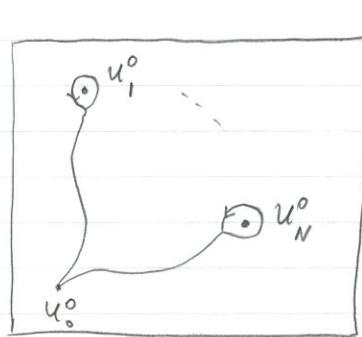


Painleve property for the Schlesinger equations

1. Schlesinger equation.

 \mathbb{C}

$$\nabla^o = d - A^o(\lambda) d\lambda$$

$$A^o(\lambda) = \sum_{i=1}^N \frac{A_i^o}{\lambda - u_i^o}$$

Fuchsian
connection on
 $\mathbb{P}^1 \times \mathbb{C}^P$

A_i^o is a $p \times p$ matrix

We assume $\sum_{i=1}^N A_i^o \neq 0 \Rightarrow \lambda = \infty$ is a sing. point too.

$$\mu: \pi_1(\mathbb{C} - \{u_1^o, \dots, u_N^o, u_o^o\}, u_o^o) \rightarrow GL_p(\mathbb{C}) \text{ monodr. repr.}$$

$\{A_i(u_1, \dots, u_N)\}_{i=1}^N$, satisfying

$$\frac{\partial A_i}{\partial u_j} = \frac{[A_j, A_i]}{u_j - u_i} \quad \text{for } 1 \leq j \neq i \leq N$$

Schlesinger equations

$$\sum_{j=1}^N \frac{\partial A_i}{\partial u_j} = 0, \quad 1 \leq i \leq N$$

$$A_i(u_1^o, \dots, u_N^o) = A_i^o, \quad 1 \leq i \leq N$$

The system

$$\partial_\lambda Y(\lambda, u) = \left(\sum_{i=1}^N \frac{B_i(u)}{\lambda - u_i} \right) Y(\lambda, u)$$

$$\partial_{u_i} Y(\lambda, u) = - \frac{B_i(u)}{\lambda - u_i} Y(\lambda, u), \quad 1 \leq i \leq N$$

is compatible if $\{A_i(u)\}_{i=1}^N$ satisfy the Schlesinger equation

- The Schlesinger equations are compatible \Rightarrow

if u_i is suff. close to u_i^* the solution always exists.

- $\nabla = d - \sum_{i=1}^n \frac{A_i(u)}{\lambda - u_i} d\lambda$ is an isomonodromic deformation.

2) Malgrange, Miwa: \exists a holom. function $T(u_1, \dots, u_N)$

s.t. if $T(u_1, \dots, u_N) \neq 0$, then
on the univ. cover
of $\{u_i \neq u_j\} \subset \mathbb{C}^n$

every local

~~the~~ solution extends to a neighb. of (u_1, \dots, u_N) .

Global deformations.

$$\mathcal{Z}_{N+1} = \left\{ u = (u_0, u_1, \dots, u_{N+1}) \in (\mathbb{P}^1)^{N+2} : u_i \neq u_j \text{ for } i \neq j \right\}$$

$u_{N+1} = \infty$

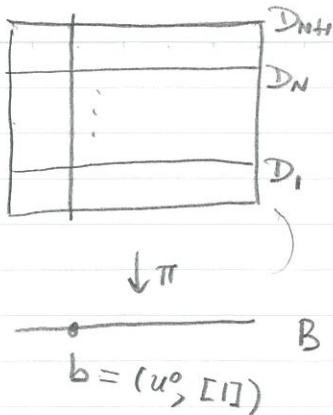
$u \in \mathcal{Z}_{N+1}$ correspond to $\mathbb{P}^1 - \{u_1, \dots, u_{N+1}\} = \mathbb{C} - \{u_1, \dots, u_N\}$
w/ a base point u_0 .

$B :=$ universal cover of \mathcal{Z}_{N+1} .

Our goal: The local solution $\{A_i(u)\}_{i=1}^N$ to the Schlesinger equations extends to a meromorphic function on B . //
2. The abstract vector bundle E .

$$D_i = \left\{ (\lambda, u, [c]) \in \mathbb{P}^1 \times B : \lambda = u_i \right\} \subset \mathbb{P}^1 \times B$$

$$\mathbb{P}^1 \times B \setminus \bigcup_{i=1}^{N+1} D_i \xrightarrow{\pi} B \quad \text{smooth fibration w/ fiber}$$



diffom. to $\mathbb{P}^1 - \{u_1^o, \dots, u_{N+1}^o\} =$

$$\mathbb{C} - \{u_1^o, \dots, u_N^o\}$$

Fact: B is ~~a~~ contractible

Rem: B is also a Stein manifold.

\Rightarrow the natural inclusion induces an Isomorphism

$$\pi_1(\mathbb{C} - \{u_1^o, \dots, u_N^o\}, u_0^o) \xrightarrow{\cong} \pi_1(\mathbb{P}^1 \times B \setminus \bigcup_{i=1}^{N+1} D_i, (u_0^o, b))$$

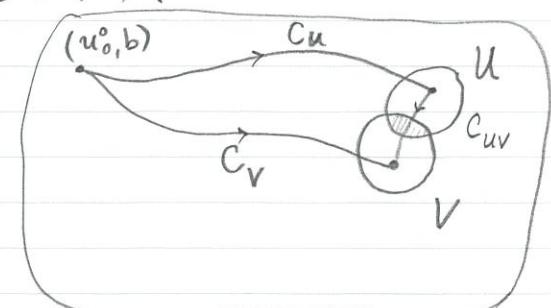
\Rightarrow we get a representation

$$\mu: \pi_1(\mathbb{P}^1 \times B \setminus \bigcup_{i=1}^{N+1} D_i, (u_0^o, b)) \rightarrow GL(\mathbb{C}^p).$$

\Rightarrow we can define a holom. v.b. $E \rightarrow \mathbb{P}^1 \times B \setminus \bigcup_{i=1}^{N+1} D_i$

equipped w/ a flat connection.

$$f = (f_1, \dots, f_p) \subset \mathbb{C}^p$$



$$(f_1^v, \dots, f_p^v) = (f_1^v, \dots, f_p^v) \cdot g_{vu}$$

flat frame
over U

$$g_{vu} = \mu(c_v^{-1} \circ c_{uv} \circ c_u)$$

3. Extending E across D_i .

Levi's theory for ∇° : $Y^\circ(\lambda)$ fundamental (solution) matrix of ∇°

If $1 \leq i \leq N$, then

$\exists S_i$ const. inv. matrix s.t.,
 ↗ holom. invertible near $\lambda = u_i$

$$Y^*(\lambda) S_i = U_i^*(\lambda) (\lambda - u_i^*)^{K_i} (\lambda - u_i^*)^{-E_i}$$

for λ near u_i^*

$$E_i = \text{Diag}(E_i^1, \dots, E_i^{P_i}) \quad \text{w/ } E_i^j = g_i^j \cdot I + \text{upper-triang.}$$

$$0 \leq \operatorname{Re} g_i^j < 1$$

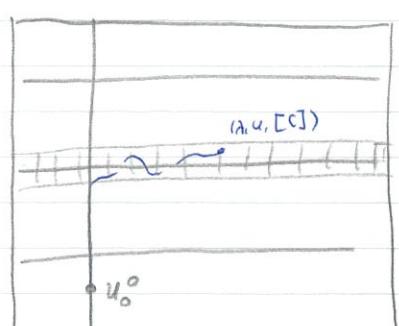
$K_i = \text{Diag}(K_i^1, \dots, K_i^{P_i})$ w/ K_i^j diagonal w/ decreasing integer entries

Remark: 1) $e^{2\pi\sqrt{-1}E_i}$ is the monodromy of $Y^*(\lambda) S_i$ around $\lambda = u_i$.

2) The entries of K_i are called exponents of the singular point.

If $i = N+1$, then same is true for $\lambda \neq \infty$ near

$$Y^*(\lambda) S_{N+1} = U_{N+1}^*(\lambda) \cdot (\lambda - u^*)^{-K_{N+1}} \lambda^{-E_{N+1}}.$$



choose $f = (f_1, \dots, f_p) \subset E_{(u^*, b)}$ basis

s.t. the monodromy coincides w/
the monodromy of $Y^*(\lambda)$

$$f^{(i)} = f \cdot S_i \quad \text{same monodromy as } Y^*(\lambda) S_i$$



$$f^{(i)} \cdot (\lambda - u_i)^{-E_i} (\lambda - u_i)^{-K_i}$$

is a

trivialization of E in ~~T_i - D_i~~ $T_i - D_i$, T_i : tubular nbhd
of D_i

$$E|_{T_i \setminus D_i} \cong (T_i \setminus D_i) \times \mathbb{C}^P \hookrightarrow T_i \times \mathbb{C}^P$$

We extend E on $\mathbb{P}^1 \times B$ so that $E|_{T_i} \cong T_i \times \mathbb{C}^P$, $\forall i$.

Properties of E :

- $E|_{\mathbb{P}^1 \times \{\infty\}}$ is trivial

$\therefore f \cdot Y^\circ(\lambda)^{-1}$ is a trivializ. of $E|_{\mathbb{P}^1 \times \{u_1^\circ, \dots, u_{N+1}^\circ\}}$

near $\lambda = u_i$ we have

$$f \cdot Y^\circ(\lambda)^{-1} = f \cdot s_i (Y^\circ(\lambda) s_i)^{-1} = \underbrace{f^{(i)} (\lambda - u_i^\circ)^{-E_i} (\lambda - u_i^\circ)^{-k_i}}_{\substack{\text{extends to a frame} \\ \text{over } \lambda = u_i^\circ}} (Y_i^\circ(\lambda))^{-1}$$

below.
invertible

$\Rightarrow f \cdot Y^\circ(\lambda)^{-1}$ is a ~~not~~ global trivializ. \square

by
Levi's
theory

- By definition $E|_{\{\infty\} \times B}$ is trivial

$\Rightarrow \exists$ an analytic subvariety $\mathcal{O} \subset B$ w/ $\mathcal{O} = \emptyset$ or

\mathcal{O} codim 1, s.t., $E|_{\mathbb{P}^1 \times (B - \mathcal{O})}$ is trivial

4. The global meromorphic solution.

$\tilde{e} = (\tilde{e}_1, \dots, \tilde{e}_p)$ trivializ. frame of $E|_{\mathbb{P}^1 \times (B - \mathcal{O})}$.

s.t. $\tilde{e}|_{\mathbb{P}^1 \times \{\infty\}} = f \cdot Y^\circ(\lambda)^{-1}$

Near $\lambda = \infty$, i.e., tubular nbhd of D_{N+1} :

$$f_{N+1} \cdot S_{N+1} \cdot \lambda^{E_{N+1}} \lambda^{K_{N+1}} = \tilde{e} \cdot \tilde{U}(\lambda, u)$$

\uparrow
holom. invertible

$$\tilde{U}(\lambda, u) = \tilde{U}_0(u) + \tilde{U}_1(u) \cdot \lambda^{-1} + \dots$$

Put $e = \tilde{e} \cdot \tilde{U}_0(u) \tilde{U}_0(u^*)^{-1}$.

Define $Y(\lambda, u, [c]) \in GL_p(\mathbb{C})$

if

multi-valued function on
 $\mathbb{P}^1 \times (B - \mathbb{D})$ w/ possible poles
 along D_i , $1 \leq i \leq N+1$

$$f = e \cdot Y(\lambda, u, [c])$$

- If λ is close to u_i , i.e. in a tubular nbhd of D_i :

$$(f S_i) \cdot (\lambda - u_i)^{-E_i} (\lambda - u_i)^{-K_i} = e \cdot U_i(\lambda, u)$$

$$\Rightarrow Y(\lambda, u, [c]) S_i = U_i(\lambda, u) \cdot (\lambda - u_i)^{r K_i} (\lambda - u_i)^{E_i}$$

- At $(u, [c]) = (u^*, [1])$ we have $e = f \cdot Y^*(\lambda)^{-1} \Rightarrow$

$$Y(\lambda, u^*, [c]) = Y^*(\lambda)$$

$$\Rightarrow dY(\lambda, u, [c]) \cdot Y(\lambda, u, [c])^{-1} = \sum_{i=1}^N \frac{A_i(u)}{\lambda - u_i} d(\lambda - u_i)$$

w/ $A_i(u)$ holom. fns on $B - \mathbb{D}$

Note that $A_i(u^*) = A_i^*$ and $\{A_i(u)\}_{i=1}^N$ satisfy the Schlesinger equation.