~terminal sing~~
- classification of $\mathbb{Q}$
- orbifold Riemann-Roch
- $-K_X = \mathcal{O}(4)$
- ~~Basket~~

Example: • $X = \mathbb{P}^3 \rightsquigarrow (-K_X)^3 = 64$

• $X = X_{66} \subset \mathbb{P}(1, 5, 6, 22, 33)$

general weighted hypersurface of deg 66.

$\rightsquigarrow$ • X: $\mathbb{Q}$-Fano 3-fold with $\rho(X) = 1$.

• $(-K_X)^3 = 1/330$

• $|-33K_X|$: birational

$|-32K_X|$: not birational.

Why these two invariants are important?

Suppose $\{X\}$: set of Fano varieties of dim n.

s.t. (1) $(-K_X)^n \le V$. $\quad$ ($\exists V, \exists m$ indep of X)

(2) $|-mK_X|$ gives a birational map

$\Longrightarrow X \overset{|-mK_X|}{\underset{(2)\Rightarrow \text{birational}}{\dashrightarrow}} Z \subset \mathbb{P}^N$

s.t. $\deg Z \le (-mK_X)^n \overset{(2)}{\le} \underbrace{m^n \cdot V}_{\text{constant}}$.

$\Rightarrow \{Z\}$: bounded.

§ General theory

Conj (Borisov - Alexeev-Borisov) Fix $n \in \mathbb{Z}_{>0}$, $\varepsilon \in (0,1)$

$\{X \mid X : \varepsilon\text{-klt Fano } n\text{-fold}\}$: bounded.

Conj (Weak BAB) Fix $n \in \mathbb{Z}_{>0}$, $\varepsilon \in (0,1)$

$\Rightarrow \exists$ ~~$M = M$~~ $V = V(n, \varepsilon)$ s.t.

if X : ε-klt Fano n-fold

then $(-K_X)^n \le V(n, \varepsilon)$

<u>Conj</u> (Birational BAB) Fix $n \in \mathbb{Z}_{>0}$, $\varepsilon \in (0,1)$

$\{X \mid X : \varepsilon\text{-klt Fano } n\text{-fold}\}$: birationally bounded.

<u>Note</u>: we must assume "$\varepsilon > 0$" in BAB & WBAB.

$X = \mathbb{P}(1,1,n)$: klt. Fano surface

$(-K_X)^2 = \frac{(n+2)^2}{n} \nearrow \infty$ as $n \nearrow \infty$.

<u>Rem</u>: ◦ BAB $\Rightarrow$ WBAB

$\Downarrow$

BBAB

◦ BAB OK for $n=2$ [Alexeev 94]

open for $n \geq 3$.

$\exists$ partial results:

boundedness of sm. Fano n-folds [KMM92]

$\mathbb{Q}$-fact. terminal Fano 3-fold with $\rho = 1$ [Kawamata 92]

Canonical Fano 3-fold [KMMT00]

toric Fano n-folds [Borisov-Borisov 94]

◦ WBAB ok for $n=2$ [Alexeev 94]

$\exists M(2, \varepsilon)$: optimal [J13]

ok for $n \geq 3$ [J14]

Open for $n \geq 4$.

◦ BBAB holds trivially true in dim 2.

even without "$\varepsilon > 0$"

($\because$ X : klt Fano surface $\Rightarrow$ X : rational $(X \overset{\text{bir}}{\sim} \mathbb{P}^2)$)

<u>But</u> in dim ≥ 3, we must assume "$\varepsilon > 0$"

due to counterexamples by [Lin 04] $n = 3$

[Okada 09] $n \geq 6$, $n \geq 3$.

BBAB holds in dim 3 [J15]

Recently Birkar claimed the following results.

- Fix $n \in \mathbb{Z}_{>0}$, $\varepsilon \in (0,1)$.
  $\exists\, m = m(n, \varepsilon)$ s.t.
  if X: ε-klt Fano n-fold
  then $|-mK_X|$: birational.
- Assume $BAB_{n-1} \Rightarrow WBAB_n$

$\rightsquigarrow$ $BAB_2 \Rightarrow WBAB_3 \Rightarrow BBAB_3$.
[Alexeev]