Open intersection numbers, matrix model, and W-constraints

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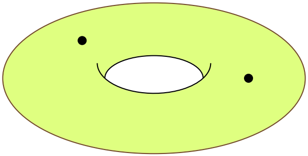


$$h = 1, n = 2$$

The compactifications of the moduli spaces of genus h Riemann surfaces with n marked points $\mathcal{M}_{h,n}$ are orbifolds of dimension

$$\dim_{\mathbb{C}} \overline{\mathcal{M}}_{h,n} = 3h - 3 + n$$

Intersection numbers of the first Chern classes $\psi_i \in H^2(\overline{\mathcal{M}}_{h,n}, \mathbb{Q})$ of the cotangent line bundles:

$$\int_{\overline{\mathcal{M}}_{h,n}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \dots \psi_n^{\alpha_n} \in \mathbb{Q}$$

Witten: Intersection theory on the moduli spaces describes 2d quantum (topological) gravity. It should be equivalent to the continuous (double scaling) limit of the Hermitian matrix model.

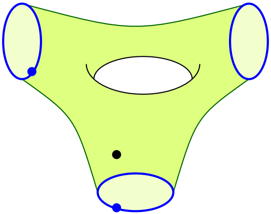
Conjecture [Witten, '91]: The generating function of the intersection numbers is a tau-function of the KdV integrable hierarchy.

or, equivalently

The generating function of the intersection numbers is annihilated by infinitely many differential operators, satisfying Virasoro commutation relations.

Proof [Kontsevich, '92]: matrix model!

- **KdV tau-function**
- **Kontsevich matrix model**
- **Virasoro constraints**



$h = 1, b = 3, k = 2, l = 1$

Open intersection numbers "$\int_{\overline{\mathcal{M}}_{h,b,k,l}} \psi_1^{\alpha_1} \dots \psi_l^{\alpha_l} \phi_{l+1}^{\beta_1} \dots \phi_{l+k}^{\beta_k}$"

Recently **[R. Pandharipande, J. Solomon and R. Tessler; A. Buryak, '14]** described (conjectured) intersection theory on $\mathcal{M}_{2h+b-1,k,l}$, that is on the combination of the moduli space of Riemann surfaces with h handles, b boundaries, k marked points on the boundary and l interior marked points

- **Tau-function? Matrix model? Virasoro (W)-constraints?**

The Kontsevich–Penner matrix integral

$$\tau_n = \det(\Lambda)^n \mathcal{C}^{-1} \int_{M\times M} [d\Phi] \exp\left(-\mathrm{Tr}\left(\frac{\Phi^3}{3!} - \frac{\Lambda^2\Phi}{2} + n\log\Phi\right)\right)$$

Tau-function of the MKP hierarchy, describes both **closed** and **open** intersection numbers.

[A.A. '14]

Parameter n counts the number of boundaries

[B. Safnuk '16]

[A.A., A. Buryak, R. Tessler '17]

n	0	arbitrary
Intersection numbers	Closed	Open
Integrable hierarchy	KdV	(M)KP
Algebra of constraints	Heisenberg+Virasoro	Virasoro+$W^{(3)}$
Cut-and-join operator	$e^{W_{KW}}\cdot 1$	"$e^{W_1+W_2/2}$" $\cdot 1$

The bilinear identity satisfied by a **tau-function** $\tau_n(\mathbf{t})$ of the **modified Kadomtsev–Petviashvili (MKP)** integrable hierarchy for $m \geq n$

$$\oint_{\infty} z^{m-n} e^{\sum_{k>0}(t_k - t_k')z^k} \tau_m(\mathbf{t} - [z^{-1}])\, \tau_n(\mathbf{t}' + [z^{-1}]) dz = 0$$

encodes all nonlinear equations of the hierarchy.

In particular, for $m = n$ we have the **KP hierarchy**. The first non-trivial **Hirota equation** contained in the KP bilinear identity is

$$(D_1^4 + 3D_2^2 - 4D_1 D_3)\tau_m \cdot \tau_m = 0$$

The second derivative of this equation with respect to t_1 gives the **KP equation** in its standard form

$$3u_{22} = (4u_3 - 12uu_1 - u_{111})_1$$

where $u = \frac{\partial^2}{\partial {t_1}^2} \log(\tau_m)$.

The **Miwa parametrization** is very convenient for matrix models

$$t_k = \frac{1}{k} \sum_{j=1}^{M} z_j^{-k}$$

for some finite M. From the boson-fermion correspondence and Wick theorem it follows that a tau-function in this parametrization is

$$\tau([Z]) := \tau\left(t_k = \frac{1}{k} \mathrm{Tr}\, Z^{-k}\right) = \frac{\det_{i,j=1}^{M} \Phi_i(z_j)}{\Delta(z)}$$

where

$$\Phi_i(z) = z^{i-1} + \sum_{j=-\infty}^{i-2} \Phi_{i,j} z^j$$

are the **basis vectors** and $\Delta(z)$ is the Vandermonde determinant. For a tau-function they describe a point of the infinite-dimensional **Sato Grassmannian**

$$\mathcal{W} = \mathrm{span}_{\mathbb{C}}\{\Phi_1(z), \Phi_2(z), \Phi_3(z), \dots\} \in \mathrm{Gr}(0)$$

Let $a \in \mathbb{C}[[z, z^{-1}, \frac{\partial}{\partial z}]]$ be a formal differential operator operator such that

$$a\mathcal{W} \subset \mathcal{W}$$

for some $\mathcal{W}$. Then, for the corresponding tau-function it holds that

$$\widehat{W}_a \tau = C \tau$$

for some constant C, where the operator $\widehat{W}_a \in \mathbb{C}[[\mathbf{t}, \frac{\partial}{\partial \mathbf{t}}]]$ can be obtaned from a by a boson-fermion correspondence.

Such operators a we call the **Kac–Schwarz operators**. These operators form an algebra. However, general properties of such an algebra for the KP tau-functions are unknown.

Sometimes it is more convenient to work with the Sato Grassmannian and KS operators!

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Let $\overline{\mathcal{M}}_{h,l}$ be the Deligne–Mumford compactification of the moduli space of genus h complex curves Σ with l marked points $x_1, \ldots, x_l$. The generating function of the intersection numbers of ψ-classes

$$\int_{\overline{\mathcal{M}}_{h,l}} \psi_1^{\alpha_1} \psi_2^{\alpha_2} \ldots \psi_l^{\alpha_l} =: \langle \tau_{\alpha_1} \tau_{\alpha_2} \ldots \tau_{\alpha_l} \rangle_h,$$

is

$$\mathcal{F}_{KW}(\mathbf{T}, \hbar) := \sum_{h=0}^{\infty} \hbar^{2h-2} \left\langle \exp\left(\hbar \sum_{m=0}^{\infty} T_m \tau_m \right) \right\rangle_h$$

It's exponentiated version is the Kontsevich–Witten tau-function of the KdV hierarchy

$$\tau_{KW}(\mathbf{T}, \hbar) = \exp\left(\mathcal{F}_{KW}(\mathbf{T}, \hbar)\right)$$

[E. Witten '91; M. Kontsevich '92]

Below we use the variables $t_{2k+1} = T_k/(2k+1)!!$, times of the KP hierarchy.

The Kontsevich–Witten tau-function is a formal series in odd times t_{2k+1} with rational coefficients. In the Miwa parametrization

$$t_k = \frac{1}{k}\mathrm{Tr}\,\Lambda^{-k}$$

it is equal to the asymptotic expansion of the **Kontsevich matrix integral** over the $M \times M$ Hermitian matrix Φ:

$$\tau_{KW}\left([\Lambda]\right) = \mathcal{C}^{-1}\int [d\Phi]\exp\left(-\frac{1}{\hbar}\mathrm{Tr}\left(\frac{\Phi^3}{3!}+\frac{\Lambda\Phi^2}{2}\right)\right)$$

All t_k can be considered as independent variables as the size of the matrices M tends to infinity and in this limit the integral yields the Kontsevich–Witten tau-function.

It is easy to show that this matrix integral defines a tau-function of the KdV integrable hierarchy.

The standard volume form on the space of hermitian matrices

$$[d\Phi] = \prod_{1\leq i<j\leq M} d\Im\Phi_{ij}\, d\Re\Phi_{ij} \prod_{k=1}^{M} d\Phi_{kk}$$

The Harish-Chandra–Itzykson–Zuber formula allows us to reduce the matrix integral to the ratio of determinants

$$\tau_{KW}\left([\Lambda]\right) = \frac{\det_{i,j=1}^{M} \Phi_i^{KW}(\lambda_j)}{\Delta\left(\lambda\right)}$$

Here λ_j are the eigenvalues of the matrix Λ and

$$\Phi_i^{KW}(\lambda) = \lambda^{i-1}\left(1 + O(\lambda^{-1})\right)$$

define a point of the **Sato Grassmannian**.

KP integrability!

The basis vectors Φ_i^{KW} are given by the integrals

$$\Phi_k^{KW}(z) = \sqrt{\frac{z}{2\pi}} e^{-\frac{z^3}{3}} \int_C dy\, y^{k-1} \exp\left(-\frac{y^3}{3!} + \frac{yz^2}{2}\right)$$

This representation allows us to find the Kac–Schwarz operators of the KW tau-function:

$$a_{KW} = \frac{1}{z}\frac{\partial}{\partial z} + z - \frac{1}{2z^2}, \qquad b_{KW} = \frac{z^2}{2}$$

The Kac–Schwarz operators a_{KW} and b_{KW} satisfy the canonical commutation relation and generate an algebra of the Kac–Schwarz operators for the KW tau-function. They allow us to construct two infinite sets of operators, which annihilate (and completely specify) the generating function

$$\frac{\partial}{\partial t_{2k}} \tau_{KW} = 0, \quad k \geq 1$$

Reduction to KdV

Consider a bosonic current on the curve $y^2 = z$ with odd boundary conditions

$$\widehat{J}_o(z) = \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \left((2k+1)\tilde{t}_{2k+1} z^{k-\frac{1}{2}} + \frac{1}{z^{k+\frac{3}{2}}} \frac{\partial}{\partial t_{2k+1}} \right)$$

where the time variables are subject to the dilaton shift

$$\tilde{t}_k = t_k - \frac{\delta_{k,3}}{3\hbar}$$

Then, we can construct

$$\widehat{\mathcal{L}}(z) = \sum_{k=-\infty}^{\infty} \frac{\widehat{\mathcal{L}}_k}{z^{k+2}} = \frac{1}{2} {}^*_* \widehat{J}_o^2(z) {}^*_* + \frac{1}{16 z^2}$$

where we use usual bosonic normal order.

These operators satisfy the Virasoro commutation relations

$$\left[\widehat{\mathcal{L}}_k, \widehat{\mathcal{L}}_m\right] = (k-m)\widehat{\mathcal{L}}_{k+m} + \frac{1}{12}k(k^2-1)\delta_{k,-m}$$

with central charge $c = 1$.

The **Virasoro constraints** follow from the boson-ferion correspondence of the KS operators $b_{KW}^{m+1} a_{KW}$

$$\hat{\mathcal{L}}_m \, \tau_{KW}(\mathbf{t}; \hbar) = 0, \qquad m \geq -1$$

completely specify the Kontsevich-Witten tau-function.

$$\widehat{\mathcal{L}}_m = \frac{1}{2}\sum_{k=1}^{\infty}(2k+1)\,\tilde{t}_{2k+1}\frac{\partial}{\partial t_{2k+2m+1}} + \frac{1}{4}\sum_{k=0}^{m-1}\frac{\partial^2}{\partial t_{2k+1}\partial t_{2m-2k-1}} + \frac{t_1^2}{4}\delta_{m,-1} + \frac{1}{16}\delta_{m,0}$$

From the Virasoro constraints it follows that the Kontsevich–Witten tau-function can be described by a cut-and-join operator **[A. A. '10]**

$$\tau_{KW}(\mathbf{t};\hbar) = e^{\hbar \widehat{W}_{KW}} \cdot 1$$

where

$$\widehat{W}_{KW} = \frac{1}{3}\sum_{k,m\geq 0}(2k+1)\,(2m+1)\,t_{2k+1}t_{2m+1}\frac{\partial}{\partial t_{2k+2m-1}} + \frac{1}{3!}\sum_{k,m\geq 0}(2k+2m+5)\,t_{2k+2m+5}\frac{\partial^2}{\partial t_{2k+1}\partial t_{2m+1}} + \frac{t_1^3}{3!} + \frac{t_3}{8}$$

Operator $\widehat{W}_{KW}$ describes a topological recursion on the level of tau-function, $\hbar^{2h-2+n}$

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The moduli spaces of open Riemann surfaces (Riemann surfaces with boundaries) were described for the disc case in **[R. Pandharipande, J. Solomon and R. Tessler '14]** and for the higher genera case in **[R. Tessler '15]**.

$$\dim_{\mathbb{R}} \mathcal{M}_{h,b,k,l} = 6h - 6 + 3b + k + 2l$$

We can consider the intersection numbers

$$\text{``}\int_{\overline{\mathcal{M}}_{h,b,k,l}} \psi_1^{\alpha_1} \dots \psi_l^{\alpha_l} \phi_{l+1}^{\beta_1} \dots \phi_{l+k}^{\beta_k}\text{''} = \left\langle \tau_{\alpha_1} \dots \tau_{\alpha_l} \sigma_{\beta_1} \dots \sigma_{\beta_k} \right\rangle_{h,b}$$

where ψ_j are the the first Chern classes of the bundles $\mathcal{L}_i$ corresponding to the interior points and ϕ_j are their analogs for the boundary points. In **[R. Pandharipande, J. Solomon and R. Tessler '14]** all intersection numbers of the form

$$\int_{\overline{\mathcal{M}}_{0,1,k,l}} \psi_1^{\alpha_1} \dots \psi_l^{\alpha_l} \phi_{l+1}^{0} \dots \phi_{l+k}^{0}$$

were constructed.

We can consider the generating function of all these intersection numbers

$$\mathcal{F}_n(\mathbf{T};\mathbf{S},\hbar) = \sum_{h=0}^{\infty}\sum_{b=0}^{\infty}\hbar^{2h-2+b}n^b\left\langle \exp\left(\hbar\sum_{k\geq 0}(T_k\tau_k + S_k\sigma_k)\right)\right\rangle_{h,b}$$

and

$$\tau_n(\mathbf{T};\mathbf{S},\hbar) = e^{\mathcal{F}_n(\mathbf{T};\mathbf{S},\hbar)}$$

In **[R. Tessler '15]** all coefficients of the generating function for $n=1$ (that is the function, to which the components of the moduli spaces with different number of boundaries contributes with the same weight) and $\mathbf{S_0} = \{S_0, 0, 0, \dots\}$ (that is without descendants on the boundary),

$$\tau_1(\mathbf{T};\mathbf{S_0},\hbar)$$

were calculated. Obtained all-genera generating function is uniquely specified by the so called **open KdV** equations and the Virasoro constraints.

In **[A. Buryak, '14]** the generating function was generalized to describe the descendants on the boundary, and the Virasoro constrains for this conjectural generalized (or extended) generating function were established.

$$\tau_1(\mathbf{T};\mathbf{S},\hbar)$$

From the definition it follows that for $n=0$ only the components without boundaries contribute, so that the generating function does not depend on S_k's and coincides with the Kontsevich-Witten tau-function

$$\tau_0(\mathbf{T};\mathbf{S},\hbar)=\tau_{KW}(\mathbf{T},\hbar)$$

$\hbar$ is not an independent variable, we can omit it

$$\tau_Q(\mathbf{T};\mathbf{S},\hbar)=\tau_Q(\mathbf{T};\mathbf{S},1)\Big|_{T_k\mapsto\hbar^{\frac{2k+1}{3}}T_k,\,S_k\mapsto\hbar^{\frac{2k+2}{3}}S_k}$$

We **unify** two infinite sets of variables T_k and S_k, corresponding to the descendants in the interior and on the boundary:

$$T_k = (2k+1)!!\, t_{2k+1}, \quad S_k = 2^{k+1}(k+1)!\, t_{2k+2}$$

Proposition: the extended generating function of open intersection numbers $\tau_n(\mathbf{t})$ is given by the matrix integral

$$\boxed{\tau_n = \mathcal{C}^{-1}\, \det(\Lambda)^n \int [d\Phi] \exp\left(-\mathrm{Tr}\left(\frac{\Phi^3}{3!} - \frac{\Lambda^2\Phi}{2} + n\log\Phi\right)\right)}$$

where

$$t_k = \frac{1}{k}\mathrm{Tr}\,\Lambda^{-k}$$

The Kontsevich-Penner model

$$\tau_n = \frac{\int [d\Phi] \det\left(1 + \frac{\Phi}{\Lambda}\right)^{-n} \exp\left(-\mathrm{Tr}\left(\frac{\Phi^3}{3!} + \frac{\Lambda\Phi^2}{2}\right)\right)}{\int [d\Phi] \exp\left(-\mathrm{Tr}\,\frac{\Lambda\Phi^2}{2}\right)}$$

This matrix integral belongs to the family of the **generalized Kontsevich models**.

$$\tau_n = \frac{\det_{i,j=1}^{M} \Phi_i^{(n)}(\lambda_j)}{\Delta(\lambda)}$$

where

$$\Phi_k^{(n)}(\lambda) = \lambda^n \Phi_{k-n}^{KW}(\lambda)$$
$$= \frac{\lambda^{n+1/2}}{\sqrt{2\pi}} e^{-\frac{\lambda^3}{3}} \int_C d\,y\, y^{k-n-1} \exp\left(-\frac{y^3}{3!} + \frac{y\lambda^2}{2}\right)$$

MKP tau-function!

The principal specialization of the tau-function coincides with the first basis vector of the Sato Grassmannian. It is annihilated by the Kac-Schwarz operator, which defines the **quantum spectral curve**

$$\left(a_n^3 - z^2 a_n + 2(n-1)\right)\Phi_1^n = 0$$

$$a_n = \frac{1}{z}\frac{\partial}{\partial z} - \left(n + \frac{1}{2}\right)\frac{1}{z^2} + z$$

After conjugation with a quasi-classical prefactor we obtain

$$\hat{A} = \hat{y}^3 - 2\hat{x}\hat{y} + 2(n-1)$$

where

$$\hat{x} = \frac{z^2}{2}, \quad \hat{y} = \frac{1}{z}\frac{\partial}{\partial z}, \quad [\hat{y}, \hat{x}] = 1$$

- $n = 0$. Closed intersection, τ_{KW}: $\hat{A} = \hat{y}\left(\hat{y}^2 - 2\hat{x}\right)$
- $n = 1$. Open intersections, τ_1: $\hat{A} = \left(\hat{y}^2 - 2\hat{x}\right)\hat{y}$

Origianal Pandharipande-Solomon-Tesser model

$$\tau_o := \tau_1$$

The Kac–Schwarz operator is

$$a_o = z\, a_{KW}\, z^{-1} = \frac{1}{z}\frac{\partial}{\partial z} - \frac{3}{2z^2} + z$$

This tau-function depends both on odd and even times, and z^2 is not a Kac–Schwarz operator anymore:

$$z^2 \Phi_1^o(z) \notin \left\{\Phi^o(z)\right\}.$$

Nevertheless,

$$l_k^o = -z^{2k+2} a_o = -z^{2k+2}\left(\frac{1}{z}\frac{\partial}{\partial z} - \frac{3}{2z^2} + z\right)$$

for $k \geq -1$ belong to the Kac–Schwarz algebra. The Virasoro commutation relations:

$$\left[l_k^o, l_m^o\right] = 2(k-m) l_{k+m}^0$$

The $W_{1+\infty}$ algebra of infinitesimal symmetries of the KP hierarchy can be described in terms of the bosonic current $\widehat{J}(z) = \sum \widehat{J}_k z^{-k-1}$, where

$$\widehat{J}_k = \begin{cases} \frac{\partial}{\partial t_k} & \text{for} \quad k > 0, \\ 0 & \text{for} \quad k = 0, \\ -kt_{-k} & \text{for} \quad k < 0 \end{cases}$$

$\widehat{J}(z)$ generates the **Heisenberg algebra**. ${}_*^*\widehat{J}(z)^2{}_*^*$ generates the **Virasoro algebra**:

$$\widehat{L}_m = \frac{1}{2} \sum_{k+l=-m} k\, l\, t_k t_l + \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+m}} + \frac{1}{2} \sum_{k+l=m} \frac{\partial^2}{\partial t_k \partial t_l}$$

${}_*^*\widehat{J}(z)^3{}_*^*$ generates the $W^{(3)}$ **algebra**:

$$\widehat{M}_k = \frac{1}{3} \sum_{a+b+c=-k} a\, b\, c\, t_a\, t_b\, t_c + \sum_{c-a-b=k} a\, b\, t_a\, t_b \frac{\partial}{\partial t_c} + \sum_{b+c-a=k} a\, t_a \frac{\partial^2}{\partial t_b \partial t_c} + \frac{1}{3} \sum_{a+b+c=k} \frac{\partial^3}{\partial t_a \partial t_b \partial t_c}$$

Using the Kac–Schwarz operators we can show that the tau-function τ_1 is an eigenfunction of the Virasoro operators:

$$\widehat{L}_k^{(1)} = \widehat{L}_{2k} + (k+2)\widehat{J}_{2k} - \widehat{J}_{2k+3} + \left(\frac{1}{8} + \frac{3}{2}\right)\delta_{k,0}, \quad k \geq -1$$

$$\widehat{M}_k^{(1)} = \widehat{M}_{2k} + 2(k+3)\widehat{L}_{2k} - 2\widehat{L}_{2k+3} - 2(k+3)\widehat{J}_{2k+3} + \left(\frac{95}{12} + 6k + \frac{4}{3}k^2\right)\widehat{J}_{2k} + \widehat{J}_{2k+6} + \frac{23\,\delta_{k,0}}{3}, \quad k \geq -2$$

These operators belong to $W_{1+\infty}$ algebra of symmetries of KP and annihilate the tau-function

$$\widehat{L}_k^{(1)}\,\tau_1 = 0, \qquad k \geq -1$$
$$\widehat{M}_k^{(1)}\,\tau_1 = 0, \qquad k \geq -2$$

$W^{(3)}$-CONSTRAINTS FOR OPEN INTERSECTION NUMBERS

In addition to the Virasoro constraints we have infinitely many higher W-constraints. Let us consider the KS operators

$$w_k^o = z^{2k+4} a_o^2, \quad k \geq -2$$

They satisfy the following commutation relations

$$\left[w_k^o, l_m^o\right] = 2(k-2m) w_{k+m}^o + 4m(m+1) l_{m+k}^o$$

and correspond to the following operators from $W_{1+\infty}$:

$$\widehat{M}_k^o = \widehat{M}_{2k} + 2(k+3)\widehat{L}_{2k} - 2\widehat{L}_{2k+3} - 2(k+3)\widehat{J}_{2k+3} + \left(\frac{95}{12} + 6k + \frac{4}{3}k^2\right)\widehat{J}_{2k} + \widehat{J}_{2k+6} + \frac{23\,\delta_{k,0}}{3}$$

$$\widehat{M}_k^o \, \tau_o = 0, \qquad k \geq -2$$

The basis vectors have an expansion

$$\Phi_k^n = z^{k-1} + \frac{12(2-p)^2-7}{24} z^{k-4}$$
$$+\left(\frac{1}{8}p^4 - \frac{5}{3}p^3 + \frac{365}{48}p^2 - \frac{55}{4}p + \frac{9241}{1152}\right) z^{k-7} + O(z^{k-10})$$

where $p = k - n$. Using the integral representation it is easy to see that

$$z^2\Phi_k^n = \Phi_{k+2}^n - 2(k-n-1)\Phi_{k-1}^n.$$

z^2 operator is not the KS operator for $n \neq 0$, because

$$z^2\Phi_1^n = \Phi_3^n + 2n\Phi_0^n \notin \{\Phi^n\}, \qquad \text{for } n \neq 0$$

However, the following operators are the KS operators

$$\mathrm{l}_{-1} = -a_n,$$
$$\mathrm{l}_0 = -z^2 a_n + n - 1,$$
$$\mathrm{l}_1 = -z^4 a_n + 2(n-1)z^2$$

Using the Kac–Schwarz description of the corresponding point of the Sato Grassmannian it is easy to show that the operators from the $W_{1+\infty}$ algebra

$$\widehat{\mathrm{L}}_{-1}^{(n)} = \widehat{L}_{-2} - \frac{\partial}{\partial t_1} + 2n\, t_2,$$
$$\widehat{\mathrm{L}}_{0}^{(n)} = \widehat{L}_{0} - \frac{\partial}{\partial t_3} + \frac{1}{8} + \frac{3n^2}{2},$$
$$\widehat{\mathrm{L}}_{1}^{(n)} = \widehat{L}_{2} - \frac{\partial}{\partial t_5} + 3n\frac{\partial}{\partial t_2}$$

satisfy the commutation relation of the subalgebra of the Virasoro algebra

$$\left[\widehat{\mathrm{L}}_{i}^{(n)}, \widehat{\mathrm{L}}_{j}^{(n)}\right] = 2(i-j)\,\widehat{\mathrm{L}}_{i+j}^{(n)}, \quad i,j = -1,0,1$$
$$\widehat{\mathrm{L}}_{k}^{(n)} \tau_n = 0, \quad k = -1,0,1$$

$k = -1$ is the **string equation**, $k = 0$ is the **dilaton equation**

Lemma: Operators

$$\widehat{\mathrm{L}}_k^{(n)} = \widehat{L}_{2k} - \frac{\partial}{\partial t_{2k+3}} + 3n\frac{\partial}{\partial t_{2k}} + \sum_{j=1}^{k-1} \frac{\partial^2}{\partial t_{2j}\partial t_{2k-2j}} + \left(\frac{1}{8} + \frac{3n^2}{2}\right)\delta_{k,0} + 2nt_2\delta_{k,-1}, \; k \geq -1$$

satisfy the Virasoro algebra commutation relations

$$\left[\widehat{\mathrm{L}}_k^{(n)}, \widehat{\mathrm{L}}_m^{(n)}\right] = 2(k-m)\widehat{\mathrm{L}}_{k+m}^{(n)}$$

annihilate the tau-function

$$\widehat{\mathrm{L}}_k^{(n)}\tau_n = 0, \quad k \geq -1$$

Remark: This is similar to the case of the Gaussian Hermitian matrix model. For this model we also have an infinite algebra of the Virasoro constraints, but only an $sl(2)$ subalgebra of it belongs to the $W_{1+\infty}$ algebra of KP symmetries. **[M. Mulase, '94]**

For arbitrary n the operators

$$
\begin{aligned}
\mathrm{m}_{-2} &= a_n^2, \\
\mathrm{m}_{-1} &= z^2 a_n^2 - (n-2)a_n, \\
\mathrm{m}_0 &= z^4 a_n^2 - 2(n-2)z^2 a_n + \frac{2}{3}(n-1)(n-2), \\
\mathrm{m}_1 &= z^6 a_n^2 - 3(n-2)z^4 a_n + 2(n-1)(n-2)z^2, \\
\mathrm{m}_2 &= z^8 a_n^2 - 4(n-2)z^6 a_n + 4(n-1)(n-2)z^4,
\end{aligned}
$$

are the KS operators. Of course, these operators are not unique KS operators with the leading terms $z^{2k-4}a_n^2$. Namely, one can add to them a combination of the above considered operators and a constant. Our choice corresponds to the commutation relations

$$
\left[\mathrm{l}_j, \mathrm{m}_k\right] = 2\,(2j-k)\mathrm{m}_{j+k}
$$

$$\widehat{\mathrm{M}}_k^{(n)} = \widehat{M}_{2k} - 2\widehat{L}_{2k+3} + \widehat{J}_{2k+6} + \left(3(k+1)n^2 + \frac{1}{4}\right)\widehat{J}_{2k}$$
$$+(k+4)n\left(\widehat{L}_{2k} - \widehat{J}_{2k+3}\right) + 2\left(n^2+\frac{1}{4}\right)n\delta_{k,0} + 4\,n^2 t_2\delta_{k,-1} + 16\,n^2 t_4\delta_{k,-2}$$
$$+(k-2)n\sum_{j=1}^{k-1}\frac{\partial^2}{\partial t_{2j}\partial t_{2k-2j}} - \frac{4}{3}\sum_{i+j+l=k}\frac{\partial^3}{\partial t_{2i}\partial t_{2j}\partial t_{2l}}$$

for $k \geq -2$.

$$\widehat{\mathrm{M}}_k^{(n)}\tau_n = 0, \quad k \geq -2$$

Commutation relations between the Virasoro and W-operators

$$\left[\widehat{\mathrm{L}}_k^{(n)}, \widehat{\mathrm{M}}_l^{(n)}\right] = 2\,(2k-l)\widehat{\mathrm{M}}_{k+l}^{(n)} - 4\left(k(k-1) - 2\delta_{k,-1}\right)\,n\,\widehat{\mathrm{L}}_{k+l}^{(n)} + 8\sum_{j=1}^{k-1} j\,\frac{\partial}{\partial t_{2k-2j}}\widehat{\mathrm{L}}_{l+j}^{(n)}$$

for $k \geq -1$ and $l \geq -2$.

$W^{(n)}$ algebra can be naturally described in terms of free bosonic fields

[A. B. Zamolodchikov '85]
[V. A. Fateev and A. B. Zamolodchikov '87]
[V. A. Fateev and S. L. Lukyanov '88]

For the case of $sl(n)$ it in can be represented in terms of the vector of $n-1$ independent bosonic currents $\vec{J} = (J_{(1)}, J_{(2)}, \ldots, J_{(n-1)})$

$$J_{(k)}(x) = \partial_x \phi_{(k)}(x) = \sum_{m=-\infty}^{\infty} J_m^{(k)} x^{-m-1}, \qquad \left[J_m^{(k)}, J_n^{(l)}\right] = m\, \delta_{k,l}\, \delta_{m,-n}$$

and is generated by

$$R_n(u) = -\,{}^*_*\prod_{m=1}^{n} (u - \vec{h}_m \cdot \vec{J})\,{}^*_*$$

Here the $\vec{h}_m$'s are the weight vectors of the fundamental representation of $sl(n)$.

In particular, for $n = 3$, the $W^{(3)}$ algebra is generated by

$$R_3(u) = -{}^*_* \prod_{m=1}^{3} (u - \vec{h}_m \vec{J}) {}^*_* = -u^3 - u\, {}^*_* \prod_{i<j} (\vec{h}_i \cdot \vec{J})(\vec{h}_j \cdot \vec{J}) {}^*_* + {}^*_* \prod_i \vec{h}_i \cdot \vec{J} {}^*_*$$
$$= -u^3 + u\, \mathcal{L}(x) + \mathcal{M}(x)$$

Then

$$\mathcal{L}(x) = \sum_{k=\infty}^{\infty} \frac{\mathcal{L}_k}{x^{k+2}} = \frac{1}{2} \left({}^*_* J_{(1)}(x)^2 + J_{(2)}(x)^2 {}^*_* \right),$$
$$\mathcal{M}(x) = \sum_{k=-\infty}^{\infty} \frac{\mathcal{M}_k}{x^{k+3}} := \frac{1}{\sqrt{6}} \left({}^*_* J_{(1)}(x)^2 J_{(2)}(x) - \frac{1}{3} J_{(2)}(x)^3 {}^*_* \right)$$

generate $W^{(3)}$ algebra with $c = 2$.

Let us introduce two bosonic currents

$$\widehat{J}_e(x) = \sum_{k=0}^{\infty} \left(\sqrt{\frac{2}{3}} k\, \tilde{t}_{2k} x^{k-1} + \sqrt{\frac{3}{2}} \frac{1}{x^{k+1}} \frac{\partial}{\partial t_{2k}} \right) + \sqrt{\frac{3}{2}} \frac{n}{x},$$

$$\widehat{J}_o(x) = \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \left((2k+1) \tilde{t}_{2k+1} x^{k-\frac{1}{2}} + \frac{1}{x^{k+\frac{3}{2}}} \frac{\partial}{\partial t_{2k+1}} \right)$$

with the dilaton shift

$$\tilde{t}_k = t_k - \frac{\delta_{k,3}}{3}$$

We see that the odd current $\widehat{J}_o(z)$ is the same as the current from the description of the Kontsevich-Witten tau-function and $\widehat{J}_e(z)$ (up to trivial rescailing of the times) is the untwisted current.

Theorem: **[A.A, '16]**

$$\widehat{\mathcal{L}}^{(n)}(x) = \sum_{k=-\infty}^{\infty} \frac{\widehat{\mathcal{L}}_k^{(n)}}{x^{k+2}} = \frac{1}{2}\left({}^*_* \widehat{J}_o(x)^2 + \frac{1}{8x^2} + \widehat{J}_e(x)^2 {}^*_* \right),$$

$$\widehat{\mathcal{M}}^{(n)}(x) = \sum_{k=-\infty}^{\infty} \frac{\widehat{\mathcal{M}}_k^{(n)}}{x^{k+3}} := \frac{1}{\sqrt{6}}\left({}^*_* \widehat{J}_e(x)\left(\widehat{J}_o(x)^2 + \frac{1}{8x^2}\right) - \frac{1}{3}\widehat{J}_e(x)^3 {}^*_* \right)$$

generate a representation of the $W^{(3)}$ algebra with central charge $c = 2$

$$\left[\widehat{\mathcal{L}}_k^{(n)}, \widehat{\mathcal{L}}_m^{(n)}\right] = (k-m)\widehat{\mathcal{L}}_{k+m}^{(n)} + \frac{1}{6}k(k^2-1)\delta_{k,-m},$$

$$\left[\widehat{\mathcal{L}}_k^{(n)}, \widehat{\mathcal{M}}_m^{(n)}\right] = (2k+m)\widehat{\mathcal{M}}_{k+m}^{(n)}$$

and

$$\left(\widehat{\mathcal{L}}^{(n)}(x)\right)_- \tau_n = 0$$

$$\left(\widehat{\mathcal{M}}^{(n)}(x)\right)_- \tau_n = 0$$

Topological expansion:

$$\tau_n(\mathbf{t};\hbar) = \exp\left(\sum_{\chi<0} \hbar^{-\chi} F_n^{(\chi)}(\mathbf{t})\right) = 1 + \sum_{k=1}^{\infty} \hbar^k \tau_n^{(k)}(\mathbf{t})$$

where

$$\chi = 2 - 2\#\text{handles} - \#\text{boundaries} - \#\text{points}$$

$\tau_n(\mathbf{t};\hbar)$ satisfies the cut-and-join type equation

$$\hbar \frac{\partial}{\partial \hbar} \tau_n(\mathbf{t},\hbar) = \left(\hbar \widehat{W}_1 + \hbar^2 \widehat{W}_2\right) \tau_n(\mathbf{t},\hbar)$$

so that $\tau_n^{(k)}$ are uniquely defined by a recursion

$$\tau_n^{(k)} = \frac{1}{k}\left(\widehat{W}_1\, \tau_n^{(k-1)} + \widehat{W}_2\, \tau_n^{(k-2)}\right)$$

with the initial conditions $\tau_n^{(0)} = 1$, $\tau_n^{(-1)} = 0$.

Operators $\widehat{\mathrm{W}}_1$ and $\widehat{\mathrm{W}}_2$ are not unique.

$$v_o(x) = \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} (2k+1) x^{k-\frac{1}{2}} t_{2k+1}$$
$$v_e(x) = \sqrt{\frac{2}{3}} \sum_{k=1}^{\infty} k\, x^{k-1} t_{2k}$$

$$\widehat{W}_1 = \frac{1}{\sqrt{2}} \frac{1}{2\pi i} \oint {}_*^* \frac{v_o(x)}{2} \left(\widehat{J}_o'(x)^2 + \frac{1}{8x^2} + \widehat{J}_e(x)^2 \right) + \frac{2 v_e(x)}{\sqrt{3}} \widehat{J}_o'(x) \widehat{J}_e(x) {}_*^* \frac{dx}{\sqrt{x}},$$
$$\widehat{W}_2 = -\frac{1}{3} \frac{1}{2\pi i} \oint v_e(x) \left({}_*^* \widehat{J}_e(x) \left(\widehat{J}_o'(x)^2 + \frac{1}{8x^2} \right) - \frac{1}{3} \widehat{J}_e(x)^3 {}_*^* \right) \frac{dx}{x^2}$$

Not from $W_{1+\infty}$!

$$
\begin{aligned}
\mathcal{F}_n(\mathbf{t}) = {} & \left(\frac{1}{8} + \frac{3}{2}\, n^2\right) t_3 + \frac{1}{6}\, {t_1}^3 + 2n\, t_1 t_2 + 6\, n t_1 t_2 t_3 + 4n \left(1 + n^2\right) t_6 \\
& + \frac{4}{3}\, n {t_2}^3 + \left(\frac{9}{4}\, n^2 + \frac{3}{16}\right) {t_3}^2 + \frac{1}{2}\, t_3 {t_1}^3 + 8\, n^2 t_2 t_4 + 4\, {t_1}^2 n t_4 + \left(\frac{15}{2}\, n^2 + \frac{5}{8}\right) t_1 t_5 \\
& + 8\, n {t_2}^3 t_3 + 15 \left(3\, n^2 + \frac{1}{4}\right) t_1 t_3 t_5 + 24\, n {t_1}^2 t_3 t_4 + 30\, n^2 {t_2}^2 t_5 + \frac{105}{8} \left(\frac{1}{16} + \frac{7}{2}\, n^2 + n^4\right) t_9 \\
& + 35 \left(n^3 + \frac{3}{4}\, n\right) t_7 t_2 + 35 \left(\frac{1}{16} + \frac{3}{4}\, n^2\right) t_7 {t_1}^2 + 32\, n \left(n^2 + 1\right) t_8 t_1 + 32 n^2\, t_1 {t_4}^2 \\
& + 48\, n^2 t_2 t_3 t_4 + 18\, n t_1 t_2 {t_3}^2 + 20\, n \left(1 + 2\, n^2\right) t_5 t_4 + 24 n \left(n^2 + 1\right) t_6 t_3 + 8\, n {t_1}^3 t_6 \\
& + \frac{3}{2}\, {t_1}^3 {t_3}^2 + 48\, n^2 t_1 t_2 t_6 + 16\, n t_1 {t_2}^2 t_4 + \frac{5}{8}\, {t_1}^4 t_5 + \left(\frac{9}{2}\, n^2 + \frac{3}{8}\right) {t_3}^3 + 15\, n {t_1}^2 t_2 t_5 + \dots
\end{aligned}
$$

Table of Contents

A complete analog of the Kontsevich–Witten description for open case.

Open and closed models are of the similar complexity: **simple!**

n	0	1	arbitrary
Intersection numbers	Closed	Open	**Refined Open**
Integrable hierarchy	KdV	KP	MKP
Algebra of constraints	Heisenberg+ Virasoro	Virasoro + $W^{(3)}$	Virasoro + $W^{(3)}$
Specified by	String	String+Dilaton	String+Dilaton
Cut-and-join operator	$e^{W_{KW}} \cdot 1$	"$e^{W_1+W_2/2}$" $\cdot 1$	"$e^{W_1+W_2/2}$" $\cdot 1$

Open questions

- Geometrical construction of the descendants on the boundary and general n cases.
- Open Hodge integrals, κ-classes
- Open topological string models for more complicated target spaces (CP^1).
- Open version of Topological recursion/Givental theory. Frobenius structure?
- Relation to CFT?