Chiral Algebras of (0, 2) Models

Beyond Perturbation Theory

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Based On

- Tan and Yagi, arXiv:0801.4782 [hep-th]
- ——, arXiv:0805.1410 [hep-th]
- ——, *Lett. Math. Phys.* **84** (2008) 257
- Yagi, Ph.D. thesis

Why $(0,2)$?

- Chiral rings of $(2,2)$ models are nice
- related to GW invariants, mirror symmetry, etc.
- quantum deformations of *finite*-dimensional cohomology
- But there are also *infinite*-dimensional structures
- Elliptic genera uses the *right-moving supersymmetry only*.
- Let's consider theories with $(0,2)$ supersymmetry.

Chiral Algebras of $(0, 2)$ Models

- are infinite-dimensional generalizations of the chiral rings
- resemble chiral algebras of CFTs (OPE algebras of holomorphic fields)
- related to geometric Langlands program, etc.
- perturbatively described by chiral differential operators.
  [Witten, *Adv. Theor. Math. Phys.* **11** (2007) 1 [hep-th/0504078]]
- Nonperturbatively, not much is understood.

$\mathcal{N} = (0, 2)$ SUSY QFT

- 2d QFT with the following symmetries:
- spacetime translations: H, P
- spacetime rotations: M
- right-moving supercharges: Q_+, $\overline{Q}_+ = Q_+^\dagger$
- right-moving R-symmetry: $Q_+ \to e^{-i\alpha} Q_+$, $\overline{Q}_+ \to e^{+i\alpha} \overline{Q}_+$
- $(0, 2)$ supersymmetry algebra:

$$Q_+^2 = \overline{Q}_+^2 = 0,$$
$$\{Q_+, \overline{Q}_+\} = H - P$$

Q-cohomology of Operators

- $Q = \overline{Q}_+$ obey $Q^2 = 0$
- Q has R-charge 1
- Q acts on operators by $[Q, \mathcal{O}]$ (bosonic) or $\{Q, \mathcal{O}\}$ (fermionic)
- The Q-action squares to zero: $\{Q, \{Q, \mathcal{O}\}\} = [Q^2, \mathcal{O}] = 0$
- and increases the R-charge by 1
- Q-cohomology of operators graded by R-charge

Chiral Algebra

- $\{Q, Q^\dagger\} = H - P \propto \partial_{\bar{z}}$
- If $\{Q, \mathcal{O}\} = 0$, then $\partial_{\bar{z}}\mathcal{O} \propto [\{Q, Q^\dagger\}, \mathcal{O}] = \{Q, [Q^\dagger, \mathcal{O}]\}$
- Q-cohomology classes vary holomorphically on Σ
- OPE: define $[\mathcal{O}_1] \cdot [\mathcal{O}_2] = [\mathcal{O}_1\mathcal{O}_2]$; then

$$[\mathcal{O}_1(z)] \cdot [\mathcal{O}_2(z')] \sim \sum_k {c_{ij}}^k(z - z')[\mathcal{O}_k(z')].$$

  ${c_{ij}}^k(z - z')$ have poles.
- Holomorphic Q-cohomology + OPE = chiral algebra

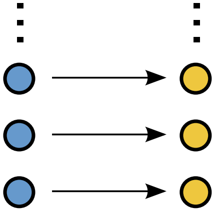
$$\mathcal{A} = \bigoplus_q \mathcal{A}^q; \quad q\text{: R-charge}$$

Vanishing "Theorem"

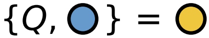
- *Nonperturbatively $\mathcal{A} = 0$ if the target space is a compact Kähler manifold with positive Ricci curvature and contains a rational curve with trivial normal bundle.* [JY, Ph.D. thesis]

  (cf. Frenkel–Losev–Nekrasov and Malikov–Arakawa [`0911.0922 [math.AG]`])

- Perturbatively, $\mathcal{A}$ has infinitely many Q-cohomology classes.
- Instantons pair all of them up:

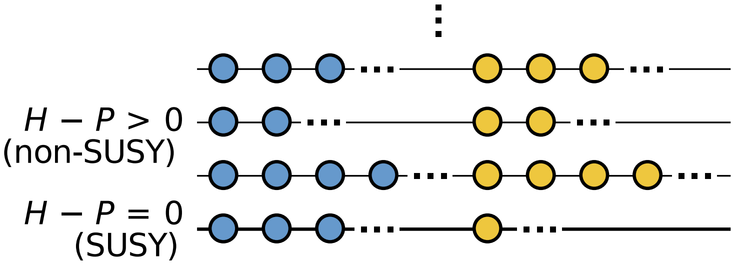


- and induce



Implication: Supersymmetry Breaking – 1/2

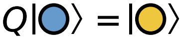
- If $\mathcal{A} = 0$, then supersymmetry is spontaneously broken
- SUSY states have $\{Q, Q^\dagger\} = H - P = 0$:



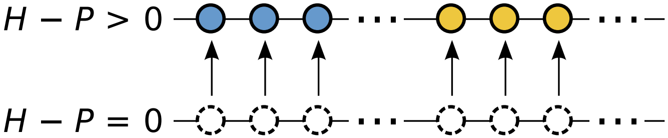
- Take space to be a circle. Then $P = 0, 1, 2, \ldots$.
- So infinite dimensional.

Implication: Supersymmetry Breaking – 2/2

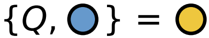
- Instantons pair up perturbative SUSY states:



- and "lift" them at once:



- Analogous to



Implication: Höhn–Stolz Conjecture

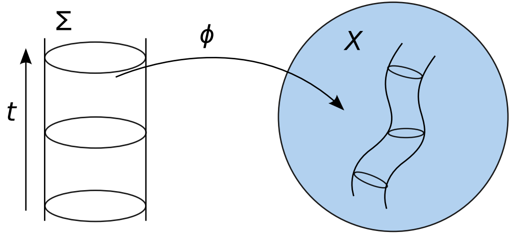
- *If M is a Riemannian spin manifold with positive Ricci curvature and $p_1(M)/2 = 0$, then the elliptic genus*

$$q^{-d/24} \sum_{n=0}^{\infty} q^n \, \mathrm{Tr}_{P=-d/24+n} (-1)^{F_R}$$

vanishes. [Stolz, Math. Ann. **304** (1996) 785]
- The elliptic genus vanishes if there are no SUSY states
- The "theorem" gives a physics proof of a special case

Implication: Geometry of Loop Spaces

- $\mathcal{L}X$: space $\mathrm{Map}(S^1, X)$ of all loops in X
- Canonical quantization gives SUSY QM on $\mathcal{L}X$:



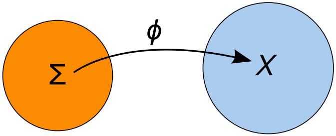
- States are spinors on $\mathcal{L}X$:

$$\{\psi_+^i(\sigma), \bar{\psi}_+^{\bar{\jmath}}(\sigma')\} = g^{i\bar{\jmath}}\delta(\sigma - \sigma') \longleftrightarrow \{\Gamma^{i\sigma}, \Gamma^{\bar{\jmath}\sigma'}\} = g^{i\sigma, \bar{\jmath}\sigma'}.$$

- SUSY states are harmonic spinors. ($Q + Q^\dagger$: Dirac operator)
- $\mathcal{L}X$ has no harmonic spinors. (cf. Lichnerowicz theorem)

$(2,2)$ Model

- Bosonic field $\phi\colon \Sigma \to X$



Σ: Riemann surface, X: Kähler manifold

- Fermionic fields

$$\psi_- \in \Gamma(K_\Sigma^{1/2} \otimes \phi^* T_X), \qquad \bar{\psi}_- \in \Gamma(K_\Sigma^{1/2} \otimes \phi^* \overline{T}_X),$$
$$\psi_+ \in \Gamma(\overline{K}_\Sigma^{1/2} \otimes \phi^* T_X), \qquad \bar{\psi}_+ \in \Gamma(\overline{K}_\Sigma^{1/2} \otimes \phi^* \overline{T}_X).$$

- Supercharges Q_+, $\overline{Q}_+$, Q_-, $\overline{Q}_-$; $\{Q_\pm, \overline{Q}_\pm\} = H \mp P$.
- Action

$$S = \int_\Sigma d^2 z (g_{i\bar{\jmath}} \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{\jmath}} + i g_{i\bar{\jmath}} \psi_+^i D_z \bar{\psi}_+^{\bar{\jmath}} + \text{terms with } \psi_-)$$

(0, 2) Model

- Kill the left-moving fermions:

$$S = \int_\Sigma d^2z(g_{i\bar{\jmath}}\partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{\jmath}} + ig_{i\bar{\jmath}}\psi^i_+ D_z\bar{\psi}^{\bar{\jmath}}_+)$$

- Supercharges Q_+ and $\overline{Q}_+$ act by

$$\begin{aligned} [Q, \phi^i] &= 0, & [Q, \phi^{\bar{\imath}}] &= \bar{\psi}^{\bar{\imath}}_+, \\ \{Q, \psi^i_+\} &= -\partial_{\bar{z}}\phi^i, & \{Q, \bar{\psi}^{\bar{\imath}}_+\} &= 0, \end{aligned}$$

  and h.c.
- R-symmetry (possibly anomalous nonperturbatively):

$$\psi_+ \to e^{-i\alpha}\psi_+, \qquad \bar{\psi}_+ \to e^{i\alpha}\bar{\psi}_+.$$

Twisting

- Essentially only one way to twist:

$$\psi^i_{\bar{z}} \in \Gamma(\overline{K}_\Sigma \otimes \phi^* T_X), \qquad \bar{\psi}^{\bar{\imath}} \in \Gamma(\phi^* \overline{T}_X).$$

- Q is now a scalar on Σ.
- $T_{z\bar{z}}$ and $T_{\bar{z}\bar{z}}$ (but not T_{zz}) are now Q-exact,

$$T_{z\bar{z}} = \{Q, \cdots\}, \qquad T_{\bar{z}\bar{z}} = \{Q, \cdots\},$$

  hence zero in the Q-cohomology.
- Antiholomorphic reparametrizations don't change Q-cohomology classes.
- A local operators $\mathcal{O}$ of dimension (n, m) transforms as

$$\mathcal{O}(0) \to \lambda^{-n}\bar{\lambda}^{-m}\mathcal{O}(0)$$

  under $z \to \lambda z$, $\bar{z} \to \bar{\lambda}\bar{z}$. If $[\mathcal{O}] \neq 0$, then $m = 0$.

Local Operators of R-charge q and Dimension $(n, 0)$

- are $(0, q)$-forms valued in a holomorphic vector bundle $V_{X,n}$.
- $\psi^i_{\bar{z}}$ and $\partial_{\bar{z}}$ cannot enter.
- $n = 0$: we have

$$\mathcal{O}_{\bar{\imath}_1 \cdots \bar{\imath}_q}(\phi, \bar{\phi}) \bar{\psi}^{\bar{\imath}_1} \cdots \bar{\psi}^{\bar{\imath}_q};$$

  thus $V_{X,0} = 1$, the trivial bundle.
- $n = 1$: we have

$$\mathcal{O}_{j\bar{\imath}_1 \cdots \bar{\imath}_q}(\phi, \bar{\phi}) \partial_z \phi^j \bar{\psi}^{\bar{\imath}_1} \cdots \bar{\psi}^{\bar{\imath}_q}, \quad \mathcal{O}^j{}_{\bar{\imath}_1 \cdots \bar{\imath}_q}(\phi, \bar{\phi}) g_{j\bar{k}} \partial_z \phi^{\bar{k}} \bar{\psi}^{\bar{\imath}_1} \cdots \bar{\psi}^{\bar{\imath}_q};$$

  thus $V_{X,1} = T_X \oplus T_X^\vee$.
- The twisted spinor bundles $S_X \oplus V_{X,n}$ enter the elliptic genus.

Classical Chiral Algebra

- Let's find how Q acts on $V_{X,n}$. Recall $[Q, \phi^{\bar{\imath}}] = \bar{\psi}^{\bar{\imath}}$.
- Take an $n = 1$ example:

$$[Q, \mathcal{O}_{j\bar{\imath}_1\cdots\bar{\imath}_q}\partial_z\phi^j\bar{\psi}^{\bar{\imath}_1}\cdots\bar{\psi}^{\bar{\imath}_q}] = \bar{\psi}^{\bar{\imath}}\partial_{\bar{\imath}}\mathcal{O}_{j\bar{\imath}_1\cdots\bar{\imath}_q}\partial_z\phi^j\bar{\psi}^{\bar{\imath}_1}\cdots\bar{\psi}^{\bar{\imath}_q}.$$

- The other type:

$$[Q, \mathcal{O}^j g_{j\bar{k}}\partial_z\phi^{\bar{k}}] = \bar{\psi}^{\bar{\imath}}\partial_{\bar{\imath}}\mathcal{O}^j g_{j\bar{k}}\partial_z\phi^{\bar{k}} + \mathcal{O}^j g_{j\bar{k}}\underbrace{D_z\bar{\psi}^{\bar{k}}}_{=0}.$$

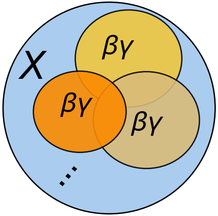
- Classically, $Q = \bar{\psi}^{\bar{\imath}}\partial_{\bar{\imath}} = \bar{\partial}$ and

$$\mathcal{A}^q \cong \bigoplus_{n=0}^{\infty} H^q_{\bar{\partial}}(X, V_{X,n}).$$

So, $\mathcal{A}$ is infinite dimensional.

Perturbative Chiral Algebra

- Perturbatively, Q is still a differential operator on $V_{X,n}$,
- but deformed: $Q = \bar{\partial} + \alpha' Q_1 + (\alpha')^2 Q_2 + \cdots$.
- Reconstructed by gluing *free* $\beta\gamma$ *systems* over X:



- More precisely,

$$\mathcal{A}^q \cong H^q(X, \mathcal{D}_X^{ch}),$$

where $\mathcal{D}_X^{ch}$ is a sheaf of chiral differential operators on X [Witten, *Adv. Theor. Math. Phys.* **11** (2007) 1 [hep-th/0504078]].

$\mathbb{CP}^1$ Model: Perturbative Chiral Algebra – 1/3

- Let's find the perturbative chiral algebra of the $\mathbb{CP}^1$ model.
- Charge 0, dimension 0: Q-closed local operators are functions on the compact X, thus constants: [1].
- Charge 0, dimension 1: $[J_-]$, $[J_3]$, $[J_+]$, where

$$J_- = g_{\phi\bar{\phi}}\partial_z\bar{\phi}, \quad J_3 = \phi g_{\phi\bar{\phi}}\partial_z\bar{\phi}, \quad J_- = \phi^2 g_{\phi\bar{\phi}}\partial_z\bar{\phi}$$

  generate the $SL(2)$ current algebra at the critical level -2.
- The list continues.

$\mathbb{CP}^1$ Model: Perturbative Chiral Algebra – 2/3

- charge 1, dimension 0: none
- charge 1, dimension 1: $[\theta]$, $\theta \propto R_{i\bar{\jmath}}\partial_z\phi^i\bar{\psi}^{\bar{\jmath}}$.
- charge 1, dimension 2: $[J_-\theta]$, $[J_3\theta]$, $[J_+\theta]$.
- Is $[\partial_z\theta]$ nonzero? Classically, yes. Perturbatively, no:

$$[Q, T_{zz}] = \partial_z\theta.$$

  Thus T_{zz} and $\partial_z\theta$ are *lifted* out of the Q-cohomology.
- The perturbative chiral algebra does not have an energy-momentum tensor whenever $c_1(X) \neq 0$.

$\mathbb{CP}^1$ Model: Perturbative Chiral Algebra – 3/3

- Compare $\mathcal{A}^0$ and $\mathcal{A}^1$:

$$
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
n = 1 & [J_-],\ [J_3],\ [J_+] & n = 2 & [J_-\theta],\ [J_3\theta],\ [J_+\theta] \\
n = 0 & [1] & n = 1 & [\theta] \\
 & & n = 0 & \text{none}
\end{array}
$$

- $[1]$ and $[\theta]$ are "ground states"
- on which elements of $\mathcal{A}^0$ acts as "creation operators."
- $\mathcal{A}^0 \cong \mathcal{A}^1$ via $[\mathcal{O}] \mapsto [\mathcal{O}\theta]$.

  [Malikov et al., *Comm. Math. Phys.* **204** (1999) 439 [`math/9803041`]]

$\mathbb{CP}^1$ Model: Nonperturbative Chiral Algebra – 1/3

- Instantons are holomorphic maps from Σ to X.
- The R-symmetry is broken to $\mathbb{Z}_{2k}$ ($2k =$ the GCD of $c_1(X)$).
- Instantons can violate the grading by dimension.
- For $X = \mathbb{CP}^1$, the R-symmetry is broken to $\mathbb{Z}_2$ by anomaly.
- Instantons "tunnel" from the "ground states" $[\theta]$ to $[1]$?
- If yes, they will induce

$$\{Q, \theta\} \sim 1.$$

The LHS has charge $2 = 0$.

$\mathbb{CP}^1$ Model: Nonperturbative Chiral Algebra – 2/3

- Compute the matrix elements of $\{Q, \theta\}$, taking $\Sigma = S^1 \times \mathbb{R}$.
- Compactify and map to correlation functions on $\mathbb{CP}^1$:

$$\langle j|\{Q, \theta(0)\}|i\rangle = \left\langle \mathcal{O}_j(\infty) \oint d\bar{z}\, G(\bar{z})\theta(1)\mathcal{O}_i(0) \right\rangle_{\mathbb{CP}^1}.$$

- Compute in the one-instanton background.
- Express the result as a zero-instanton quantity.
- Found:

$$\left\langle \mathcal{O}_j(\infty)\{Q, \theta(1)\}\mathcal{O}_i(0)\right\rangle_{\mathbb{CP}^1} \sim \left\langle \mathcal{O}_j(\infty)\mathcal{O}_i(0)\right\rangle_{\mathbb{CP}^1}$$

- Therefore, $\{Q, \theta\} \sim 1$.

$\mathbb{CP}^1$ Model: Nonperturbative Chiral Algebra – 3/3

- More precisely,

$$\{Q, \theta\} \sim e^{-t}(1 + Q\text{-exact local operator}).$$

- $[\theta]$ recovers in the chiral algebra as $t \to \infty$.
- $[1] = 0$, thus

$$[\mathcal{O}] = [1] \cdot [\mathcal{O}] = 0 \cdot [\mathcal{O}] = 0.$$

- In other words, any Q-closed local operator is Q-exact.
- Therefore, the chiral algebra is nonperturbatively *trivial*.

Supersymmetry Breaking

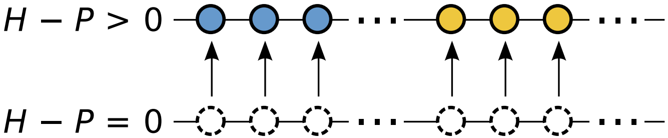
- Since $Q^2 = 0$, we can also define the Q-cohomology of states.
- It is isomorphic to the space of SUSY states
- and a module over the chiral algebra:

$$[\mathcal{O}] \cdot [|\Psi\rangle] = [\mathcal{O}|\Psi\rangle].$$

- If the chiral algebra is trivial, then

$$[|\Psi\rangle] = [1] \cdot [|\Psi\rangle] = 0 \cdot [|\Psi\rangle] = 0.$$

- Therefore, the Q-cohomology of states is trivial.
- Supersymmetry is spontaneously broken:



Vanishing “Theorem”

- X is a compact Kähler manifold
- X is spin and $p_1(X)/2 = 0$ (X is a valid target space)
- has positive Ricci curvature (the model is asymptotically free)
- contains an instanton (rational curve) $C \cong \mathbb{CP}^1$
- the normal bundle $N_{C/X}$ is trivial, where

$$0 \longrightarrow T_C \longrightarrow T_X|_C \longrightarrow N_{C/X} \longrightarrow 0.$$

- Then $\mathcal{A} = 0$ nonperturbatively.

“Proof”

- Say $\{Q, \theta\} \sim e^{-t}\mathcal{O}$.
- $\mathcal{O}$ is a classically Q-closed local operator.
- The charge 0 and dimension 0 part of $\mathcal{O}$ is a holomorphic function on X. (Recall $Q = \bar{\partial}$ classically.)
- Since X is compact, this function is constant.
- Compute the contribution to $\{Q, \theta\}$ from $C \subset X$:

  $$\{Q, \theta\} \sim e^{-t}(\chi_C + \text{other instantons}),$$

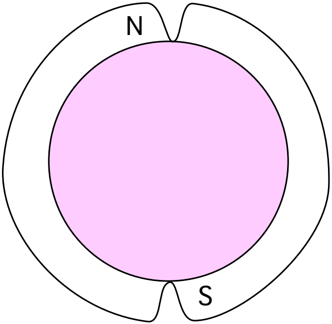
  where $\chi_C = 1$ on C and 0 elsewhere.
- Thus the constant part of $\mathcal{O}$ is nonzero:

  $$\{Q, \theta\} \sim e^{-t}(1 + \cdots).$$

- Therefore, the chiral algebra is trivial.

Holomorphic Morse Theory on Loop Space

- Morse function on $X = \mathbb{CP}^1$ introduces a potential:

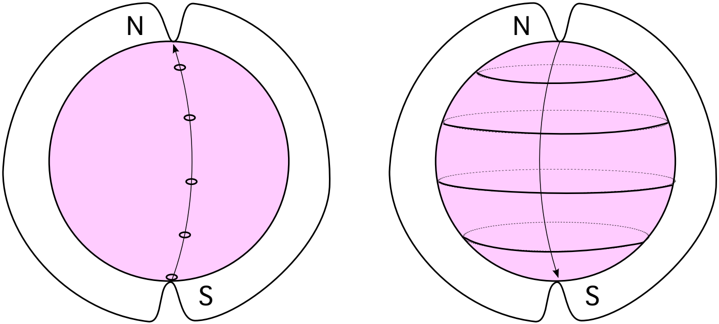


Morse index equal to 0 at S and 1 at N

- Approximate supersymmetric states of fermion number 0 at S
- Approximate supersymmetric states of fermion number 1 at N

Instantons in the Deformed $\mathbb{CP}^1$ Model

- Instantons



- Worldline instantons induce $Q|S\rangle \sim |N\rangle$ and lift all the states that do not enter the classical cohomology.
- Perturbative corrections lift some of the states
- Worldsheet instantons induce $Q|N\rangle \sim |S\rangle$ and lift the rest

Outlook

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