# From Phase Space to Integrable Representations and Level-Rank Duality

Arghya Chattopadhyay

Indian Institute of Science Education and Research, Bhopal

June 19, 2018

# Matrix Models: Quick Overview

- A Random Matrix Model(UMM) is characterised by a matrix ensemble $E$, and a probability measure $d\mu(M)$.

$$\mathcal{Z} = \int_E d\mu(M)$$

- We are interested in the form

$$\mathcal{Z} = \int_E dM e^{-\mathrm{Tr}V(M)}$$

with $dM = \prod_{i=1}^{N} dM_{i,i} \prod_{1\leq i<j\leq N} d\mathrm{Re}M_{i,j}\, d\mathrm{Im}M_{i,j}$ and $V$ is called the potential.

# Matrix Models: Quick Overview

- A Random Matrix Model(UMM) is characterised by a matrix ensemble $E$, and a probability measure $d\mu(M)$.

$$\mathcal{Z} = \int_E d\mu(M)$$

- We are interested in the form

$$\mathcal{Z} = \int_E dM e^{-\mathrm{Tr} V(M)}$$

with $dM = \prod_{i=1}^{N} dM_{i,i} \prod_{1\le i<j\le N} d\mathrm{Re}M_{i,j}\ d\mathrm{Im}M_{i,j}$ and $V$ is called the potential.

- If $M$ is a Hermitian $N \times N$ matrix then the partition function can be re-written as

$$\mathcal{Z} = \int \prod_{i=1}^{N} d\lambda_i \prod_{i<j}^{N} (\lambda_i - \lambda_j)^2 e^{-\frac{1}{g_s}\sum_{i=1}^{N} W(\lambda_i)}$$

- Or equivalently $\mathcal{Z} = \int \prod_{i=1}^{N} d\lambda_i\ e^{-\frac{1}{g_s} W_{eff}(\lambda)}$, with $W_{eff}(\lambda_i) = W(\lambda_i) - \frac{N g_s}{N} \sum_{j\neq i}^{N} \ln|\lambda_i - \lambda_j|$

# Unitary Matrix Models: Overview

# Unitary Matrix Models: Overview

- A Unitary Matrix Model(UMM) is a statistical ensemble of unitary matrices $(U)$ defined by the partition function

$$\mathcal{Z} = \int [\mathcal{D}U] \exp[-V(U)] \; ; \quad \text{with } V(U) \to \text{potential function;} \quad \mathcal{D}U \to \text{Haar measure.}$$

- Going to a diagonal basis $U = \{e^{i\theta_i}\}$, for $i \in [1, N]$. In this basis the Haar measure takes the form

$$\int [\mathcal{D}U] = \prod_{i=1}^{N} \int_{-\pi}^{\pi} d\theta_i \prod_{i<j} \sin^2\left(\frac{\theta_i - \theta_j}{2}\right)$$

- Therefore $V_{eff}(\{\theta_i\}) = V(\{\theta_i\}) - \sum_{i<j}^{N} \ln \sin^2\left(\frac{\theta_i - \theta_j}{2}\right)$

## Unitary Matrix Models: Overview

- A Unitary Matrix Model(UMM) is a statistical ensemble of unitary matrices $(U)$ defined by the partition function

$$\mathcal{Z} = \int [\mathcal{D}U] \exp[-V(U)] \; ; \quad \text{with } V(U) \to \text{potential function;} \quad \mathcal{D}U \to \text{Haar measure.}$$

- Going to a diagonal basis $U = \{e^{i\theta_i}\}$, for $i \in [1, N]$. In this basis the Haar measure takes the form

$$\int [\mathcal{D}U] = \prod_{i=1}^{N} \int_{-\pi}^{\pi} d\theta_i \prod_{i<j} \sin^2\left(\frac{\theta_i - \theta_j}{2}\right)$$

- Therefore $V_{eff}(\{\theta_i\}) = V(\{\theta_i\}) - \sum_{i<j}^{N} \ln\sin^2\left(\frac{\theta_i - \theta_j}{2}\right)$

- The thermal partition function of a YM theory on $S^p$ is given by the Euclidean path integral on $S^p \times S^1$. Integrating out all the massive modes this path integral reduces to an integral

$$\mathcal{Z}_{YM} = \int [\mathcal{D}U] \exp\left[-V_{YM}(U)\right] = \prod_{i=1}^{N} \int_{-\pi}^{\pi} d\theta_i \prod_{i<j} \sin^2\left(\frac{\theta_i - \theta_j}{2}\right) e^{-V_{YM}(U)}$$

where $U = e^{i\beta\alpha}$ and $\alpha = \frac{1}{V_p}\int_{S^p} A_0$.

## Unitary Matrix Models: A Phase Space Formulation

## Unitary Matrix Models: A Phase Space Formulation

- Solution of UMMs in large $N$ limit renders a distribution of eigenvalues $\rho(\{\theta_i\})$ of Unitary matrices for different phases.
- Different phases of the solution is characterised by number of *cuts* on the unit circle.

# Unitary Matrix Models: A Phase Space Formulation

- Solution of UMMs in large $N$ limit renders a distribution of eigenvalues $\rho(\{\theta_i\})$ of Unitary matrices for different phases.
- Different phases of the solution is characterised by number of *cuts* on the unit circle.
- Phases can also be equivalently captured by Young diagrams corresponding to different representations of $U(N)$. Therefore one can associate one additional information of box number distribution $\rightarrow u(\{h_i\})$.
- UMM also has an interpretation in the language of free fermions where eigenvalues behaves like position of the free fermions and the number of boxes in Young diagram behaving like momentum.

$$\rho(\{\theta_i\}) \rightarrow \text{Position distribution}$$
$$u(\{h_i\}) \rightarrow \text{Momentum distribution}$$

- Thus a relation between them defines a Fermi surface (phase space distribution/ droplet) in two dimensions for different phases of the UMM under consideration.
- Different phases are distinguished by different shapes of fermi surfaces/droplets.

# Why phase space description?

## Goal of our programme

- Phase space/Young diagrams provide a useful labelling of different operators in the gauge theory.
- According to AdS/CFT correspondence, these operators have a description in a dual bulk geometry.
- Goal is to understand how dual string geometry emerges from phase droplets in spirit of LLM.

## Goal of this talk

- To provide a phase space description for different phases of CS-matter theory on $S^2 \times S^1$.
- To show how phase space construction imposes constraints on large $N$ representations.
- Level-rank duality in phase space.

# Outline

# Chern-Simons matter systems: The Matrix Model

# Chern-Simons matter systems: The Matrix Model

- Consider CS theory on $S^2 \times S^1$ interacting with matter in fundamental representations. Partition function of this theory is

$$\mathcal{Z} = \int [\mathcal{D}A][\mathcal{D}\mu] e^{i\frac{k}{4\pi}\mathrm{Tr}\int (AdA+\frac{2}{3}A^3) - S_{matter}} \qquad \mathcal{D}\mu \rightarrow \text{the matter field measure}$$

- Result of Integrating out the matter fields [S. Jain et al, JHEP09(2013)009]

$$\mathcal{Z} = \int [\mathcal{D}A] e^{i\frac{k}{4\pi}\mathrm{Tr}\int (AdA+\frac{2}{3}A^3) - S_{eff}(U(x))}$$

where $U(x) \rightarrow$ 2 dimensional holonomy fields around the thermal circle $S^1$.

- The thermal partition function for Chern-Simons matter system on $S^2 \times S^1$ can be written as a discretized sum over the holonomy matrix $U$ as

$$\mathcal{Z}_{\mathrm{CS}} = \prod_{m=1}^{N} \sum_{n_m=-\infty}^{\infty} \left[ \prod_{l\neq m} 2\sin\left(\frac{\theta_l(\vec{n}) - \theta_m(\vec{n})}{2}\right) e^{-V(U)} \right]$$

$$V(U) = T^2 V_2\, v(U); \;\; \theta_m(\vec{n}) = \frac{2\pi n_m}{k}, \;\; n_m \in \mathbb{Z}$$

- Discretization interval between two allowed eigenvalues is $\dfrac{2\pi}{k}$.

# Chern-Simons matter systems: Eigenvalue distribution

- Discretization interval of $\frac{2\pi}{k}$ implies that the number of eigenvalues between $\theta$ and $\theta + \triangle\theta$ is

$$\triangle x \leq \frac{\triangle\theta}{2\pi/k}$$

.

## Chern-Simons matter systems: Eigenvalue distribution

- Discretization interval of $\frac{2\pi}{k}$ implies that the number of eigenvalues between $\theta$ and $\theta + \triangle\theta$ is

$$\triangle x \leq \frac{\triangle\theta}{2\pi/k}$$

.

- Eigenvalue density in large $N$ limit

$$\rho(\theta) = \lim_{\triangle x \to 0} \frac{1}{N} \frac{\triangle x}{\triangle \theta}$$

has a maximum value

$$\boxed{0 \leq \rho(\theta) \leq \frac{1}{2\pi\lambda}; \qquad \lambda = \frac{N}{k}}$$

# Chern-Simons matter systems: Gross-Witten model

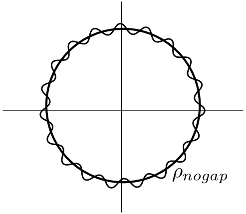
- We study CS matter theory on $S^2 \times S^1$ with

$$V(U) = -N\beta_1(\text{Tr}U + \text{Tr}U^\dagger) \quad \rightarrowtail \text{(Gross-Witten potential)}$$

# Chern-Simons matter systems: Gross-Witten model

- GWW model exhibits two phases when there is no restriction on its spectral density.
- For $\beta_1 < 1/2$ The eigenvalues are distributed between $(-\pi, \pi]$ (no gaps in the solution).

$$\rho(\theta)_{ng} = \frac{1}{2\pi}(1 + 2\beta_1 \cos\theta).$$

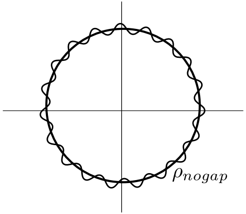


# Chern-Simons matter systems: Gross-Witten model

- GWW model exhibits two phases when there is no restriction on its spectral density.
- For $\beta_1 < 1/2$ The eigenvalues are distributed between $(-\pi, \pi]$ (no gaps in the solution).

$$\rho(\theta)_{ng} = \frac{1}{2\pi}(1 + 2\beta_1 \cos\theta).$$

No Gap solution

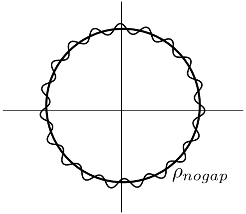


# Chern-Simons matter systems: Gross-Witten model

- GWW model exhibits two phases when there is no restriction on its spectral density.
- For $\beta_1 < 1/2$ The eigenvalues are distributed between $(-\pi, \pi]$ (no gaps in the solution).

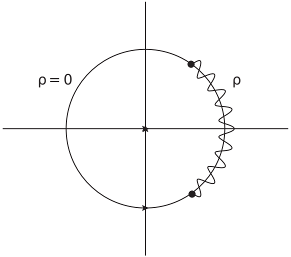
$$\rho(\theta)_{ng} = \frac{1}{2\pi}(1 + 2\beta_1 \cos\theta).$$

No Gap solution




- As the repulsion coming from the Haar measure is constant, the system undergoes a third order phase transition at $\beta_1 = 1/2$. For $\beta_1 = 1/2$ the eigenvalue density

$$\rho(\theta)_{lg} = \frac{2\beta_1}{\pi}\sqrt{\frac{1}{2\beta_1} - \sin^2\frac{\theta}{2}}\ \cos\frac{\theta}{2}; \text{ for } \sin^2\frac{\theta}{2} < \frac{1}{2\beta_1}$$
$$= 0. \quad \text{for } \sin^2\frac{\theta}{2} > \frac{1}{2\beta_1}$$

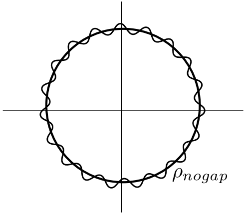


# Chern-Simons matter systems: Gross-Witten model

- GWW model exhibits two phases when there is no restriction on its spectral density.
- For $\beta_1 < 1/2$ The eigenvalues are distributed between $(-\pi, \pi]$ (no gaps in the solution).

$$\rho(\theta)_{ng} = \frac{1}{2\pi}(1 + 2\beta_1 \cos\theta).$$

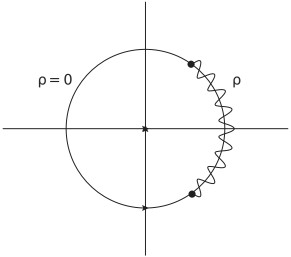
No Gap solution




- As the repulsion coming from the Haar measure is constant, the system undergoes a third order phase transition at $\beta_1 = 1/2$. For $\beta_1 = 1/2$ the eigenvalue density

$$\rho(\theta)_{lg} = \frac{2\beta_1}{\pi}\sqrt{\frac{1}{2\beta_1} - \sin^2\frac{\theta}{2}}\ \cos\frac{\theta}{2};\ \text{for}\ \sin^2\frac{\theta}{2} < \frac{1}{2\beta_1}$$
$$= 0. \quad \text{for}\ \sin^2\frac{\theta}{2} > \frac{1}{2\beta_1}$$

One Gap solution



## Chern-Simons matter systems: Gross-Witten model

## Chern-Simons matter systems: Gross-Witten model

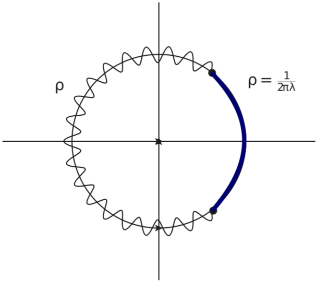
- Spectral density have an upper bound $0 \leq \rho(\theta) \leq \frac{1}{2\pi\lambda}$
- No gap phase is allowed if
  - $\beta_1 < \frac{1}{2\lambda} - \frac{1}{2}$ for $\lambda > \frac{1}{2}$.
  - $\beta_1 < \frac{1}{2}$ for $\lambda < \frac{1}{2}$.
- Similarly the lower gap phase exists in the parameter range
  - $\beta_1 < \frac{1}{8\lambda^2}$ and $\lambda \leq \frac{1}{2}$
  - For $\lambda > \frac{1}{2}$ the solution does not exists.
- Capping the allowed value of $\rho(\theta)$ introduces two new phases in the system.

# Chern-Simons matter systems: Gross-Witten model

## New Phase 1: Upper cap

$$\rho(\theta) = \frac{1}{2\pi\lambda} - 2\beta_1 \frac{|\sin\frac{\theta}{2}|}{\pi}\sqrt{\frac{\frac{1}{\lambda}-1}{2\beta_1} - \cos^2\frac{\theta}{2}}$$

$$= \frac{1}{2\pi\lambda}, \quad \text{for } \cos^2\frac{\theta}{2} > \frac{\frac{1}{\lambda}-1}{2\beta_1}.$$

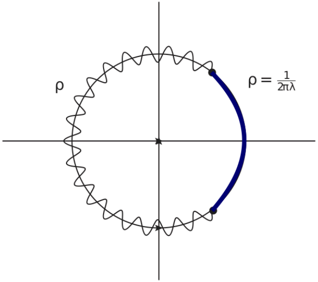


## Chern-Simons matter systems: Gross-Witten model

### New Phase 1: Upper cap

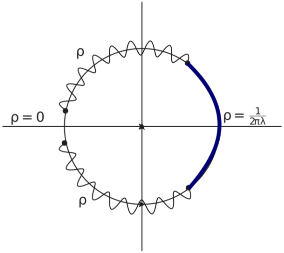
$$\rho(\theta) = \frac{1}{2\pi\lambda} - 2\beta_1 \frac{|\sin\frac{\theta}{2}|}{\pi}\sqrt{\frac{\frac{1}{\lambda}-1}{2\beta_1} - \cos^2\frac{\theta}{2}}$$

$$= \frac{1}{2\pi\lambda}, \quad \text{for } \cos^2\frac{\theta}{2} > \frac{\frac{1}{\lambda}-1}{2\beta_1}.$$




### New Phase 2: Upper cap with Lower Gap

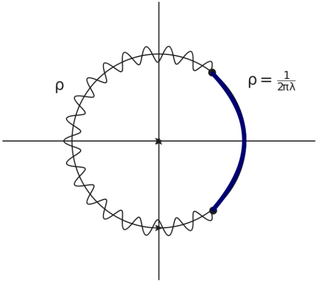
$$\rho(\theta) = \frac{|\sin\theta|}{4\pi^2\lambda}\sqrt{(\sin^2\frac{\theta}{2} - \sin^2\frac{a}{2})(\sin^2\frac{b}{2} - \sin^2\frac{\theta}{2})}\; I(\theta, a, b)$$



# Chern-Simons matter systems: Gross-Witten model

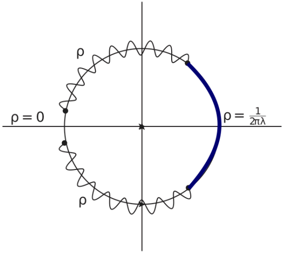
## New Phase 1: Upper cap

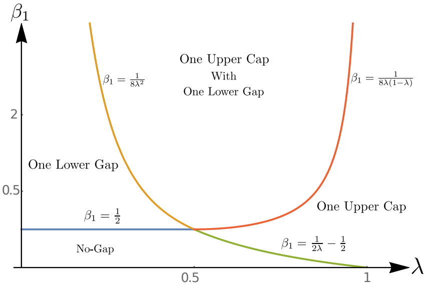
$$\rho(\theta) = \frac{1}{2\pi\lambda} - 2\beta_1 \frac{|\sin\frac{\theta}{2}|}{\pi}\sqrt{\frac{\frac{1}{\lambda}-1}{2\beta_1} - \cos^2\frac{\theta}{2}}$$
$$= \frac{1}{2\pi\lambda}, \quad \text{for } \cos^2\frac{\theta}{2} > \frac{\frac{1}{\lambda}-1}{2\beta_1}.$$




## New Phase 2: Upper cap with Lower Gap

$$\rho(\theta) = \frac{|\sin\theta|}{4\pi^2\lambda}\sqrt{(\sin^2\frac{\theta}{2} - \sin^2\frac{a}{2})(\sin^2\frac{b}{2} - \sin^2\frac{\theta}{2})}\ I(\theta, a, b)$$



# Chern-Simons matter system: Level Rank Duality

- The level-rank duality in terms of renormalised level $k$ and rank $N$ is given by $N \to k - N$ and $k \to k$.

- Level rank duality maps Wilson loop in the representation labelled by the Young tableaux $Y$ to the Wilson loop in the representation labelled by the Young tableaux $\tilde{Y}$, where $Y$ and $\tilde{Y}$ are related by transposition.

- Under level-rank duality 't Hooft coupling constant transforms as

$$\lambda^D = \frac{k-N}{k} = 1 - \lambda,$$

- Demanding partition function is invariant under level-rank duality, the second constant in the system $\beta_1$ also transforms under level-rank duality as

$$\beta_1^D = \frac{\lambda}{1-\lambda}\beta_1$$

# Chern-Simons matter system: Level Rank Duality

- The level-rank duality in terms of renormalised level $k$ and rank $N$ is given by $N \to k - N$ and $k \to k$.

- Level rank duality maps Wilson loop in the representation labelled by the Young tableaux $Y$ to the Wilson loop in the representation labelled by the Young tableaux $\tilde{Y}$, where $Y$ and $\tilde{Y}$ are related by transposition.

- Under level-rank duality 't Hooft coupling constant transforms as

$$\lambda^D = \frac{k - N}{k} = 1 - \lambda,$$

- Demanding partition function is invariant under level-rank duality, the second constant in the system $\beta_1$ also transforms under level-rank duality as

$$\beta_1^D = \frac{\lambda}{1 - \lambda}\beta_1$$

- Under this duality the eigenvalues of lower-gap and upper-cap phase are related by

$$\rho_{uc}(\lambda^D, \beta_1^D, \theta) = \frac{\lambda}{1 - \lambda}\left[\frac{1}{2\pi\lambda} - \rho_{lg}(\lambda, \beta_1, \theta + \pi)\right]$$

# From Partition function to Young Diagram

# From Partition function to Young Diagram

- Partition function

$$\mathcal{Z} = \int [\mathcal{D}U] \exp\left[N \sum_{n=1}^{\infty} \frac{\beta_n}{n} (\text{Tr}U^n + \text{Tr}U^{\dagger n})\right]$$

- Expanding the exponential

$$\mathcal{Z} = \int [\mathcal{D}U] \sum_{\vec{k}} \frac{\epsilon(\vec{\beta}, \vec{k})}{z_{\vec{k}}} \Upsilon_{\vec{k}}(U) \sum_{\vec{l}} \frac{\epsilon(\vec{\beta}, \vec{l})}{z_{\vec{l}}} \Upsilon_{\vec{l}}(U)$$

where

$$\epsilon(\vec{\beta}, \vec{k}) = \prod_{n=1}^{\infty} N^{k_n} \beta_n^{k_n}, \quad z_{\vec{k}} = \prod_{n=1}^{\infty} k_n! n^{k_n}, \text{ and } \Upsilon_{\vec{k}} = \prod_n (\text{Tr}U^n)^{k_n}.$$

- Here, $n$ runs from over positive integers and $\vec{k} = (k_1, k_2, \cdots)$. $k_n$ can be $0$ or any positive integer.

## From Partition function to Young Diagram

- $\Upsilon_{\vec{k}}(U)$ can be rewritten in terms of characters of the conjugacy class of the permutation group $S_k$ using the Unitary and Symmetric group duality

$$\Upsilon_{\vec{k}}(U) = \sum_R \chi_R(C(\vec{k}))\mathsf{Tr}_R[U]$$

- Using the orthogonality relation between characters of representation of $U(N)$

$$\mathcal{Z} = \sum_R \sum_{\vec{k}} \frac{\varepsilon(\vec{\beta},\vec{k})}{z_{\vec{k}}} \sum_{\vec{l}} \frac{\varepsilon(\vec{\beta},\vec{l})}{z_{\vec{l}}} \chi_R(C(\vec{k}))\chi_R(C(\vec{l})).$$

- Sum over $R$ can be decomposed as

$$\sum_R \longrightarrow \sum_{K=1}^{\infty} \sum_{\{\lambda_i\}} \delta\left(\sum_{i=1}^{N} \lambda_i - K\right) \quad \text{with} \quad K = \sum_\alpha \alpha k_\alpha; \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0.$$

# From Partition function to Young Diagram

- $\Upsilon_{\vec{k}}(U)$ can be rewritten in terms of characters of the conjugacy class of the permutation group $S_k$ using the Unitary and Symmetric group duality

$$\Upsilon_{\vec{k}}(U) = \sum_R \chi_R(C(\vec{k}))\mathrm{Tr}_R[U]$$

- Using the orthogonality relation between characters of representation of $U(N)$

$$\mathcal{Z} = \sum_R \sum_{\vec{k}} \frac{\varepsilon(\vec{\beta},\vec{k})}{z_{\vec{k}}} \sum_{\vec{l}} \frac{\varepsilon(\vec{\beta},\vec{l})}{z_{\vec{l}}} \chi_R(C(\vec{k}))\chi_R(C(\vec{l})).$$

- Sum over $R$ can be decomposed as

$$\sum_R \longrightarrow \sum_{K=1}^{\infty} \sum_{\{\lambda_i\}} \delta\left(\sum_{i=1}^{N} \lambda_i - K\right) \quad \text{with} \quad K = \sum_\alpha \alpha k_\alpha; \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0.$$

- Hence the partition function

$$\mathcal{Z} = \sum_{\vec{\lambda}} \sum_{\vec{k},\vec{l}} \frac{\varepsilon(\vec{\beta},\vec{k})\varepsilon(\vec{\beta},\vec{l})}{z_{\vec{k}} z_{\vec{l}}} \chi_{\vec{\lambda}}(C(\vec{k}))\chi_{\vec{\lambda}}(C(\vec{l}))\, \delta\left(\sum_n n k_n - \sum_i \lambda_i\right) \delta\left(\sum_n n l_n - \sum_i \lambda_i\right).$$

# From Partition function to Young Diagram

- In the large $N$ limit one can in principle extremize the partition function w.r.t $\lambda_i$'s and find dominant representations.
- Characters are difficult to find.
- For simpler model (Gross-Witten model) where $\beta_1 \neq 0$ and $\beta_{n>1} = 0$, we are only left with the trivial conjugacy class, for which character is just the dimension of the representation

$$\chi_{\vec{h}}(C_{k_1}) = \frac{k_1!}{\prod_{i=1}^{N} h_i!} \prod_{i<j} (h_i - h_j) \text{ where } h_i = \lambda_i + N - i$$

- Defining the young tableaux density in the Large $N$ limit

$$u(h) = -\frac{\partial x}{\partial h}, \qquad \frac{h_i}{N} = h(x), \qquad x = \frac{i}{N} \quad \text{with} \quad x \in [0,1].$$

- In the large $N$ limit the saddle point equation is given by

$$⨍_{h_L}^{h_U} dh' \frac{u(h')}{h - h'} = \ln\left[\frac{h}{\beta_1}\right].$$

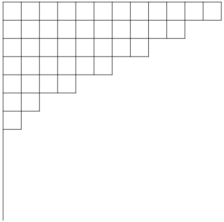
Now the task is to find $u(h)$ for the most dominant representation.

# Young Distribution for GW potential

# Young Distribution for GW potential : Uncapped case

[S. Dutta, R.Gopakumar, JHEP0803:011,2008]

- for $\beta_1 < 1/2$, $u(h)$ saturates the maximum value in a finite range of $h$



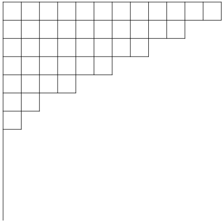
$$\begin{aligned} u(h) &= 1, & 0 \leq h \leq 1 - 2\beta_1. \\ &= \frac{1}{\pi}\cos^{-1}\left[\frac{h-1}{2\beta_1}\right], & 1 - 2\beta_1 \leq h \leq 1 + 2\beta_1 \end{aligned}$$

# Young Distribution for GW potential : Uncapped case

[S. Dutta, R.Gopakumar, JHEP0803:011,2008]

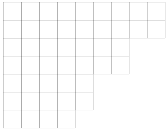
- for $\beta_1 < 1/2$, $u(h)$ saturates the maximum value in a finite range of $h$

$$u(h) = 1, \qquad 0 \le h \le 1 - 2\beta_1.$$
$$= \frac{1}{\pi}\cos^{-1}\left[\frac{h-1}{2\beta_1}\right], \quad 1 - 2\beta_1 \le h \le 1 + 2\beta_1$$



- for $\beta_1 > 1/2$, $u(h)$ never saturates the upper bound

$$u(h) = \frac{2}{\pi}\cos^{-1}\left[\frac{h + \beta_1 - 1/2}{2\sqrt{\beta_1 h}}\right]$$
$$\beta_1 + \frac{1}{2} - \sqrt{2\beta_1} \le \ h \ \le \beta_1 + \frac{1}{2} + \sqrt{2\beta_1}$$



- There exists a Douglas-Kazakov type phase transition between these saddle points as one varies the parameter $\beta_1$.

# Relation Between the Eigenvalue and Young Tableaux side

**[(S. Dutta, R.Gopakumar, 2008 ),(S. Dutta, P. Dutta 2016), (AC, S. Dutta and P. Dutta,2017)]**

# Relation Between the Eigenvalue and Young Tableaux side

**[(S. Dutta, R.Gopakumar, 2008 ),(S. Dutta, P. Dutta 2016), (AC, S. Dutta and P. Dutta,2017)]**

- Inverting the no-gap distribution

$$h = 1 + 2\beta_1 \cos(\pi u(h))$$

- Eigenvalue distribution of this phase

$$2\pi\rho(\theta) = 1 + 2\beta_1 \cos(\theta)$$

- Thus we have the identification

$$u(h) = \frac{\theta}{\pi}, \quad \rho(\theta) = \frac{h}{2\pi}$$

- For one gap phase we have

$$h^2 - (1 + 2\beta_1 \cos \pi u(h))h + (\beta_1 - \frac{1}{2})^2 = 0$$

- We have the following Identity

$$\boxed{\pi u(h) = \theta, \;\; h_+ - h_- = 2\pi\rho(\theta)}$$

$h_+$ and $h_-$ are the solutions of the above equation.

# Relation Between the Eigenvalue and Young Tableaux side

**[(S. Dutta, R.Gopakumar, 2008 ),(S. Dutta, P. Dutta 2016), (AC, S. Dutta and P. Dutta,2017)]**

- This allows one to write both the eigenvalue and the young tableaux distribution in terms of a single constant phase space distribution function $\omega(h,\theta)$ such that

$$\rho(\theta) = \int_0^\infty \omega(h,\theta)dh, \qquad u(h) = \int_{-\pi}^{\pi} \omega(h,\theta)d\theta$$

# Relation Between the Eigenvalue and Young Tableaux side

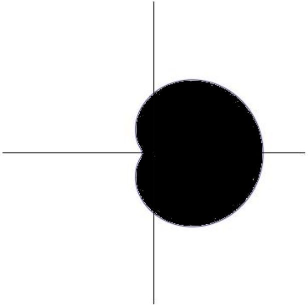
**[(S. Dutta, R.Gopakumar, 2008 ),(S. Dutta, P. Dutta 2016), (AC, S. Dutta and P. Dutta,2017)]**

- This allows one to write both the eigenvalue and the young tableaux distribution in terms of a single constant phase space distribution function $\omega(h,\theta)$ such that

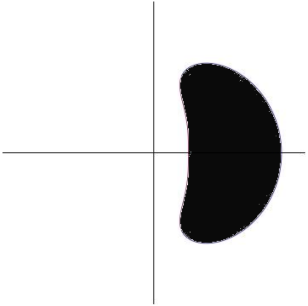
$$\rho(\theta) = \int_0^\infty \omega(h,\theta)dh, \qquad u(h) = \int_{-\pi}^{\pi} \omega(h,\theta)d\theta$$

$\omega(h,\theta)$ is a distribution in two dimensions,

$$\omega(h,\theta) = \frac{1}{2\pi}\Theta(h - h_-(\theta))\Theta(h_+(\theta) - h).$$



**(a)** Phase space distribution for no-gap phase.



**(b)** Phase space distribution for one-gap phase.

# Constraint on Young diagrams for Chern-Simons matter theory

**[(AC, S. Dutta and P. Dutta,2018)]**

# Constraint on Young diagrams for Chern-Simons matter theory

**[(AC, S. Dutta and P. Dutta,2018)]**

$$\rho(\theta) = \frac{h_+ - h_-}{2\pi} \leq \frac{1}{2\pi\lambda}$$

# Constraint on Young diagrams for Chern-Simons matter theory

**[(AC, S. Dutta and P. Dutta,2018)]**

$$\rho(\theta) = \frac{h_+ - h_-}{2\pi} \leq \frac{1}{2\pi\lambda}$$

- We see that the eigenvalue density is related to spread (support) of $u(h)$ or the width of the Young distribution

Claim: For CS-matter theory, dominant reps have a distribution function with max spread $\frac{1}{\lambda}$.

# Constraint on Young diagrams for Chern-Simons matter theory

**[(AC, S. Dutta and P. Dutta,2018)]**

**What does it mean??**

- Using the relation between number of boxes $n_i$ and $h_i$ : $h_i = n_i + N - 1$, we see that upper cap phase corresponds to those diagrams for which

$$h_1 < k \text{ which implies } h(0) < \frac{1}{\lambda} \quad (h(0) = h_1/N)$$

- Integrable representation:An integrable representation of $SU(N)_k$ is characterised by an Young diagram which has maximum $k - N$ number of boxes in the first row.

# Constraint on Young diagrams for Chern-Simons matter theory

**[(AC, S. Dutta and P. Dutta,2018)]**

**What does it mean??**

- Using the relation between number of boxes $n_i$ and $h_i$ : $h_i = n_i + N - 1$, we see that upper cap phase corresponds to those diagrams for which

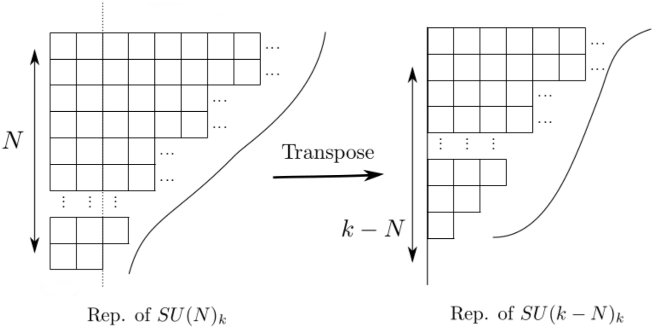
$$h_1 < k \text{ which implies } h(0) < \frac{1}{\lambda} \quad (h(0) = h_1/N)$$

- Integrable representation:An integrable representation of $SU(N)_k$ is characterised by an Young diagram which has maximum $k - N$ number of boxes in the first row.

- Thus from phase space identification we see that our claim implies that putting a cap on eigenvalue distribution constraints the corresponding representation to be integrable.

# Level Rank duality and our claim

- We use the level rank duality to find the dominant representation for the upper cap, and verify our claim.
- Level rank duality maps Wilson loop in the representation $Y$ to the Wilson loop in the representation $\tilde{Y}$, where $Y$ and $\tilde{Y}$ are related by transposition.
- Lower gap is mapped to upper cap via level rank duality. **[S. Jain et al, JHEP09(2013)009]**
- The dominant representations for upper cap can be obtained by transposing the dominant representations for the lower gap.




$$\begin{aligned} u(h) &= 1 \qquad 0 < h < p \\ &= 0 \qquad p < h < q \\ &= \tilde{u}(h) \qquad q < h < r \end{aligned}$$

# Representation dual to lower gap

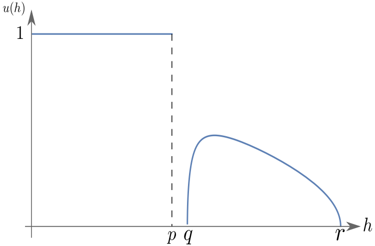
**[(AC, S. Dutta and P. Dutta,2018)]**

- Solving the saddle point equation with this ansatz we have

$$\boxed{\tilde{u}(h) = 1 - \frac{1}{\pi}\cos^{-1}\left[1 - \frac{(h-q)(r-h)}{2\beta_1(h-p)}\right]}$$

where,

$$\begin{aligned} p &= 2 - \frac{1}{\lambda} \\ q &= \beta_1 - \frac{1}{2\lambda} + \frac{3}{2} - \sqrt{\frac{2\,\beta_1}{\lambda}(1-\lambda)}, \\ r &= \beta_1 - \frac{1}{2\lambda} + \frac{3}{2} + \sqrt{\frac{2\,\beta_1}{\lambda}(1-\lambda)}. \end{aligned}$$




- According to our claim $p < \frac{1}{\lambda}$ and $r - q < \frac{1}{\lambda}$. Imposing $p < q$, our claim is satisfied if

$$\lambda < 1, \quad \beta_1 < \frac{1}{8\lambda(1-\lambda)}, \quad \beta_1 > \frac{1}{2\lambda} - \frac{1}{2}$$

# Representation dual to lower gap

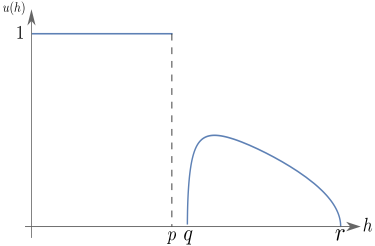
[(AC, S. Dutta and P. Dutta,2018)]

- Solving the saddle point equation with this ansatz we have

$$\tilde{u}(h) = 1 - \frac{1}{\pi}\cos^{-1}\left[1 - \frac{(h-q)(r-h)}{2\beta_1(h-p)}\right]$$

where,

$$\begin{aligned} p &= 2 - \frac{1}{\lambda} \\ q &= \beta_1 - \frac{1}{2\lambda} + \frac{3}{2} - \sqrt{\frac{2\,\beta_1}{\lambda}(1-\lambda)}, \\ r &= \beta_1 - \frac{1}{2\lambda} + \frac{3}{2} + \sqrt{\frac{2\,\beta_1}{\lambda}(1-\lambda)}. \end{aligned}$$




- According to our claim $p < \frac{1}{\lambda}$ and $r - q < \frac{1}{\lambda}$. Imposing $p < q$, our claim is satisfied if

$$\boxed{\lambda < 1, \quad \beta_1 < \frac{1}{8\lambda(1-\lambda)}, \quad \beta_1 > \frac{1}{2\lambda} - \frac{1}{2}}$$

# Level-Rank duality

- Level rank duality implies $\rightarrow \lambda^D = \frac{k-N}{k} = 1 - \lambda, \qquad \beta_1^D = \beta_1 \frac{\lambda}{1-\lambda}$
- As under the transposition total number of boxes is unchanged, thus

$$\boxed{h^D = \frac{\lambda}{1-\lambda}\left(h - 2 + \frac{1}{\lambda}\right)}$$

# Level-Rank duality

- Level rank duality implies $\rightarrow \lambda^D = \frac{k-N}{k} = 1 - \lambda, \qquad \beta_1^D = \beta_1 \frac{\lambda}{1-\lambda}$
- As under the transposition total number of boxes is unchanged, thus

$$\boxed{h^D = \frac{\lambda}{1-\lambda}\left(h - 2 + \frac{1}{\lambda}\right)}$$

- Under this duality one can check that

$$\boxed{u_{\text{lower gap}}\left(\lambda^D, \beta_1^D, h^D\right) = u_{\text{upper cap}}\left(\lambda, \beta_1, h\right)}$$

# Level-Rank duality

- Level rank duality implies $\rightarrow \lambda^D = \dfrac{k-N}{k} = 1-\lambda, \qquad \beta_1^D = \beta_1 \dfrac{\lambda}{1-\lambda}$
- As under the transposition total number of boxes is unchanged, thus

$$\boxed{h^D = \frac{\lambda}{1-\lambda}\left(h - 2 + \frac{1}{\lambda}\right)}$$

- Under this duality one can check that

$$\boxed{u_{\text{lower gap}}\left(\lambda^D, \beta_1^D, h^D\right) = u_{\text{upper cap}}\left(\lambda, \beta_1, h\right)}$$

- As before, identifying $\pi u(h) = \theta$ and $h_+ - h_- = 2\pi\bar{\rho}(\theta)$

$$\bar{\rho}(\theta) = \frac{1}{2\pi}\left(2 - \frac{1}{\lambda}\right) + \frac{\beta_1}{2\pi}\sqrt{\frac{\frac{1}{\lambda}-1}{2\beta_1} - \sin^2\frac{\theta}{2}}\cos\frac{\theta}{2}.$$

- This eigenvalue distribution is related to the upper cap eigenvalue distribution obtained before by,

$$\rho(\theta) = \frac{1}{\pi}\left(\frac{1}{\lambda} - 1\right) + \bar{\rho}(\theta + \pi).$$

# Droplet picture for the upper cap phase

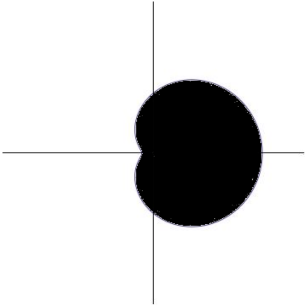
[(S. Dutta, R.Gopakumar),(AC, S. Dutta and P. Dutta,2018)]

- Define the phase space density

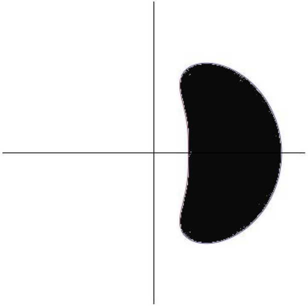
$$\omega(h,\theta) = \frac{1}{2\pi}\Theta(h - h_-(\theta))\Theta(h_+(\theta) - h).$$

such that

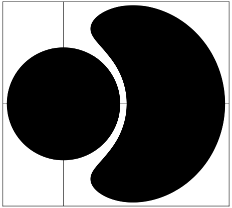
$$\rho(\theta) = \int_0^\infty \omega(h,\theta)dh, \qquad u(h) = \int_{-\pi}^{\pi} \omega(h,\theta)d\theta.$$



**(a)** Phase space distribution for no-gap phase.



**(b)** Phase space distribution for one-gap phase.



**(c)** Phase space distribution for upper-cap phase.

# Summary

- We discussed large $N$ phases of Chern-Simons matter theories on $S^2 \times S^1$.
- Lower gap phase and upper cap phase are related to each other by level-rank duality.
- We also write down the partition function for CS-m theory in Young diagram basis and showed that large $N$ phases can be equivalently classified in terms of young tableaux density.
- In large $N$ limit, different phases are dominated by different representations of $SU(N)$.
- We see that putting a cap on eigenvalue density restricts the dominant representations to be integrable representations.
- Representations of lower and upper-cap are related to each other by level-rank duality.


*Image Source:https://www.youtube.com/user/minutephysics