

Boundedness results on Fano varieties. / C

§ Fano varieties.

X : normal projective variety with mild singularities.

$\omega_X := \bigwedge^{\dim X} \Omega_X$: canonical sheaf.
 $\cong \mathcal{O}_X(K_X)$ canonical divisor.

Def: X : Fano $\iff -K_X$: ample.

$(\iff i: X \hookrightarrow \mathbb{P}^n$
 $\mathcal{O}_X(-mK_X) \cong i^* \mathcal{O}_{\mathbb{P}^n}(1)$
 $\exists m \in \mathbb{Z} > 0$)

Example: (1) \mathbb{P}^n : Fano

(2) $X_d \subset \mathbb{P}^n$: smooth hypersurface of deg d
 Fano iff $d < n+1$.

(3) X : smooth Fano curve $\iff X \cong \mathbb{P}^1$

(4) X : smooth Fano surface $\iff X \cong \begin{cases} \mathbb{P}^1 \times \mathbb{P}^1 & \text{or "general"} \\ S_d: \text{blowup } \sqrt{d} \text{ pts on } \mathbb{P}^2 & \end{cases}$
 ("del Pezzo") $0 \leq d \leq 8$

(5) $\mathbb{P}(a_0, a_1, \dots, a_n)$: weighted projective space.
 is Fano with "klt" singularities.

§. Birational geometry

X, Y : algebraic varieties

Def: $X \stackrel{\text{bir}}{\sim} Y$: birational equivalent

$\iff \exists U \subset X, V \subset Y$: open Zariski subset
 s.t. $U \cong V$.

Goal of birational geometry:

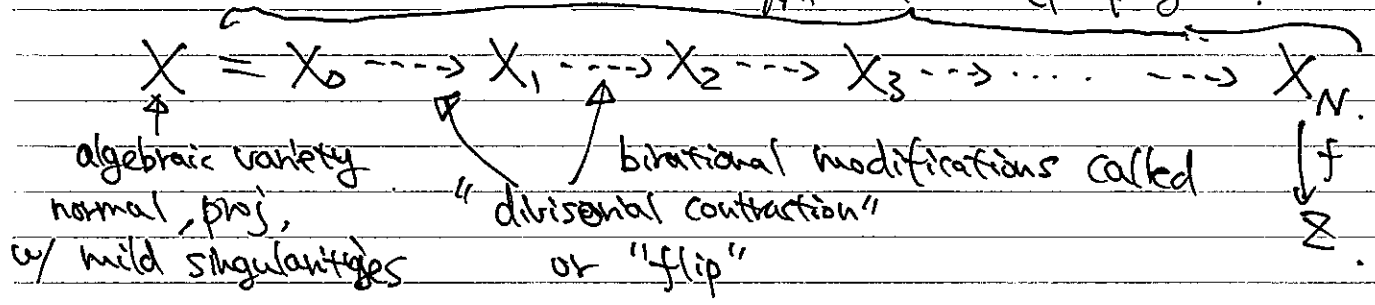
classify all algebraic varieties up to birational equivalence.

To do this, we need:

Step 1 Find some "good model" in each birational equivalence class \leftarrow Minimal model program

Step 2 Classify all "good models". \leftarrow Moduli problem.

(Conj) (Good Minimal Model)



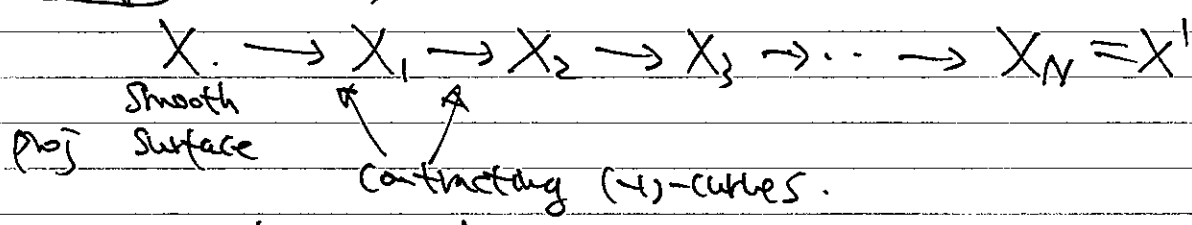
Sit. (1) if X is not uniruled (covered by \mathbb{P}^1 's)

then K_{X_N} : semiample, i.e., $X_N \xrightarrow{f} \mathbb{Z}$
 $K_{X_N} = f^* H$. H : ample div. on \mathbb{Z} .

(2) if X is uniruled

then $X_N \xrightarrow{f} \mathbb{Z}$: Mori fiber space.
 (general fiber = Fano)

Example (surface)



Sit. either (1) $K_{X'}$: semiample

or (2) $X' \rightarrow \mathbb{Z}$: MFS

- (2-1) $X' \rightarrow \mathbb{Z} \rightarrow \text{Sm. curve}$: ruled surface
- (2-2) $X' \cong \mathbb{P}^2$.

~~Study~~ bir geom of

Study of $X \leftarrow$ Study of $X \xrightarrow{F} Z$

\leftarrow Study of general fibers \mathcal{X} of Z .

blocks of algebraic varieties in birational geometry form $\left\{ \begin{array}{l} (1) K_F: \text{ample } (K_F > 0) \leftrightarrow \text{general type} \\ (2) K_F \equiv 0 \quad (K_F = 0) \leftrightarrow \text{Calabi-Yau} \\ (3) -K_F: \text{ample } (K_F < 0) \leftrightarrow \text{Fano} \end{array} \right.$

Our interest: classify Fano varieties.

Hope: \exists "finitely many" Fano varieties.

Evidence: curves, surfaces. = "boundedness"

Thm (Kollar-Miyaoka-Mori '92) Fix $n \in \mathbb{Z}_{>0}$

$\{X \mid X: \text{smooth Fano } n\text{-fold}\}$: bounded.

Def: $\{X\}_{\text{algebraic}}$: set of algebraic varieties is bounded

def $\exists X \rightarrow S$: projective morphism between Noetherian schemes

s.t. $\forall t \in T, \exists s \in S$ s.t. $X_t \cong X_s$ d.pt

Note: We must consider "singularities" of ~~var~~ X

because even we start from smooth X .

In MMP we get singular X_X (with terminal singularities)

Singularities in MMP

X : normal projective variety

K_X : canonical div. s.t. $K_X: \mathbb{Q}$ -Cartier (ie. $\mathcal{O}_X(nK_X)$: line bundle)

$\exists m \in \mathbb{Z}$

Take log resolution $Y \xrightarrow{f} X$ (= a proper birational morphism s.t. Y is smooth, $Ex(f)$ is divisor)

We may write

$$K_Y = f^* K_X + \sum a_i E_i$$

the set where f is not biregular. \swarrow SNC \searrow $\sum E_i$

X is terminal if $a_i > 0 \forall i$. (the best sing we may expect in MMP)
 Canonical if $a_i \geq 0 \forall i$

Kawamata log terminal (klt) if $a_i > -1 \forall i$

log canonical if $a_i \geq -1$ (the worst sing we should have in MMP)

fix

$\epsilon \in [0, 1]$ ϵ -klt if $a_i > -1 + \epsilon \forall i$

Note: Mori classified terminal sing in dim 3.

in general, we don't have explicit description for these singularities.

two directions $\left\{ \begin{array}{l} \text{explicit study of terminal Fano 3-folds.} \\ \text{general theory of klt Fano varieties.} \end{array} \right.$

Explicit results on terminal Fano 3-folds.

Thm [Kawamata, Kollar-Miyasaka-Mori-Takagi'00] \mathbb{Q} -Fano 3-folds.
 $\{ \mathbb{Q}$ -Fano 3-folds $\}$: bounded.

Explicit boundedness results on \mathbb{Q} -Fano 3-folds: X .

$\frac{1}{330} \leq (-K_X)^3 \leq 64$ (" \leq ": Prokhorov, " \geq ": J.A. Chen - M. Chen)

$| -mK_X |$ gives a birational map $\forall m \geq 97$ (M. Chen - J)

$X \xrightarrow{| -mK_X |} \mathbb{P}^{N = \dim(-mK_X)}$ ($\forall m \geq 39$ if $\rho(X) = 1$)

Tools for classification of ^{3-dim} terminal sing
orbifold Riemann-Roch ~~Basket~~
 $-K_X = O(4)$

Example: $X = \mathbb{P}^3 \rightarrow (-K_X)^3 = 64$

$X = X_{66} \subset \mathbb{P}(1, 5, 6, 22, 33)$

general weighted hypersurface of deg 66

$\rightarrow X: \mathbb{Q}$ -Fano 3-fold with $\rho(X) = 1$.

$(-K_X)^3 = \frac{1}{330}$

$| -33K_X |$: birational

$| -32K_X |$: not birational.

Why these two invariants are important?

Suppose $\{X\}$: set of Fano varieties of dim n .

s.t. (1) $(-K_X)^n \leq V$.

(2) $| -mK_X |$ gives a birational map

($\exists V, \exists m$ indep of X)

$\Rightarrow X \xrightarrow{| -mK_X |} Z \subset \mathbb{P}^N$
(1) \Rightarrow birational

s.t. $\deg Z \leq (-mK_X)^n \leq \frac{m^n \cdot V}{\text{constant}}$

$\Rightarrow \{Z\}$: bounded.

§ General theory

Conj (Borisov - Alexeev - Borisov) Fix $n \in \mathbb{Z}_{>0}$, $\epsilon \in (0, 1)$

$\{X \mid X: \epsilon\text{-klt Fano } n\text{-fold}\}$: bounded.

Conj (Weak BAB) Fix $n \in \mathbb{Z}_{>0}$, $\epsilon \in (0, 1)$

~~for~~ $\Rightarrow \exists V = V(n, \epsilon)$ s.t.

if $X: \epsilon\text{-klt Fano } n\text{-fold}$

then $(-K_X)^n \leq V(n, \epsilon)$

Conj (Birational BAB) Fix $n \in \mathbb{Z}_{\geq 0}$, $\epsilon \in (0, 1)$

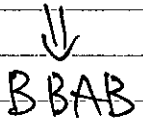
$\{X \mid X: \epsilon\text{-klt Fano } n\text{-fold}\}$: birationally bounded.

Note: we must assume " $\epsilon > 0$ " in BAB & WBAB

$X = \mathbb{P}(1, 1, n)$: klt. Fano surface

$$(K_X)^2 = \frac{(n+2)^2}{n} \nearrow \infty \text{ as } n \nearrow \infty.$$

Rem: \circ BAB \Rightarrow WBAB



\circ BAB \circ OK for $n=2$ [Alexeev 94]
open for $n \geq 3$.

\exists partial results:

boundedness of sm. Fano n -folds [KMM92]

Qfact. terminal Fano 3-fold with $\rho=1$ [Kawamata 92]

canonical Fano 3-fold [KMMT00]

toric Fano n -folds [Borisov-Borisov 94]

\circ WBAB OK for $n=2$ [Alexeev 94]

$\exists M(z, \epsilon)$: optimal [J13]

OK for $n \geq 3$ [J14]

Open for $n \geq 4$.

\circ BBAB holds trivially true in dim 2

even without " $\epsilon > 0$ "

($\because X$: klt Fano surface $\Rightarrow X$: rational ($X \stackrel{\text{bit}}{\sim} \mathbb{P}^2$))

But in dim ≥ 3 , we must assume \circ " $\epsilon > 0$ "

due to counterexamples by [Lin 04] $n=3$

[Okada 09] $n \geq 6, n \geq 3$

BBAB holds in dim 3 [JIS]

Recently Birkar claimed the following results:

o Fix $n \in \mathbb{Z}_{>0}$. $\Sigma \in \{0,1\}$.

$\exists m = m(n, \epsilon)$ s.t.

if $X: \Sigma$ -klt Fano n -fold

then $| -mK_X |$: birational.

o Assume $BAB_{n-1} \Rightarrow WBAB_n$

$\rightsquigarrow BAB_2 \Rightarrow WBAB_3 \Rightarrow BBAB_3$.

[Alexeev]