Fuzzy spheres and M2-M5 systems in ABJM

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November 10, 2009

based on HN, C. Papageorgakis, S. Ramgoolam, arXiv:0903.3966 and
HN and C. Papageorgakis, arXiv:0908.3263

PART I

- String theory and Matrix theory

- The ABJM model

- Fuzzy 2-spheres vs. fuzzy 3-spheres

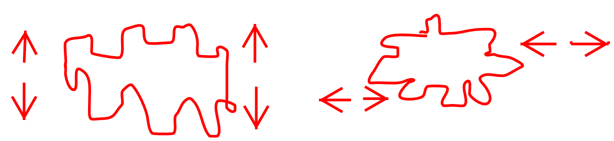
1. String theory and Matrix theory

•Quantum gravity unification: at Planck energy (dim. analysis)

$$E_{Pl} = (\frac{\hbar c^5}{G})^{1/2} \sim 10^{19} GeV$$

for a relativistic (c) quantum ($\hbar$) gravity (G) theory.

•At this energy, matter particles (leptons, quarks, Higgs and exotics) and gauge bosons (force particles) are all tiny vibrating strings.



•Different vibration modes $\longrightarrow$ different particles (fields): Quantum theory of different particles.

•Also, classical spin-mass relation $J = \alpha_0 + \alpha' M^2$ gets quantized.

•Each $(J_n, M_n) \longrightarrow$ different particle species: infinite number or species.

•Perturbative treatment: S matrices well defined. Need 10 dimensions.

•Nonperturbatively: No full definition.

•Coupling g_s becomes 11th dimension. Nonperturbative objects: Dp-branes (extended in p spatial dimensions), In particular, D-particles.

•Spacetime and gravity should emerge from theory.

•**Matrix theory**: partial attempt:

•Quantum mechanics of N D-particles as $N \to \infty$.

•Physics of N D0's: matrices: $N \times N$ d.o.f. $(X^i)^{ab}(t)$ and super-partners. $a, b = 1, .., N$; $i = 1, ..., 9$. (transverse coordinates of an object moving in 11th dimension).

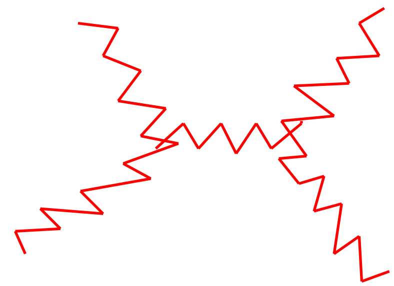
•Spacetime is emergent. Position of classical objects 1,2

$$X^i = \begin{pmatrix} x_1^i \mathbb{1}_{N_1 \times N_1} & 0 & 0 \\ 0 & x_2 \mathbb{1}_{N_2 \times N_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

•Classical, perturbative, gravitational interaction from quantum interactions of Matrix theory. E.G., 1-loop Matrix fluctuations $\to$ classical interaction of 2 gravitons in flat space.

•1 graviton = collection of N_1 D-particles.,

$$V^{1-loop}_{(qu.,Matrix} = V^{class.}_{grav.-grav.}$$



(quantum theory version of Newtonian interaction)

•At quantum level, space= fuzzy: off-diagonal matrix d.o.f. $(X_i)^{ab}$, $a \neq b$ (instead of N positions, we have N^2 d.o.f).

•Extended classical objects (Dp-branes) also from matrices.

•e.g. torus T^2: $X_1 \propto p$, $X_2 \propto q$, where p, q are large-N matrices satisfying $[p, q] = 2\pi i/N \to 0$.

•$\to$ become classical coordinates (p, q) on T^2.

2. The ABJM model

•Nonperturbative string theory should be 11d: "M theory"

•It contains M2-branes (membranes) as fundamental objects, M5-branes as solitons. Theory of M2-branes?

•Action of one membrane: known.

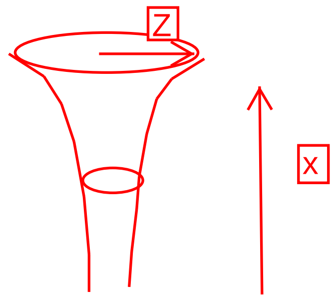
•ABJM (2008) $\rightarrow$ action on 3d worldvolume of N M2-branes, at large distances

•Like Matrix theory, contains $N \times N$ matrices. In particular, 8 transverse coordinates become $(Z^I)^{a,b}(x_1, x_2, t)$ matrices.

•However, now $(Z^I)^{a,b}$ are $U(N) \times U(\bar{N})$ bifundamental fields, and $\exists\ A_\mu^{(1)aa'}(x_1, x_2, t)$: $U(N)$ adjoint and $A_\mu^{(2)bb'}(x_1, x_2, t)$: $U(\bar{N})$ adjoint $\rightarrow$ 2 types of gauge bosons (force particles).

•Perhaps rules similar to Matrix theory?

•At least a stable solution looks like a fuzzy worldvolume, Matrix theory-style: "fuzzy funnel": sphere with radius= function of x_1.



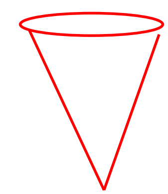
•The section $x_1 =$ constant: "fuzzy sphere" $\longrightarrow$ defined by some $SU(2)$ generator J_i:

$$[J_i, J_j] = 2i\epsilon_{ijk}J_k$$

•Extra classical coordinates emerge from membrane theory!

•Matrix theory: derived from theory of N D-particles in flat space (yet real spacetime + gravity: emergent)

•ABJM model: derived from theory of N M2-branes on C^4/Z_k cone (cone of $2\pi/k$ deficit angle)



•$U(N) \times U(\bar{N})$ matrices with integer $k = 1/\text{coupling}$.

•Caveat: perturbation theory in $N/k \Rightarrow$ need large k for most calculations $\rightarrow$ not flat space.

3. Fuzzy 2-spheres vs. fuzzy 3-spheres

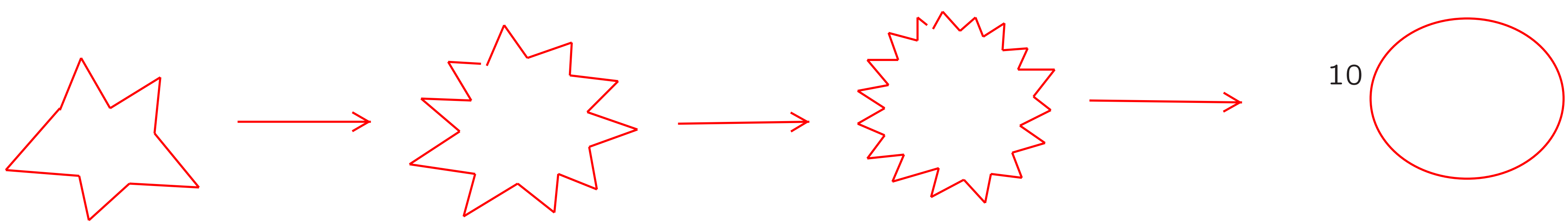
- M theory has M5-branes $\Rightarrow$ natural guess: fuzzy 3-sphere transverse to N M2's = M5 on S^3. Seems to be correct.

- We show that in fact: 2-sphere.

- Interpretation: 2-sphere $= S^3/Z_k$. Locally, $S^3 \simeq S^2 \times S^1$: $S^3/Z_k \simeq S^2 \times S^1/Z_k$.

- Effectively, reduce to 10 dimensional string theory

- But: fuzzy space= approx. to classical space as $N \to \infty$.



•Finite $k \Rightarrow$ need finite N for perturbation in N/k.

•Yet we find that even at finite N fuzzy space looks like 2-sphere.

•Hard to see how to obtain a 3-sphere.

•New formulation of fuzzy 2-sphere in terms of fuzzy Killing spinors= fermionic square root of classical coordinates on S^2

$$X_i \sim \bar{\eta}^I (\gamma_i)_{IJ} \eta^J; \quad X_i X_i = 1$$

BREAK

PART II

- 1.BLG and ABJM
- 2.Fuzzy 2-spheres and fuzzy 3-spheres
- 3.Fuzzy 2-sphere structure of ABJM
- 4.Action for fluctuations
- 5.Interpretation and fuzzy sphere equivalence
- 6.Fuzzy Killing spinors
- 7.Supersymmetry and twisting the fields

1. BLG and ABJM

•$\mathcal{N} = 8$ susy BLG theory of multiple M2's developped so that the BPS fuzzy S^3-funnel defined by ($X^m \perp M2$; $S^3 \in M5$ by $X^m X^m = 1$)

$$\frac{dX^m}{ds} = ik\epsilon^{mnpq}[X^n, X^p, X^q]$$

for $X^m = \bar{X}^m/\sqrt{s}$, for fields living in 3-algebra

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d$$

defined by $X^M = X^M_a T^a$; $M = 1, ..., 8$.

•Commuting fields X^m_a satisfy

$$\partial_s X^m_a = ik\epsilon^{mnpq} f^{bcd}{}_a X^n_b X^p_c X^q_d$$

•Only consistent solution has $SO(4)$ invariance

$$f^{abcd} = f^{abc}{}_e Tr(T^d T^e) = \tilde{f}\epsilon^{abcd}$$

•van Raamsdonk: reformulate it in terms of bifundamentals in Lie algebra of $SO(4) = SU(2) \times SU(2)$.

$$X^M = \frac{1}{2} X_a^M \tau^a = \frac{1}{2}(X_4^M \mathbb{1} + i X_i^M \sigma^i)$$

•ABJM generalizes $SU(2) \times SU(2)$ to $SU(N) \times SU(N)$ theory and $\mathcal{N} = 8$ susy $\rightarrow \mathcal{N} = 6$ susy.

•$A_\mu^{(1)}, A_\mu^{(2)}$ for $U(N) \times U(\bar{N})$; bifundamentals $C^I = (Z^\alpha, W^{\alpha\dagger})$.

•1/2 susy BPS fuzzy funnel equation

$$\partial_s Z^\alpha = -4a(Z^\beta Z_\beta^\dagger Z^\alpha - Z^\alpha Z_\beta^\dagger Z^\beta)$$
$$Z^\alpha = \frac{1}{\sqrt{8as}} G^\alpha$$

•G^α gives fuzzy sphere equation

$$G^\alpha = G^\beta G^\dagger_\beta G^\alpha - G^\alpha G^\dagger_\beta G^\beta$$

•Massive deformation (GRVV): preserves maximal $\mathcal{N} = 6$ susy ($SU(4)$-invariant susy rules), but R symmetry of $\mathcal{L}$ is broken to $SU(2) \times SU(2)$.

•In split $C^I = (R^\alpha, Q^\alpha)$, fully susy vacuum is

$$Q^\alpha = 0; \quad R^\alpha = fG^\alpha; \quad f^2 = \frac{\mu k}{2\pi}$$

2. Fuzzy 2-spheres and fuzzy 3-spheres

•Fuzzy 2-sphere: defined by $SU(2)$ algebra:

$$[X^i, X^j] = 2i\epsilon^{ijk} X^k$$

with the sphere condition $X_i X_i = R^2$ satisfied by any $N = 2j+1$ dimensional representation of $SU(2)$

$$X^i X^i = N^2 - 1 = 4j(j+1)$$

•Is explicitly $SU(2) = SO(3)$ invariant. All $N \times N$ matrices can be decomposed in spherical harmonics made up of X^i's,

$$Y_{lm}(X^i) = a_{lm}^{(i_1 \ldots i_l)} (X^{(i_1} \ldots X^{i_l)} - \text{traces})$$

•Fluctuations of $N \times N$ theories around this solution can be expanded in $Y_{lm}(X^i)$.

•Tends to field theory on classical S^2 as $N \to \infty$.

•**Fuzzy 3-sphere**: based on $SO(4)$-invariant Guralnik-Ramgoolam formulation in terms of $Sym(\Gamma_m)^{\otimes r}$ (simplified by Ramgoolam, 2002 and HN, 2004), formulate it for comparison to BLG/ABJM:

$$\epsilon^{mnpq} X_n^+ X_p^- X_q^+ \ = \ 2\Big(\frac{(r+1)(r+3)+1}{r+2}\Big) X_m^+$$
$$\epsilon^{mnpq} X_n^- X_p^+ X_q^- \ = \ 2\Big(\frac{(r+1)(r+3)+1}{r+2}\Big) X_m^-$$

with the constraints and sphere condition

$$X_m^+ X_n^+ = X_m^- X_n^- = 0; \quad X_m^+ X_m^- + X_m^- X_m^+ = N$$

•BLG fuzzy funnel $X_m \propto \bar{X}_m/\sqrt{s} \Rightarrow$

$$R^2 \bar{X}^m = ik\epsilon^{mnpq}[\bar{X}^n, \bar{X}^p, \bar{X}^q]$$

•For the commuting fields X_m^a ($\bar{X}_m = X_m^a T_a$)

$$R^2 X_a^m = ik\epsilon^{mnpq} f^{bcd}{}_a X_b^n X_c^p X_d^q$$

•Only for the $SO(4)$ case, $f^{abcd} = \tilde{f}\epsilon^{abcd}$, similar to fuzzy $S^3 \rightarrow$ a index is algebra one: analog of m index for S^3.

•More precisely, van Raamsdonk formulation:

$$R^2 X^m = -ik\epsilon^{mnpq} X^n X^{\dagger p} X^q$$

•So: $X_m^+ \to X_m$, $X_m^- \to X_m^\dagger$.

•ABJM (fuzzy funnel, or massive: fuzzy sphere)

$$\text{Re}\{G^\alpha = G^\beta G_\beta^\dagger G^\alpha - G^\alpha G_\beta^\dagger G^\beta\}$$

•For $r = 1$, i.e. when $X_m = \Gamma_m$, it matches $\epsilon^{mnpq} X_n^+ X_p^- X_q^+ =$ [...]X_m^+, for $r > 1$ ($\sim (\Gamma_m)^{\otimes r}$), it doesn't.

•So, for $r > 1$ ($SU(N) \times SU(N)$ with $N > 2$), not a (usual?) fuzzy S^3!

3. Fuzzy 2-sphere structure of ABJM

•Prove that it is a fuzzy S^2 (except for $r = 1 \leftrightarrow N = 2 \leftrightarrow \mathcal{A}_4$ BLG), which can **also** be interpreted as (**most** fuzzy) S^3.

•Massive ABJM: $Q^\alpha = 0, R^\alpha = f(\mu)G^\alpha$; ABJM funnel:$Z^\alpha = f(s)G^\alpha$.

•Explicit matrices

$$(G^1)_{m,n} = \sqrt{m-1}\ \delta_{m,n}$$
$$(G^2)_{m,n} = \sqrt{(N-m)}\ \delta_{m+1,n}$$
$$(G_1^\dagger)_{m,n} = \sqrt{m-1}\ \delta_{m,n}$$
$$(G_2^\dagger)_{m,n} = \sqrt{(N-n)}\ \delta_{n+1,m}$$

•Satisfy $\sum_{p=1}^{p} X_p X^p = G^\alpha G_\alpha^\dagger = N-1 \rightarrow$ suggests fuzzy S^3?

•BUT (first hint) $G^1 = G_1^\dagger \rightarrow X_2 = 0$?

•Still, G^α could be transformed to different basis.

•We have two $U(2)$ algebras defined by

$$J^{\alpha}{}_{\beta} = G^{\alpha} G^{\dagger}_{\beta}; \quad \bar{J}_{\alpha}{}^{\beta} = G^{\dagger}_{\alpha} G^{\beta}$$

splitting into two $SU(2)$ algebras

$$J_i = (\sigma_i^T)^{\alpha}{}_{\beta} J^{\beta}{}_{\alpha}; \quad [J_i, J_j] = 2i\epsilon_{ijk} J_k$$
$$\bar{J}_i = (\sigma_i^T)^{\alpha}{}_{\beta} \bar{J}_{\alpha}{}^{\beta} \Rightarrow [\bar{J}_i, \bar{J}_j] 2i\epsilon_{ijk} \bar{J}_k$$

and two $U(1)$ generators proportional to the identity in a subspace

$$J \equiv J^{\alpha}{}_{\alpha} = (N-1) 1\!\!1_{N\times N}$$
$$(\bar{J})_{mn} = (\bar{J}_{\alpha}{}^{\alpha})_{mn} = N\delta_{mn} - N\delta_{m1}\delta_{n1} = N(1\!\!1_{(N-1)\times(N-1)})_{mn}$$

•It would seem that we have $SO(4) = SU(2) \times SU(2)$ invariance, as for fuzzy S^3, but **they are not independent**!

•Physical fluctuations: bifundamentals of $U(N) \times U(\bar{N})$: $G^\alpha; G^\dagger_\alpha$, transform as

$$\begin{aligned} J_i G^\alpha - G^\alpha \bar{J}_i &= (\tilde{\sigma}_i)^\alpha_\beta G^\beta \\ G^\dagger_\alpha J_i - \bar{J}_i G^\dagger_\alpha &= G^\dagger_\beta (\tilde{\sigma}_i)^\beta_\alpha \end{aligned}$$

.

$\rightarrow$ only a combination of $U(N)$ and $U(\bar{N})$ $\rightarrow$ fields on a **single** S^2.

•Example of fuzzy S^3: Ishii, Ishiki, Shimasaki, Tsuchiya, 2008 $\rightarrow$ BMN plane wave matrix model.

$X_i \rightarrow N_s \times N_t$ subsets $X_i^{(s,t)}$. Then $SU(2)$ in j_s representations $L_i^{(j_s)}$ act separately on two sides, by

$$L_i \circ X_i^{(s,t)} = L_i^{(j_s)} X_i^{(s,t)} - X_i^{(s,t)} L_i^{(j_t)}$$

•Fields of fuzzy S^2 expanded in fuzzy spherical harmonics:

$$Y_{lm}(J_i) = \sum_i f_{lm}^{(i_1 \dots i_l)} J_{i_1} \dots J_{i_l}$$

•Matrices (fluctuations) in adjoint of $U(N)$ and in adjoint of $U(\bar{N})$ expand as:

$$A = \sum_{l=0}^{N-1} a^{lm} Y_{lm}(J_i)$$

.

$$\bar{A} = \bar{a}_0 \bar{E}_{11} + \sum_{l=0}^{N-2} \bar{a}_{lm} Y_{lm}(\bar{J}_i) + \sum_{k=2}^{N} b_k g_{\bar{1}\bar{k}} + \sum_{k=2}^{N} \bar{b}_k g_{\bar{k}\bar{1}}$$

where

$$\begin{aligned} \bar{E}_{11} &= |e_{\bar{1}} >< e_{\bar{1}}| \\ \bar{E}_{1k} &= |e_{\bar{1}}\rangle\langle e_{\bar{k}}| \equiv g_{\bar{1}\bar{k}} \\ \bar{E}_{k1} &= |e_{\bar{k}}\rangle\langle e_{\bar{1}}| \equiv g_{\bar{k}\bar{1}} \\ Y_{lm}(\bar{J}_i) &= \sum_i f_{lm}^{(i_1 \ldots i_l)} \bar{J}_{i_1} \ldots \bar{J}_{i_l} \end{aligned}$$

•Fluctuating bifundamental matrices (e.g. δR^α) expand as

$$r^\alpha = r^\alpha_\beta G^\beta + \sum_{k=1}^{N} t^\alpha_k \hat{E}_{k1} = r G^\alpha + s^\alpha_\beta G^\beta + T^\alpha$$

and $r, {s^\alpha}_\beta, t^\alpha_k$ are expanded in $Y_{lm}(J_i)$

•**Note that we could have expanded in $Y_{lm}(\bar{J}_i)$!**

5. Action for fluctuations

.

•ABJM → massive μ deformation changes the potential in $C^I = (R^\alpha, Q^\alpha)$ and gives a mass μ to the fermions.

•Fluctuations

$$
\begin{array}{ll}
R^\alpha = fG^\alpha + r^\alpha \,, & R^\dagger_\alpha = fG^\dagger_\alpha + r^\dagger_\alpha \\
Q^{\dot\alpha} = q^{\dot\alpha} \,, & Q^\dagger_{\dot\alpha} = q^\dagger_{\dot\alpha} \\
A_\mu = A_\mu \,, & \psi^{\dagger I} = \psi^{\dagger I}
\end{array}
$$

•At finite N, $[J_i, .]$ acts as a "fuzzy derivative" operator ($\sim \partial_i$) acting on $Y_{lm}(J_i)$ and the "fuzzy Laplacean" is $[J_i,[J_i,.]]$.
•Fermions: subtle. We could decompose

$$\psi^\alpha = G^\alpha\psi + G^\beta\tilde{U}^\alpha_\beta = \tilde{\psi}G^\alpha + U_\alpha{}^\beta G^\beta$$

$$\chi_{\dot\alpha} = \chi_{\dot\alpha\beta}G^\beta; \quad (\chi_{\dot\alpha\alpha})^\dagger = \chi^{\dot\alpha\alpha}$$

where $\psi, U_\alpha{}^\beta, \chi_{\alpha\dot\alpha}$ are expanded in $Y_{lm}(J_i)$, or keep $\psi^\alpha, \chi_{\dot\alpha}$ as they are.

• The bosonic potential gives at finite N

$$
\begin{aligned}
V &= V_6^{r^2} + V_6^{s^2} + V_6^{\perp} + V_6^{\text{r-s}} + V_2^{r\text{-}s\text{-}q} + V_4^{r\text{-}s} \\
&= -\frac{4\pi^2}{3}\frac{f^4}{k^2}\left[3(N-4)\mathrm{Tr}\,([J_i,r][J_i,r]) - 45(N-1)\mathrm{Tr}\,(r^2)\right] \\
&\quad -\frac{4\pi^2}{3}\frac{f^4}{k^2}\Big[\frac{3}{4}(N-1)\mathrm{Tr}\,(s_is_i) - \frac{3}{4}(N+1)\mathrm{Tr}\,([J_i,s_i]^2) + \frac{3}{4}(4N-1)i\epsilon_{ijk}\mathrm{Tr}\,(s_iJ_ks_k) \\
&\quad -\frac{3}{4}N\mathrm{Tr}\,(s_i\Box s_i) + \frac{3}{8}i\epsilon_{ijk}\mathrm{Tr}\,(J_is_j\Box s_k + s_iJ_j\Box s_k)\Big] + \frac{4\pi^2 f^4}{3k^2}6\mathrm{Tr}\,\left(\mathcal{L}_\alpha^\gamma(q^{\dot\beta})^\dagger\mathcal{L}_\gamma^\alpha(q^{\dot\beta})\right) \\
&\quad -\frac{4\pi^2}{3}\frac{f^4}{k^2}\Big[-\frac{9}{4}\mathrm{Tr}\,(r(\Box s_iJ_i + J_i\Box s_i)) + \frac{3}{4}\mathrm{Tr}\,(r\Box(J_is_i + s_iJ_i)) - \frac{9}{2}\mathrm{Tr}\,(r(s_iJ_i + J_is_i))\Big] \\
&\quad -\mu^2\Big[\frac{1}{4}(N-1)\mathrm{Tr}\,(s_is_i) + (N-1)\mathrm{Tr}\,(r^2) + \frac{1}{2}\mathrm{Tr}\,(r(s_iJ_i + J_is_i)) - \frac{1}{4}i\epsilon_{ijk}\mathrm{Tr}\,(s_iJ_js_k) \\
&\quad +q^{\dot\alpha}q^\dagger_{\dot\alpha}\Big] + \frac{8\pi\mu f^2}{k}\Big[-i\epsilon_{ijk}\mathrm{Tr}\,(rJ_is_jJ_k) + \frac{1}{2}\mathrm{Tr}\,(r(s_iJ_i + J_is_i))\Big]
\end{aligned}
$$

• The fermions give at finite N

$$
\begin{aligned}
&i\mu\int \mathrm{Tr}\,(\tilde\sigma_i)^\alpha{}_\beta[\psi^{\dagger\beta}J_i - \bar J_i\psi^{\dagger\beta}]\psi_\alpha \; - \int \mathrm{Tr}\,[\psi^{\dagger\alpha}\gamma^\mu\partial_\mu\psi_\alpha + i\mu\psi^{\dagger\alpha}\psi_\alpha] \\
&+\frac{\mu}{2}\epsilon_{jik}\mathrm{Tr}\,\left[(\tilde\sigma_k)^\alpha{}_\beta J_j\chi^{\dot\delta\beta}J_i\chi_{\dot\delta\alpha}\right] + \int \mathrm{Tr}\,[-\chi^{\dagger\dot\alpha}\gamma^\mu D_\mu\chi_{\dot\alpha} + i\mu\chi^{\dagger\dot\alpha}\chi_{\dot\alpha}]
\end{aligned}
$$

Classical S^2 in large N limit

- We describe it in terms of J_i, but we could use $\bar{J}_i$ as well.

$$(J_i)^2 = N^2 - 1; \quad (\bar{J}_i)^2 = (N-1)^2 - 1$$

- The classical coordinates differ at subleading order

$$x_i = \frac{J_i}{\sqrt{N^2-1}}; \quad \bar{x}_i = \frac{\bar{J}_i}{\sqrt{(N-1)^2-1}}$$

- Then

$$x_i^2 = 1; \quad [x_i, x_j] = \frac{2i}{\sqrt{N^2-1}} \epsilon_{ijk} x_k \to 0$$

$$\frac{1}{N}\mathrm{Tr} \to \int d^2\sigma \sqrt{\hat{h}}$$

•Selected bosonic objects in the Lagrangean

.

$$[J_i, \cdot\,] = -2i\epsilon_{ijk}x_j\partial_k = -2iK_i = -2iK_i^a\partial_a$$

$$\Box = [J_i,[J_i,\cdot]] = -4\frac{1}{\sqrt{\hat{h}}}\partial_a(\sqrt{\hat{h}}\,\partial^a) \equiv -4\hat{\Box}$$

$$K_i^aK_i^b = h^{ab}$$

$$\epsilon_{ijk}x_iK_j^aK_k^b = \omega^{ab} = \frac{\epsilon^{ab}}{\sin\theta}$$

•We can also define the adjoint action of K_i on G^α by

$$-2iK_i^a\partial_a(G^\alpha) = -2iK_i(G^\alpha) \equiv N(x_iG^\alpha - G^\alpha\bar{x}_i) = [(\tilde{\sigma}_i)^\alpha_\beta - x_i\delta^\alpha_\beta]G^\beta$$

such that $K_i(x_j) = \epsilon_{ijk}x_k$ as we need, and obtain

$$\partial_a(G^\alpha) = \frac{1}{-2i}h_{ab}K_i^b(\tilde{\sigma}_i)^\alpha_\beta G^\beta$$

which is a fermionic object, to be defined later.

•In the parallel scalar fluctuation $r^\alpha = rG^\alpha + s^\alpha_\beta G^\beta, s_i = (\sigma_i)^\alpha{}_\beta s^\beta{}_\alpha$ is decomposed in the classical limit as

$$s_i = K_i^a A_a + x_i \phi$$

•But then **at large N (!)**, we have

$$r^\alpha = rG^\alpha + s^\alpha_\beta G^\beta \Rightarrow ... \to K_i^a A_a \frac{(\tilde{\sigma}_i)^\alpha_\beta}{2} G^\beta + \frac{2r + \phi}{2} G^\alpha$$

•At finite N, the expansion of s_i (thus of r^α) is not clear: $\exists$ inconsistencies, though they could be $\sim$ gauge transformations.
•The transverse scalars are decomposed as

$$q^{\dot{\alpha}} = Q^{\dot{\alpha}}_\alpha G^\alpha = \sum_{l=0}^{N-1} (Q^{\dot{\alpha}}_\alpha)_{lm} Y_{lm}(J_i) G^\alpha$$

•Higgs mechanism for CS-Higgs system

$$A_\mu^{(1,2)\,(ij)}(x) \to \mathbf{A}^{(1,2)}(x;\sigma)$$

•Define linear combinations of the $U(1)$ gauge fields on the S^2:

$$\begin{aligned} A_\mu &= \frac{1}{2}(\mathbf{A}_\mu^{(1)} + \mathbf{A}_\mu^{(2)}) \\ B_\mu &= \frac{1}{2}(\mathbf{A}_\mu^{(1)} - \mathbf{A}_\mu^{(2)}) \end{aligned}$$

•Then

$$S_{\text{CS}} \to N\frac{k}{2\pi}\int d^3x\, d^2\sigma\sqrt{\hat{h}}\Big(\epsilon^{\mu\nu\rho}B_\mu F_{\nu\rho}\Big)$$

$$-\int d^3x\, \text{Tr}\left(D_\mu C_I^\dagger D^\mu C^I\right) \to f^2N^2\int d^3x\, d^2\sigma\sqrt{\hat{h}}\Big(\frac{1}{N^2}\mathbf{A}_\mu^{(1)}\hat{\Box}\mathbf{A}^{(2)\mu} - (\mathbf{A}_\mu^{(1)} - \mathbf{A}_\mu^{(2)})^2\Big)$$

•Subtlety: $1/N^2$ correction kept

$$G^\alpha Y_{lm}(\bar{x}_i)G_\alpha^\dagger = (N-1)Y_{lm}(x_i) - \frac{l(l+1)}{2N}Y_{lm}(x_i) + \mathcal{O}\Big(\frac{1}{N^2}\Big)$$

•Although a priori different whether we use x_i or $\bar{x}_i$ $\to$ same result for $1/N^2$ corrections to action.

•Then B_μ is auxiliary and **subleading**

$$B^\mu = \frac{1}{8f^2N}\frac{k}{2\pi}\epsilon^{\mu\nu\lambda}F_{\nu\lambda}$$

$\rightarrow$ reason why we needed to keep $1/N^2$ terms.

$$\int d^3x\, d^2\sigma\sqrt{\hat{h}}\left(N\frac{k}{2\pi}\epsilon^{\mu\nu\rho}B_\mu F_{\nu\rho} - 4f^2N^2B_\mu B^\mu + f^2A_\mu\hat{\Box}A^\mu - f^2B_\mu\hat{\Box}B^\mu\right)$$
$$= \int d^3xd^2\sigma\, \sqrt{\hat{h}}\left(-f^2\partial^a A_\mu\partial_a A^\mu - \left(\frac{k}{2\pi}\right)^2\frac{1}{8f^2}F^{\mu\nu}F_{\mu\nu}\right)$$

•Final action

$$S_{phys} = \frac{1}{g_{YM}^2}\int d^3xd^2\sigma\sqrt{h}\Big[-\frac{1}{4}F_{AB}F^{AB} - \frac{1}{2}\partial_A\Phi\partial^A\Phi - \frac{\mu^2}{2}\Phi^2 - \partial^M q^\dagger_{\dot\alpha}\partial_M q^{\dot\alpha} + \frac{\mu}{2}\,\omega^{ab}F_{ab}\Phi$$
$$+\left(\frac{1}{2}\bar\Upsilon^{\dot\alpha}\tilde{D}_5\Upsilon_{\dot\alpha} + \frac{i}{2}\mu\bar\Upsilon^{\dot\alpha}\Upsilon_{\dot\alpha} + h.c.\right) - (\psi S)\tilde{D}_5(S^{-1}\psi^\dagger) + \frac{i}{2}\mu(\psi S)(S^{-1}\psi^\dagger)\Big]$$

where S is a fixed rotation matrix for spinors and $\Upsilon^\alpha_{\dot\alpha} = (P_- S^{-1}\chi_{\dot\alpha})^\alpha$.

5. Interpretation and fuzzy sphere equivalence

•Fuzzy BPS funnel of pure ABJM $\rightarrow$ same unrescaled bosonic action, with

.

$$R^\alpha = f(s)G^\alpha; \quad f(s) = \sqrt{\frac{k}{4\pi s}}$$

•Thus, only replace $\mu \leftrightarrow 1/(2s)$.

•M5 compactified to D4 due to large k (large N: classical, but N/k fixed)

•Radius of sphere is

$$R_{\mathrm{ph}}^2 = \frac{2}{N}\mathrm{Tr}\,(X^I X_I^\dagger) = 8\pi^2 f^2 N l_p^3$$

•Energy in M5/D4 picture: $S^3/\mathbb{Z}_k$ vs. S^2:

$$E = \frac{T_2^2}{2\pi}\int \frac{2\pi^2}{k} R_{\mathrm{ph}}^3 dR_{\mathrm{ph}} dx_1 = T_4 \int 4\pi\left(\frac{R_{\mathrm{ph}}}{2}\right)^2 dR_{\mathrm{ph}} dx_1$$

•Indeed,

$$S^3/\mathbb{Z}_k \stackrel{k\to\infty}{\to} S^2_{R/2} \times S^1_{R/k}; Z^i \to Z^i e^{2\pi i/k} \simeq Z^i + 2\pi i Z^i/k$$

•If $Z^1 = v + i0$; $Z^{2,3,4} = 0$, then X^2 has radius v/k.

•$\mathbb{Z}_k$ reduces k times the S^1 in the $S^3 : S^2$ over S^1 Hopf fibration.

•Classical Hopf fibration:

$$x_i = (\sigma_i^T)^\alpha{}_\beta Z^\beta Z^*_\alpha \Rightarrow x_i x_i = 1$$

•Note that the S^1 is: $Z^\alpha \to e^{i\theta} Z^\alpha \Rightarrow$ this is in fact a map between S^2 and CP^1.

$$CP^1 : \{Z^\alpha \leftrightarrow \lambda Z^\alpha\} \Leftrightarrow \{Z^\alpha \leftrightarrow e^{i\theta} Z^\alpha, \sum_\alpha |Z^\alpha|^2 = 1\}$$

Fuzzy sphere equivalence

•Classically, we saw that the Hopf fibration

$$x_i = (\tilde{\sigma}_i)^\alpha{}_\beta g^\beta g^\dagger_\alpha; \quad \bar{x}_i = (\tilde{\sigma}_i)^\alpha{}_\beta g^\dagger_\alpha g^\beta$$

means that if we define the S^3 coordinates g^α modulo a $U(1)$ phase (the Hopf fiber), the resulting $\tilde{g}^\alpha$ are some coordinates on S^2

•Fuzzy version: $G^\alpha = U\tilde{G}^\alpha$ or $G^\alpha = \hat{\tilde{G}}^\alpha \hat{U}$, with $(U, \hat{U})$ unitary $\rightarrow e^{i\alpha(x)}$. Then $\tilde{G}^\alpha$ satisfy the same GRVV algebra

$$-\tilde{G}^\alpha = \tilde{G}^\beta \tilde{G}^\dagger_\beta \tilde{G}^\alpha - \tilde{G}^\alpha \tilde{G}^\dagger_\beta \tilde{G}^\beta$$

$\rightarrow$ its representations should be $\Leftrightarrow SU(2)$.

•Irreps: Indeed, the $J_i \rightarrow$ matrices of the general $|jm>$ rep. of $SU(2)$ and $\bar{J}_i \rightarrow |j-1/2, m>$ rep. of $SU(2)$, and

$$J = (N-1) 1\!\!1_{N\times N}; \quad \bar{J} = N 1\!\!1_{(N-1)\times(N-1)}$$

•Reducible: Casimir $J_i J_i$: is diagonal and

$$J = diag(J_{N_1}, J_{N_2}, ...); \quad \bar{J} = diag(\bar{J}_{N_1}, \bar{J}_{N_2}, ...)$$

•GRVV algebra $\Rightarrow SU(2)$ algebras in general.

•Reversely, in the classical limit

$$g^\alpha = \begin{pmatrix} g^1 \\ g^2 \end{pmatrix} = \frac{e^{i\phi}}{\sqrt{2(1+x_3)}} \begin{pmatrix} 1 + x_3 \\ x_1 - i x_2 \end{pmatrix} = e^{i\phi} \tilde{g}^\alpha$$

•On the fuzzy sphere

.
$$G^\alpha = \begin{pmatrix} G^1 \\ G^2 \end{pmatrix} = \begin{pmatrix} J + J_3 \\ J_1 - iJ_2 \end{pmatrix} \frac{T^{-1}}{2} U_{N\times\bar{N}} = \tilde{G}^\alpha U_{N\times N}$$
$$G^\alpha = \begin{pmatrix} G^1 \\ G^2 \end{pmatrix} = \hat{U}_{N\times\bar{N}} \frac{\tilde{T}^{-1}}{2} \begin{pmatrix} \bar{J} + \bar{J}_3 \\ \bar{J}_1 - i\bar{J}_2 \end{pmatrix} = \hat{U}\tilde{\bar{G}}^\alpha$$

and we define $V_N^+ \to V_{N-1}^- \oplus V_1^-$ and $J = (N-1)\,1\!\mathrm{l}_{N\times N}$, $\bar{J} = N(1 - E_{11})\,1\!\mathrm{l}_{N\times N}$ for irreps and similar for reducible reps.

•Fuzzy superalgebra

$$\mathbf{J}_i = \begin{pmatrix} J_i & 0 \\ 0 & \bar{J}_i \end{pmatrix} \quad \text{and} \quad \mathbf{J}_\alpha = \begin{pmatrix} 0 & \sqrt{N}\tilde{G}_\alpha \\ -\sqrt{N}\tilde{G}_\alpha^\dagger & 0 \end{pmatrix}$$
$$\begin{aligned} [\mathbf{J}_i, \mathbf{J}_j] &= 2i\epsilon_{ijk}\mathbf{J}_k \\ [\mathbf{J}_i, \mathbf{J}_\alpha] &= (\tilde{\sigma}_i)_{\alpha\beta}\mathbf{J}^\beta \\ \{\mathbf{J}_\alpha, \mathbf{J}_\beta\} &= -(\tilde{\sigma}_i)_{\alpha\beta}\mathbf{J}_i = -(i\tilde{\sigma}_2\tilde{\sigma}_i)_{\alpha\beta}\mathbf{J}_i \end{aligned}$$

is $OSp(1|2)$ for the fuzzy supersphere: trivial, i.e. $\Leftrightarrow$ bosonic supersphere.

6. Fuzzy Killing spinors

- Classical objects on S^3 modulo a phase

.
$$\tilde{g}^\alpha = \frac{1}{\sqrt{2(1+x_3)}} \begin{pmatrix} 1+x_3 \\ x_1 - ix_2 \end{pmatrix}$$

$\rightarrow$ (Majorana) spinors of the $SO(2)_{l.L.}$ $\rightarrow$ related to Killing spinor

- Killing spinor on S^n satisfy

$$D_\mu \eta(x) = \pm \frac{i}{2} m \gamma_\mu \eta(x)$$

satisfy orthonormality, completeness and modified Majorana spinor conditions

- In terms of Killing spinors

$$x_i = (\sigma_i)^I{}_J (\eta^I)^\dagger \gamma_3 \eta^J = (\tilde{\sigma}_i)^I{}_J \left(\sqrt{2} P_+ \eta^I\right)^\dagger \left(\sqrt{2} P_+ \eta^J\right)$$

suggesting the identification

$$\frac{\tilde{G}^\alpha}{\sqrt{N}} \equiv \tilde{g}^\alpha \leftrightarrow \tilde{g}^I \equiv \sqrt{2} P_+ \eta^I$$

•On S^2, $\exists$ matrices ${S^\alpha}_\beta$ that takes us between spherical and Euclidean spinors on S^2, giving in particular

$$(S\Gamma_3 S^{-1})^\alpha{}_\beta = -x_i(\tilde{\sigma}_i)^\alpha{}_\beta; \quad (S\Gamma_a S^{-1})^\alpha{}_\beta = -h_{ab}K_i^b(\tilde{\sigma}_i)^\alpha{}_\beta$$

•One gets the Killing spinor

$$\frac{1}{\sqrt{2}}(S^{-1})^\alpha{}_\beta \epsilon^{\beta I} = \eta_+^{\alpha I}$$

•Comparing $\partial_a \tilde{g}^\alpha$ and $\partial_a(\sqrt{2}P_+\eta^I)$, we see an extra term

$$\partial_a(\sqrt{2}P_+\eta^I) = -\frac{i}{2}(S\Gamma_a S^{-1})^I{}_J(\sqrt{2}P_+\eta^J) + \tilde{T}_a(\sqrt{2}P_+\eta^I)$$

since $\tilde{g}^\alpha$ and η^I are only identified up to a phase.

•Other Hopf fibrations suggests generalizations to $S^7 \to S^4$, $S^{15} \to S^8$, $S^7 \to CP^3$, but details of fuzzy constructions need to be worked out.

7. Supersymmetry and twisting the fields

.

•$\tilde{G}^{\alpha} \longrightarrow$ fuzzy Killing spinors (fermionic) $\longrightarrow$ redefining the fields by $\tilde{G}^{\alpha}$ amounts to twisting the fields.

•Fuzzy spinorial spherical harmonics can be obtained from $\tilde{G}^{\alpha}$
OR
•(usual): spinorial spherical harmonics $\Xi^{\pm\alpha}_{lm}$ arise from coefficients of exp. in usual spherical harmonics.

•Now, $q^{\dot{\alpha}}$ and ψ_{α} expansion still contains $\tilde{G}^{\alpha}$.

•Large N limit of susy rules is subtle $\longrightarrow$ simpler to twist:

$$q^{\dot{\alpha}} = Q^{\dot{\alpha}}_{\alpha}\tilde{G}^{\alpha}$$

$$\psi_{\alpha} = \tilde{\psi}\tilde{G}_{\alpha} + U_{\alpha}{}^{\beta}\tilde{G}_{\beta} \quad U_{\alpha}{}^{\beta} = \frac{1}{2}U_i(\tilde{\sigma}_i)_{\alpha}{}^{\beta}$$

$$U_i = K^a_i g_a + \widehat{\psi} x_i$$

•To preserve susy on D-branes with curved worldvolume $\Rightarrow$ need to twist D-brane fields: On S^2, embed the S^2 spin connection ($SO(2) = U(1)$ valued) into R-symmetry $\Rightarrow$ max $\mathcal{N} = 1$ after dimensional reduction.

•3d $\mathcal{N} = 6$ theory at large $N \Rightarrow$ 5d $U(1)$ theory on $\simeq$ classical S^2. After dim. red. back to 3d, at most $\mathcal{N} = 1$. Actually, $\mathcal{N} = 0$.

•Action for $q^{\dot\alpha}$ and ψ_α becomes after twisting

$$N^2 \int d^3x d^2\sigma \sqrt{\hat h}\Big[\frac{1}{2}\bar\Xi^{\dot\alpha}(-i2\mu\hat\nabla_{S^2})^2\Xi_{\dot\alpha} - \frac{1}{2}\partial_\mu\bar\Xi^{\dot\alpha}\partial^\mu\Xi_{\dot\alpha} - 3\mu^2\bar\Xi^{\dot\alpha}\Xi_{\dot\alpha}\Big]$$
$$+N^2 \int d^2\sigma\sqrt{\hat h}[\frac{1}{4}\bar\Lambda\partial\!\!\!/\Lambda + \frac{1}{4}\bar g_a \partial\!\!\!/ g^a + \frac{i\mu}{2}\hat\omega^{ab}\bar G_{ab}\Lambda + i\mu\bar\Lambda\Lambda]$$

where $\Lambda \equiv 2(\tilde\psi - \frac{1}{2}\hat\psi)$, and Ξ is the modified Majorana spinor

$$\Xi^{\dot\alpha}_\alpha \equiv (Q^{\dot\alpha}SP_+)_\alpha + i(Q^{\dot\alpha}SP_-) = (C^{\dot\alpha}, iD^{\dot\alpha})$$

•"Fermonic Higgs mechanism": $\tilde\psi + \hat\psi/2$ disappears.

Conclusions

•Although the hope of BLG/ABJM was to see M-theory, and obtain M2-M5 systems on a fuzzy S^3, the solution is a fuzzy S^2

•Large N classical limit and M5 theory perturbative treatment in N/k effectively force large k, reducing $S^3/Z_k \to S^2$.

•Two potential $SU(2)$'s act in tandem on ABJM fields, giving an $SO(3) = SU(2)$ invariance at finite $N \Rightarrow$ fuzzy S^2,

•Fuzzy S^2 construction in terms of G^α is $\Leftrightarrow SU(2)$ construction, and G^α is a fuzzy Killing spinor.

•The susy action of D4 on S^2 is obtained.

•Twisting the fields simplifies the analysis. G^α relates twisted to untwisted fields.