Geometry of isomonodromy deformations

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Introduction

The main target of this talk is the **discrete Painlevé equations and their generalizations**. First, in this introduction, I will look at the **differential case** to explain some backgrounds and motivations.

▲ There are 6 **Painlevé differential equations** (or 8 equations in geometric classification)

$$\begin{array}{ccccccccc}
P_{\mathrm{VI}} & \to & P_{\mathrm{V}} & \to & P_{\mathrm{III}} & \to & (P_{\mathrm{III}'}) & \to & (P_{\mathrm{III}''}) \\
 & & & \searrow & & \searrow & & \searrow & \\
 & & & & P_{\mathrm{IV}} & \to & P_{\mathrm{II}} & \to & P_{\mathrm{I}}
\end{array}$$

• Each equation P_{J} can be written in Hamiltonian form:

$$\frac{dq}{dt} = \frac{\partial H_{\mathsf{J}}}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_{\mathsf{J}}}{\partial q},$$

where

$$H_{\mathrm{I}} = \frac{1}{2}p^2 - 2q^3 - tq,$$

$$H_{\mathrm{II}} = \frac{1}{2}p^2 - (q^2 + \frac{t}{2})p - \alpha q,$$

$$\cdots$$

$$H_{\mathsf{VI}} = \frac{q(q-1)(q-t)}{t(t-1)}\Big\{p^2 - \Big(\frac{\alpha_4}{q} + \frac{\alpha_3}{q-1} + \frac{\alpha_0 - 1}{q-t}\Big)p + \frac{\alpha_2(\alpha_1+\alpha_2)}{q(q-1)}\Big\}. \quad (\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1)$$

- H_{J} depends explicitly on t (**non-autonomous** system).
- $H_{\mathsf{J}(\neq \mathrm{III}'')}$ is a **polynomial** in (p, q).
- $H_{\mathsf{J}(\neq \mathrm{I},\mathrm{III}'')}$ has some **parameters** α_i.

▲ The Painlevé differential equations have at least **three origins**:

(1) **Painlevé property.** [P.Painlevé, ($\sim$ 1900)]

(2) **Isomonodromic deformation (IMD).** [R.Fuchs (1905)]

(3) **Space of initial conditions.** [K.Okamoto (1979)]

• We want to clarify the relations among these aspects for **continous and discrete** Painlevé equations.

First, we will review these aspects in differential case.

▲ **Origin (1) Painlevé property**

• A singularity of solutions of a differential equation is said **"movable"** if its location can move depending on the initial condition.

• For **nonlinear** equations, there may be a **movable singularity**

$$\text{e.g.} \quad y = \sqrt{t - t_0} \quad \text{for} \quad 2y\frac{dy}{dt} = 1.$$

• For some special cases, nonlinear equations can have the following property (**Painlevé property**):

all the movable singularities are only poles.

Typical examples are the equations for the elliptic functions, The Painlevé equations are certain deformations of them.

- The Weierstrass $\wp$-function $y = \wp(t)$:

$$(y')^2 = 4y^3 - g_2 y - g_3 \quad \text{or} \quad y'' = 6y^2 - \frac{g_2}{2},$$

$$y = \frac{1}{u^2} + \frac{g_2}{20}u^2 + \frac{g_3}{28}u^4 + \frac{g_2^2}{1200}u^6 + \cdots . \quad (u = t - t_0)$$

- Non-autonomous deformation $\rightarrow$

The P_{I} equation: $q'' = 6q^2 + t$,

$$q = \frac{1}{u^2} - \frac{t_0}{10}u^2 - \frac{1}{6}u^3 + Cu^4 + \frac{t_0^2}{300}u^6 + \cdots . \quad (u = t - t_0)$$

- Search for this kind of solution gives a useful test to detect integrability:

Painlevé-test [Kowalevski(1889)].

▲ Origin (2) **Isomonodromic deformation (IMD)**

• 2nd order equation (with rational coefficients $a(x), b(x)$)

$$L: \qquad Y_{xx} + a(x)Y_x + b(x)Y = 0.$$

Solutions $Y_1(x), Y_2(x)$ may have nontrivial **monodromy** :

$$Y_i(x) \underset{\text{analytic continuation}}{\longrightarrow} C_{i1}Y_1(x) + C_{i2}Y_2(x).$$

• A deformation L is **isomonodromic deformation (IMD)**

$\Leftrightarrow$ The monodromy C_{ij} is independent of the deformation parameter t

$\Leftrightarrow$ **compatibility** of L with a deformation equation

$$B: \quad Y_t = r(x)Y_x + s(x)Y,$$

where $r(x), s(x)$ are rational functions in x.

- **Example**. Lax pair for P_{VI}.

$L: \quad Y_{xx} + a(x)Y_x + b(x)Y = 0.$

(i) Local exponents:

x	0	1	t	∞	q
exp.	0	0	0	α_2	0
	α_4	α_3	α_0	$\alpha_1+\alpha_2$	2

$$\Rightarrow \begin{cases} a(x) = \dfrac{1-\alpha_4}{x} + \dfrac{1-\alpha_3}{x-1} + \dfrac{1-\alpha_0}{x-t} + \dfrac{-1}{x-q}, \\ b(x) = \dfrac{1}{x(x-1)}\left\{\dfrac{q(q-1)p}{x-q} - \dfrac{t(t-1)H}{x-t} + \alpha_2(\alpha_1+\alpha_2)\right\}. \end{cases}$$

(ii) $x = q$ is **apparent singularity**: (solutions are regular)

$\Rightarrow$ determine the parameter $H = H_{\mathrm{VI}}(q, p)$.

$B: \quad \frac{t(t-1)}{q-t}Y_t + \frac{x(x-1)}{q-x}Y_x + \frac{pq(q-1)}{x-q}Y = 0.$

- P_{VI} is a prototype of IMD. There are many other IMDs.

▲ **Origin (3) Space of initial conditions**

• Okamoto constructed a surface X_{J} which parametrize the solutions of P_{J} [Okamoto(1979)].

▲ **Example.** P_{IV} **case**: $H_{\mathrm{IV}} = pq(p-q-t) - a_1 p - a_2 q$.

$$\{\text{Solutions}\} \sim \{\text{Initial values } (q,p) \in \mathbb{C}^2 \text{ at } t = t_0\}.$$

• However there may be additional solutions s.t. $q \to \infty$ and/or $p \to \infty$ $(t \to t_0)$. To include them, define a surface

$$X_{\mathrm{IV}} = \{(q,p)\} \cup \{(q_1,p_1)\} \cup \{(q_2,p_2)\} \cup \{(q_2,p_2)\},$$

patched by

$$(*) \quad \begin{aligned} (q,p) &= (a_1 p_1 + q_1 p_1^2, \tfrac{1}{p_1}) = (\tfrac{1}{q_2}, -a_2 q_2 + q_2^2 p_2) \\ &= (\tfrac{1}{q_3}, \tfrac{1}{q_3} + t - a_0 q_3 - q_3^2 p_3), \end{aligned}$$

$(a_0 + a_1 + a_2 = 1)$.

• The P_{IV} equation extended to X_{IV} has the following properties:

(i) $(*)$ are **symplectic** $\rightarrow$ Hamiltonian system on each chart.

(ii) $(*)$ are **bi-rational** $\rightarrow$ transformed Hamiltonians may have poles.

However, **they are still polynomial**! and moreover

(iii) This property determines the P_{IV} equation **uniquely** [Takano et. al (1997)].

Geometry knows Painlevé equations!

• Since the Lax pair has more information than equation, it is better to know not only the equation but also its Lax pair.

Question. Can we obtain the Lax pair also from the geometry?

▲ The geometry related to our main example: nine points blowup of $\mathbb{P}^2$

- The surface $\mathrm{Bl}_9(\mathbb{P}^2)$ ($\cong$ $\mathrm{Bl}_8(\mathbb{P}^1 \times \mathbb{P}^1)$) has infinitely many (-1) curves [Nagata (1960)].
- It has **affine Weyl group** symmetry of type $E_8^{(1)}$, whose translation part $\mathbb{Z}^8$ gives the **elliptic difference Painlevé equation** [Sakai (2001)].

0	$\mathbb{P}^2$	$\{1\}$	0
1	$\mathrm{Bl}_1(\mathbb{P}^2)$	A_1	1
2	$\mathrm{Bl}_2(\mathbb{P}^2)$	$A_1 \times A_1$	3
3	$\mathrm{Bl}_3(\mathbb{P}^2)$	$A_2 \times A_1$	6
4	$\mathrm{Bl}_4(\mathbb{P}^2)$	A_4	10
5	$\mathrm{Bl}_5(\mathbb{P}^2)$	D_5	16
6	$\mathrm{Bl}_6(\mathbb{P}^2)$	E_6	27
7	$\mathrm{Bl}_7(\mathbb{P}^2)$	E_7	56
8	$\mathrm{Bl}_8(\mathbb{P}^2)$	E_8	240
9	$\mathrm{Bl}_9(\mathbb{P}^2)$	$E_8^{(1)}$	∞

Question. Can we obtain the Lax pair of IMD from such geometry?

▲ **Ans.** Yes. we can construct IMDs from geometry. (**"Geometric engineering" of IMD**).

- Plan:

  (1) From geometry to discrete Painlevé equations

  (2) Lax formulation

  (3) Generalizations

- Our conclusion will be

Geometry knows not only the Painlevé equations but also various generalizations of them together with the Lax form.

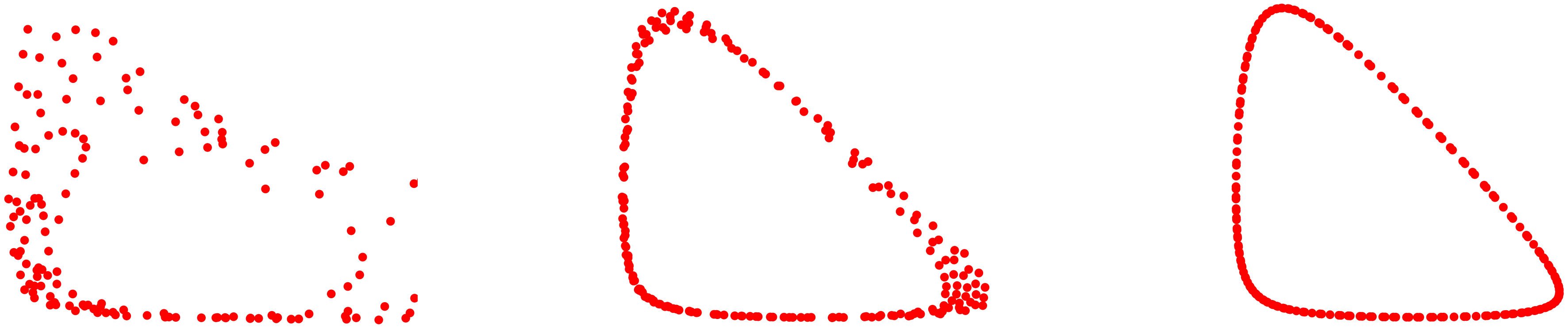
(various = continuous/discrete, higher order, ...)

1. From geometry to discrete Painlevé equations

• **Example 1.** Consider a **discrete dynamical system** (non-autonomous system on $\mathbb{C}^2 = \{x, y\}$) generated by the mapping:

$$T : (a, x, y) \mapsto \left(qa,\ a\frac{1 + xy}{x},\ \frac{1}{xy}\right).$$

• Plot of orbit in (x, y) plane:



$q = 1.01$ $\quad$ $q = 1.001$ $\quad$ $q = 1$

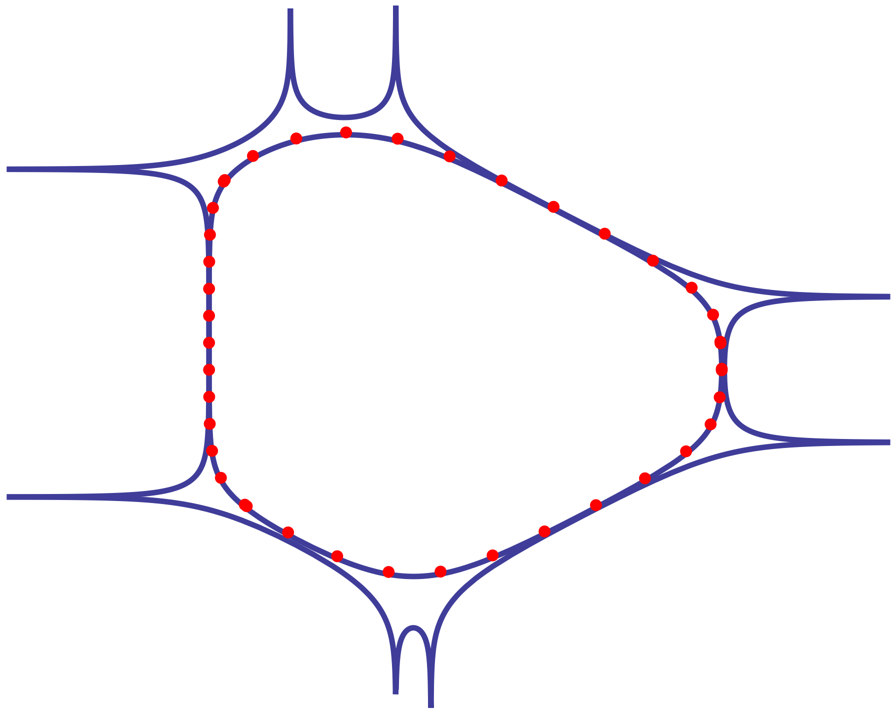
- **Example 2.** Consider two involutions:

$$i_x : (x, y) \to (\tilde{x}, y), \quad \tilde{x} = \frac{ab}{x}\frac{(y+t)(y+u)}{(y+r)(y+s)},$$

$$i_y : (x, y) \to (x, \tilde{y}), \quad \tilde{y} = \frac{rs}{y}\frac{(x+c)(x+d)}{(x+a)(x+b)},$$

where $abtu = cdrs$.

- Iteration of $T = i_x \circ i_y$ (or $T^{-1} = i_y \circ i_x$) gives a discrete integrable system.



• **Conserved curves**:

type	conserved curve
$A_0^{(1)}$	$H=\frac{y}{x}+x+\frac{1}{y}$
$A_1^{(1)}$	$H=\frac{x}{a}+\frac{1}{xy}+\frac{1}{x}+y$
$A_{1+1}^{(1)}$	$H=\frac{1}{abxy}+\frac{1}{abx}+\frac{1}{aby}+\frac{x}{a}+y$
$A_{2+1}^{(1)}$	$H=\frac{y}{abx}+\frac{1}{abx}+\frac{y}{b}+\frac{cx}{y}+\frac{c}{y}+x$
$A_4^{(1)}$	$H=\frac{x}{b_1b_3b_4y}+\frac{y}{b_4x}+\frac{1}{b_1b_4xy}+\frac{b_1+1}{b_1b_4x}+\frac{b_3+1}{b_1b_3b_4y}+y+b_2x,$
$D_5^{(1)}$	$H=\frac{1}{xy}((x+a)(x+b)y^2+\{(r+s)x^2+ab(t+u)\}y$ $+rs(x+c)(x+d))$
$E_6^{(1)}$	$\cdots$

$\leftrightarrow$ 5d, $\mathcal{N}=2$, $SU(2)$ Seiberg-Witten curve.

• A remarkable progress in spectral theory for corresponding **quantum operators** $\widehat{H}$ [Hatsuda, Marino,...].

▲ 2nd order Painlevé equations [Sakai(2001)]

ell. $E_8^{(1)}$

$$\nearrow A_1^{(1)}$$

mul. $E_8^{(1)} \to E_7^{(1)} \to E_6^{(1)} \to D_5^{(1)} \to A_4^{(1)} \to A_{2+1}^{(1)} \to A_{1+1}^{(1)} \to A_1^{(1)} \to A_0^{(1)}$

add. $E_8^{(1)} \to E_7^{(1)} \to E_6^{(1)} \to D_4^{(1)} \to A_3^{(1)} \to A_{1+1}^{(1)} \to A_1^{(1)} \to A_0^{(1)}$

$$\searrow \quad \searrow \quad \searrow$$

$$A_2^{(1)} \to A_1^{(1)} \to A_0^{(1)}$$

- Cases in blue/magenta admit discrete/continuous flows.
- The same diagram arises in gauge theory for $d = 4, 5, 6$.

▲ **Simple geometric construction of integrable mappings on $\mathbb{P}^2$**

- **bi-degree $(2,2)$ curve**: $C : \varphi(x,y) = 0$

  $\to$ involutions $i_x : (x,y) \mapsto (\tilde{x}, y)$ and $i_y : (x,y) \mapsto (x, \tilde{y})$

  $\to T = i_x \circ i_y$ (or $T^{-1} = i_y \circ i_x$): (an addition formula on C)

- Apply this construction to **a pencil of $(2,2)$ curves**:

$$\varphi(x,y) = F(x,y) - hG(x,y) = 0$$

  $\to$ **The QRT mapping** $T : \mathbb{P}^2 \to \mathbb{P}^2$

  $\to$ conserved quantity $H(x,y) = \frac{F(x,y)}{G(x,y)} = h$.

[Quispel-Roberts-Tompson (1989)], [Tsuda(2004)]

▲ **Example 1'**. For $H(x,y) = \frac{x}{a} + \frac{1}{xy} + \frac{1}{x} + y$, we have

$$i_x : x \mapsto \tilde{x} = \frac{a}{x}(1 + \frac{1}{y}), \quad i_y : y \mapsto \tilde{y} = \frac{1}{xy}.$$

The composition $T = i_y \circ i_x$ gives **Example 1** ($q = 1$).

▲ The pencil of the bi-degree (2,2) curves $F(x,y) - HG(x,y) = 0$ has 8 common points **in a special position**: $\mathrm{Bl}_8(\mathbb{P}^1 \times \mathbb{P}^1) = \frac{1}{2}K3$:

Config.	bi-degree $(2,2)$ curve	evolution equation
special	1-parameter family	QRT mapping
non-special	unique	Painlevé equation

• The discrete Painlevé equation is a **deautonomization** of the QRT.

It has no longer any integral but the degree grows gently, i.e.

(degree of mapping) $\sim$ ($\sharp$ iteration)^2.

▲ Deautonomization of Example 2

$\rightarrow q\text{-}P_{\mathrm{VI}}$ equation [Jimbo-Sakai(1996)] ($D_5^{(1)}$ symmetry)

$$\overline{f}f = v_3v_4\frac{(g-\frac{v_5}{\kappa_2})(g-\frac{v_6}{\kappa_2})}{(g-\frac{1}{v_1})(g-\frac{1}{v_2})}, \quad \underline{g}g = \frac{1}{v_1v_2}\frac{(f-\frac{\kappa_1}{v_7})(f-\frac{\kappa_1}{v_8})}{(f-v_3)(f-v_4)}.$$

▲ Up/down shift notations for discrete (difference) equation:

- Evolution map: $T(*)=\overline{*}$, $T^{-1}(*)=\underline{*}$.
- Parameters: $\kappa_1,\kappa_2,v_1,\cdots,v_8$: $q=\kappa_1^2\kappa_2^2/(v_1\cdots v_8)$.

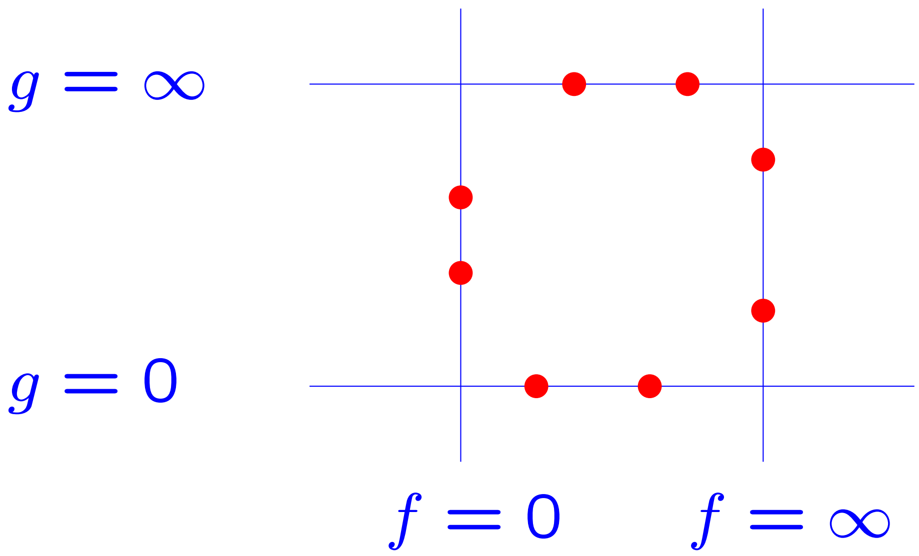
$$\overline{\kappa_1}=q^{-1}\kappa_1, \quad \overline{\kappa_2}=q\kappa_2, \quad \overline{v_i}=v_i.$$

- Dependent variables: f, g.

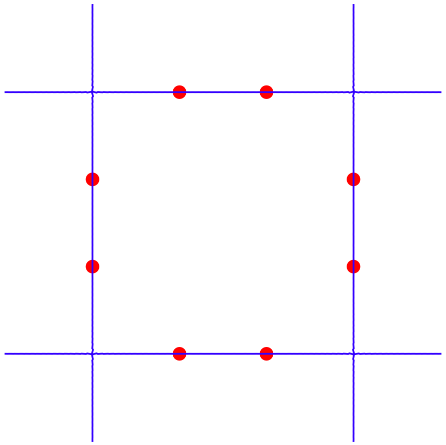
▲ **The singular points of q-$P_{\rm VI}$:**

$$(f,g)=(\infty,\frac{1}{v_1}),\ (\infty,\frac{1}{v_2}),\ (v_3,\infty),\ (v_4,\infty),$$

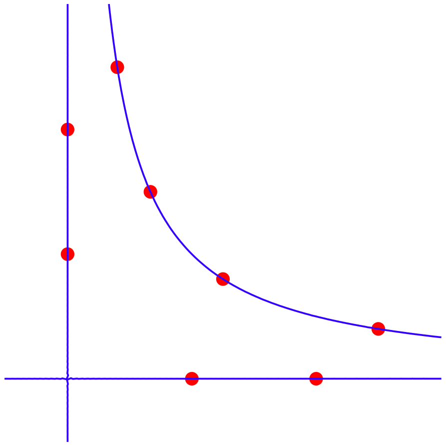
$$(0,\frac{v_5}{\kappa_2}),\ (0,\frac{v_6}{\kappa_2}),\ (\frac{\kappa_1}{v_7},0),\ (\frac{\kappa_1}{v_8},0).$$



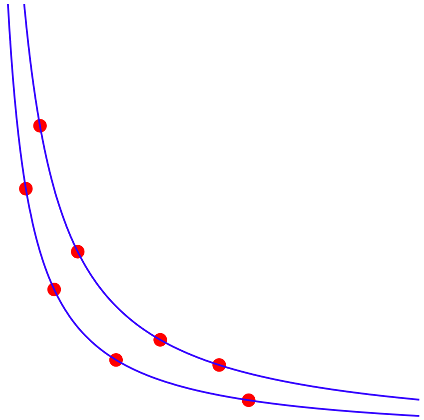
▲ Other cases



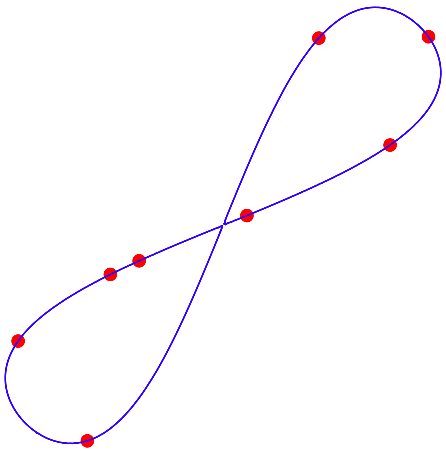
q-$D_5^{(1)}$



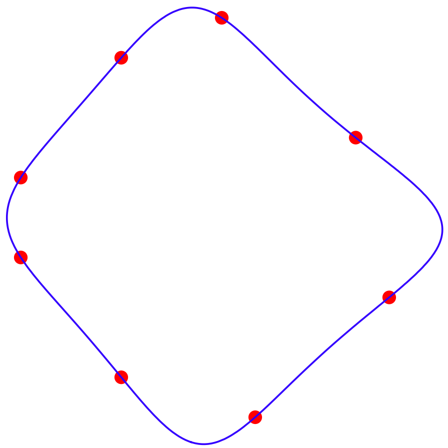
q-$E_6^{(1)}$



q-$E_7^{(1)}$



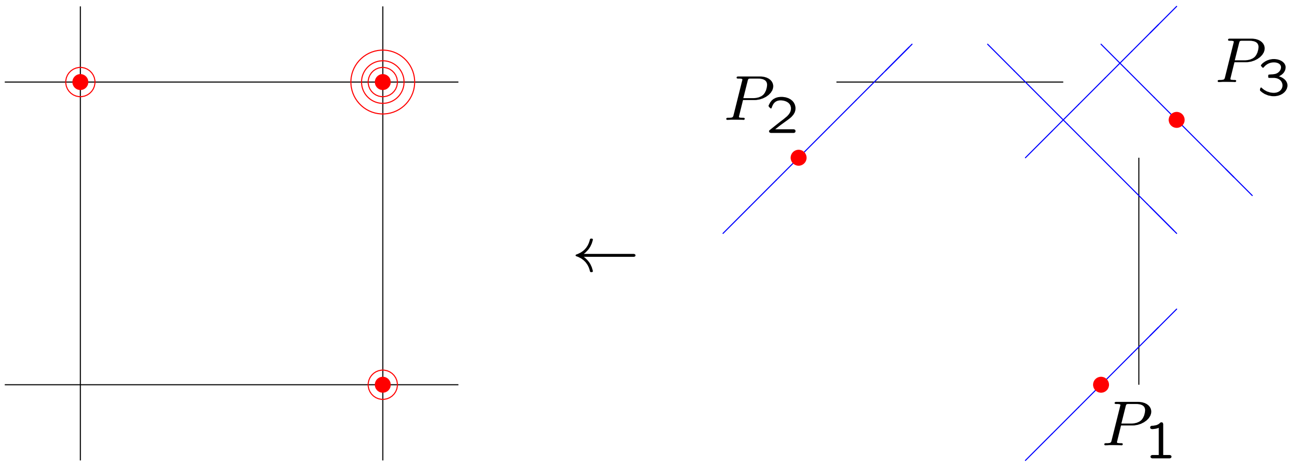
q-$E_8^{(1)}$



ell-$E_8^{(1)}$

• **More degenerate cases**: multiple blowing-up points.

e.g. $P_{\rm IV}$ case:



$$(P_1)_{\text{double}}: \quad (\infty, 0) \leftarrow (q,p) = (a_1 p_1 + q_1 p_1^2, \frac{1}{p_1}),$$

$$(P_2)_{\text{double}}: \quad (0, \infty) \leftarrow (q,p) = (\frac{1}{q_2}, -a_2 q_2 + q_2^2 p_2),$$

$$(P_3)_{\text{quadruple}}: \quad (\infty, \infty) \leftarrow (q,p) = (\frac{1}{q_3}, \frac{1}{q_3} + t - a_0 q_3 - q_3^2 p_3).$$

2. Lax formulation

▲ The scalar Lax pair for q-P_{VI} ($\Leftrightarrow$ matrix form [Jimbo-Sakai (1996)])

$$L_1 : \left\{ \frac{z \prod_{i=1}^{2} (g v_i - 1)}{q g} - \frac{\prod_{i=1}^{4} v_i \prod_{i=5}^{6} \left(g - \frac{v_i}{\kappa_2} \right)}{f g} \right\} Y(z)$$

$$+ \frac{v_1 v_2 \prod_{i=3}^{4} \left(\frac{z}{q} - v_i \right)}{f - \frac{z}{q}} \left\{ g Y(z) - Y(\tfrac{z}{q}) \right\} + \frac{\prod_{i=7}^{8} \left(\frac{\kappa_1}{v_i} - z \right)}{q(f - z)} \left\{ Y(qz) - \frac{1}{g} Y(z) \right\} = 0,$$

$$L_2 : \left\{ 1 - \frac{f}{z} \right\} \overline{Y}(z) + Y(qz) - \frac{1}{g} Y(z) = 0.$$

The compatibility of L_1, L_2 gives the q-P_{VI}.

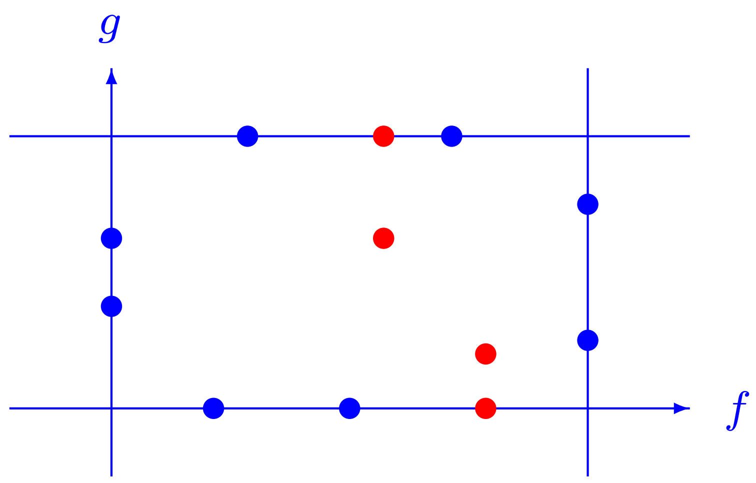
- **Basic property of L_1:**

As an algebraic curve in f,g, the equation L_1 for q-$P_{\rm VI}$ is uniquely characterized by the following conditions:

(1) polynomial of bi-degree $(3, 2)$.

(2) passing through the following 12 points:

$$\left(\infty, \frac{1}{v_i}\right)_{i=1}^{2}, \quad \left(v_i, \infty\right)_{i=3}^{4},$$
$$\left(0, \frac{v_i}{\kappa_2}\right)_{i=5}^{6}, \quad \left(\frac{\kappa_1}{v_i}, 0\right)_{i=7}^{8},$$
$$(z, \infty), \quad \left(\frac{z}{q}, 0\right),$$
$$\left(z, \frac{Y(z)}{Y(qz)}\right), \quad \left(\frac{z}{q}, \frac{Y(z/q)}{Y(z)}\right).$$



▲ This property is **universal** for almost all the Painlevé equations.

The linear equation L_1 can be determined by the conditions:

(1) polynomial in (f, g) of bi-degree $(3, 2)$,

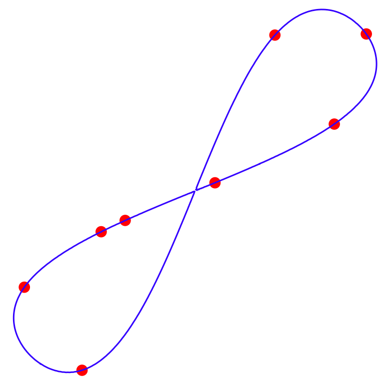
(2) vanishes at 12 points: $P_1, \ldots, P_8, P(x), P(x'), Q_1, Q_2$.

- $P_1, \cdots, P_8$ are given by specifying the type of equation.
- $P(x')$ is determined from $P_1, \cdots, P_8, P(x)$ (Abel's relation).
- **How to choose the points Q_1, Q_2 ?**

They must determine the $Y(qx)$, $Y(x)$, $Y(x/q)$ dependence of L_1.

L_1 should be linear in $Y(qx)$, $Y(x)$, $Y(x/q) \to$ determine Q_1, Q_2.

• **Example 3.** q-$E_8^{(1)}$ case.



Parameterization of a nodal curve:

$$P(x) = \Big(F(x), G(x)\Big) = \Big(x + \frac{\kappa_1}{x}, x + \frac{\kappa_2}{x}\Big).$$

The 12 points: $P(v_1), \ldots, P(v_8)$, $P(x)$, $P(\frac{\kappa_1 q}{x})$, and Q_1, Q_2,

$$\text{where} \quad Q_1 : \quad f = F(x), \quad \frac{g - G(x)}{g - G(\frac{\kappa_1}{x})} = \frac{Y(qx)}{Y(x)},$$

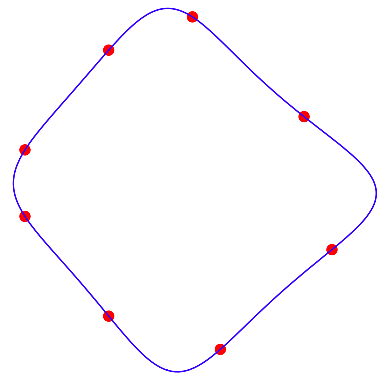
and $Q_2 = Q_1|_{x \to \frac{x}{q}}$. Then L_1 is linear in $Y(qx), Y(x), Y(x/q)$.

• A Lax pair is given by L_1 and

$$L_2 : \{g - G(x)\}Y(x) - \{g - G(\tfrac{\kappa_1}{x})\}Y(qx)$$
$$+ C(x - \tfrac{\kappa_1}{x})\{f - F(x)\}\overline{Y}(x) = 0,$$

where C is a constant.

• **Example 4.** Elliptic $E_8^{(1)}$ case:



Parametrization of the generic (2,2) curve:

$$(f,g)=\left(\frac{F_b(x)}{F_a(x)},\frac{G_b(x)}{G_a(x)}\right),$$

where $F_a(x)=[\frac{a}{x}][\frac{\kappa_1}{ax}],\quad G_a(x)=[\frac{a}{x}][\frac{\kappa_2}{ax}],$

$$[x]=\sum_{n\in\mathbb{Z}}(-1)^n x^{n+\frac{1}{2}}p^{\frac{n(n+1)}{2}}.\ [px]=-\frac{p^{-\frac{1}{2}}}{x}[x],\quad [\tfrac{1}{x}]=-[x].$$

We put

$$\mathcal{F}(f,x)=F_a(x)f-F_b(x),\quad \mathcal{G}(g,x)=G_a(x)g-G_b(x).$$

- **The Lax pair** for elliptic $E_8^{(1)}$ equation:

$$L_2 : \mathcal{G}(g,x)Y(x) - \mathcal{G}(g,\tfrac{\kappa_1}{x})Y(qx) + \mathcal{F}(f,x)\overline{Y}(x) = 0,$$

$$L_3 : \mathcal{G}(g,x)U(\tfrac{\kappa_1}{qx})\overline{Y}(qx) - \mathcal{G}(g,\tfrac{\kappa_1}{qx})U(x)\overline{Y}(x)$$

$$+w\overline{\mathcal{F}}(\overline{f},x)[\tfrac{x^2}{\kappa_1},\tfrac{qx^2}{\kappa_1}]Y(qx) = 0, \quad U(x) = \Pi_{i=1}^{8}[\tfrac{v_i}{x}]$$

Compatibility $\Rightarrow$

$$\frac{\mathcal{F}(f,\frac{\kappa_2}{x})\overline{\mathcal{F}}(\overline{f},\frac{\kappa_2}{x})}{\mathcal{F}(f,x)\overline{\mathcal{F}}(\overline{f},x)} = \frac{U(\frac{\kappa_2}{x})}{U(x)} \quad \text{for } \mathcal{G}(g,x) = 0,$$

$$\frac{\mathcal{G}(g,\frac{\kappa_1}{x})\underline{\mathcal{G}}(\underline{g},\frac{\kappa_1}{x})}{\mathcal{G}(g,x)\underline{\mathcal{G}}(\underline{g},x)} = \frac{U(\frac{\kappa_1}{x})}{U(x)} \quad \text{for } \mathcal{F}(f,x) = 0.$$

• **Example 5.** L_1 for differential P_{VI}: $(f = q, g = qp)$:

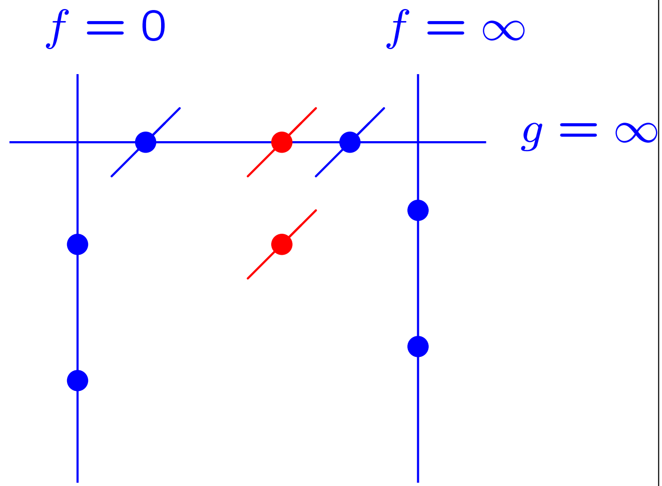
(1) bi-degree $(3, 2)$.

(2) passing through the following 12 points:

$$\left(1 + \alpha_3\epsilon, \frac{1}{\epsilon}\right)_{\text{double}}, \ \left(t + \alpha_0\epsilon, \frac{t}{\epsilon}\right)_{\text{double}},$$

$$(0, 0), \ (0, \alpha_4), \ (\infty, -\alpha_2), \ (\infty, -\alpha_1 - \alpha_2),$$

$$\left(x + \epsilon, -\frac{x}{\epsilon}\right)_{\text{double}}, \ \left(x + \epsilon, \frac{y'(x + \epsilon)}{y(x + \epsilon)}\right)_{\text{double}}$$



• Degeneration from q-P_{VI} configuration : confluence of two lines at $g = 0$ and $g = \infty$.

▲ Two characterizations of L_1 for P_{VI}

(i) In (x, ∂_x):

- the local exponents (Riemann scheme),
- apparent condition at $x = q$ where $Y'(x) = pY(x)$.

(ii) In (q, p):

- vanishing conditions at the 8 points,
- extra 4 vanishing conditions at $\left(x+\epsilon, -\frac{x}{\epsilon}\right)_{\text{double}}$ and $\left(x+\epsilon, \frac{y'(x+\epsilon)}{y(x+\epsilon)}\right)_{\text{double}}$.

- These two characterizations give the same L_1 (due to the symmetry $(x, \partial_x) \leftrightarrow (q, p)$).

3. Genalralizations

▲ Garnier system

• 2nd order Fuchsian differential equation on $\mathbb{P}^1$ with $N+3$ regular singular points at $x = t_1, \ldots, t_{N+3}$.

$$\psi_{xx} + u(x)\psi = 0,$$

$$u(x) = \sum_{a=1}^{N+3} \Big\{ \frac{\Delta_a}{(x-t_a)^2} - \frac{H_a}{x-t_a} \Big\} + \sum_{i=1}^{N} \Big\{ \frac{-\frac{3}{4}}{(x-q_i)^2} + \frac{p_i}{x-q_i} \Big\}.$$

IMD → **Garnier system** [Garnier (1912)]

$$\frac{\partial q_i}{\partial t_a} = \frac{\partial H_a}{\partial p_i}, \quad \frac{\partial p_i}{\partial t_a} = -\frac{\partial H_a}{\partial q_i}.$$

→ System of $2N$ unknown variables: $N=1$ case is P_{VI}.

• **A scalar Lax pair for q-Garnier system** [Nagao-Y(2016)]

$$L_2: \ F(x)\overline{y}(x) + G(x)y(x) - A(x)y(qx) = 0,$$

$$L_3: \ qx\overline{F}(x)y(qx) + G(x)\overline{y}(qx) - qtB(x)\overline{y}(x) = 0.$$

$$A(x) = \prod_{i=1}^{N+1}(x-a_i), \quad B(x) = \prod_{i=1}^{N+1}(x-b_i),$$

$$F(x) = \sum_{i=0}^{N} f_i\, x^i, \quad G(x) = ct + \sum_{i=1}^{N} g_i\, x^i + x^{N+1}.$$

• **Parameters**: $\overline{(a_i, b_i, c, t)} = (a_i, b_i, c, qt)$.

• **Dynamical variables**: the coefficients f_i, g_i. ($\sharp = 2N+1$, but only the ratios $f_0 : f_1 : \cdots : f_N$ are important $\to \sharp_{\text{eff}} = 2N$).

• From L_2 and L_3, we have

$$L_1: \quad A(x)F(\frac{x}{q})y(qx) - R(x)y(x) + tB(\frac{x}{q})F(x)y(\frac{x}{q}) = 0,$$

where $R(x)$ is a polynomial of degree $2N+1$.

• Compatibility of L_2, L_3 or $L_1 \to q$-**Garnier system**:

$$xF(x)\overline{F}(x) = tA(x)B(x) \quad \text{for} \quad G(x) = 0,$$
$$G(x)\underline{G}(x) = tA(x)B(x) \quad \text{for} \quad F(x) = 0.$$

• To ses the geometric meaning of this equation, we consider the autonomous limit.

• **Autonomous limit** of L_1 equation:

$$\left[A(x)F(\frac{x}{q})T_x - R(x) + tB(\frac{x}{q})F(x)T_x^{-1}\right]y(x) = 0,$$

where $T_x x = qxT_x$. For $q \to 1$, we obtain an algebraic equation:

$$\boxed{C: \ A(x)T_x - U(x) + \frac{tB(x)}{T_x} = 0.}$$

= **spectral curve** for autonomous q-Garnier system

= hyperelliptic curve of bi-degree $(N+1, 2)$ in (x, T_x)

= SW curve for 5d, $\mathcal{N} = 1$, $SU(N)$, $N_f = 2N$

We will use the notation $y = T_x$ in the followings.

▲ **Meaning of the polynomials $F(x), G(x)$**

• Dynamical variables of q-Garnier system

= a pair of polynomials $F(x)/\mathbb{C}^*, G(x)$

= set of N-points $\{Q_i = (x_i, y_i)\}$ on spectral curve C

$$F(x_i) = 0, \quad y_i = G(x_i).$$

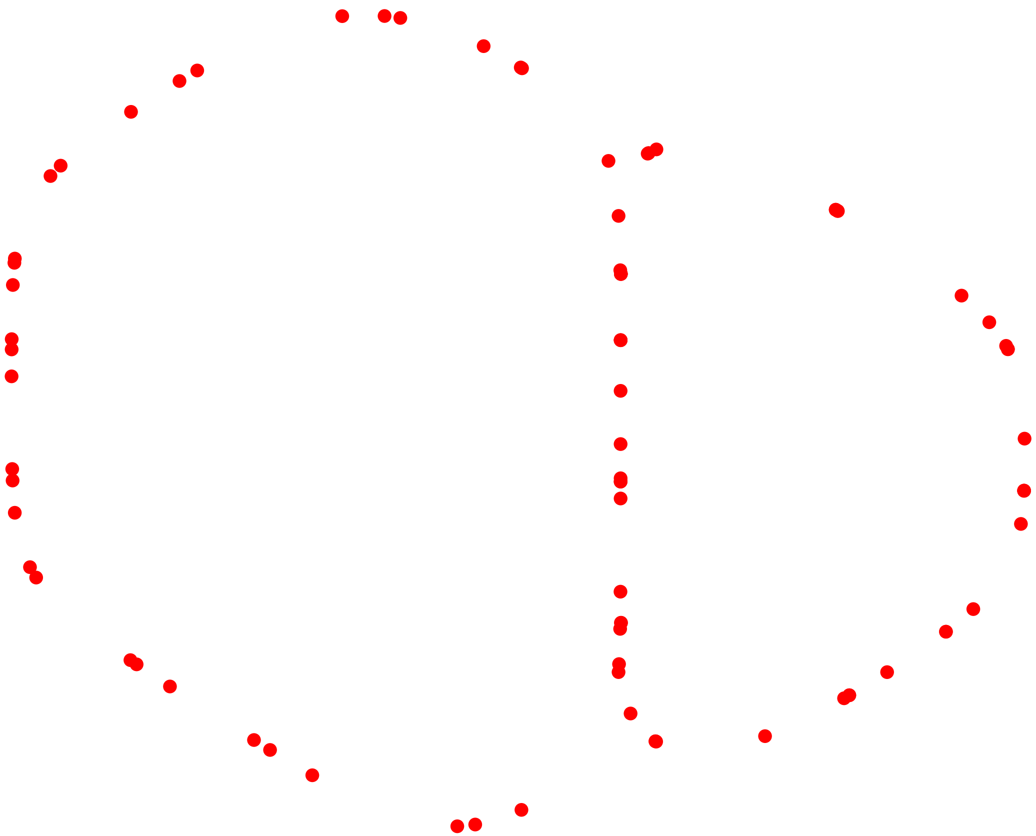
• The evolution = an **addition formula** on C.

• For $N > 1$, the addition formula for $\{Q_i\}$ are **not** bi-rational.

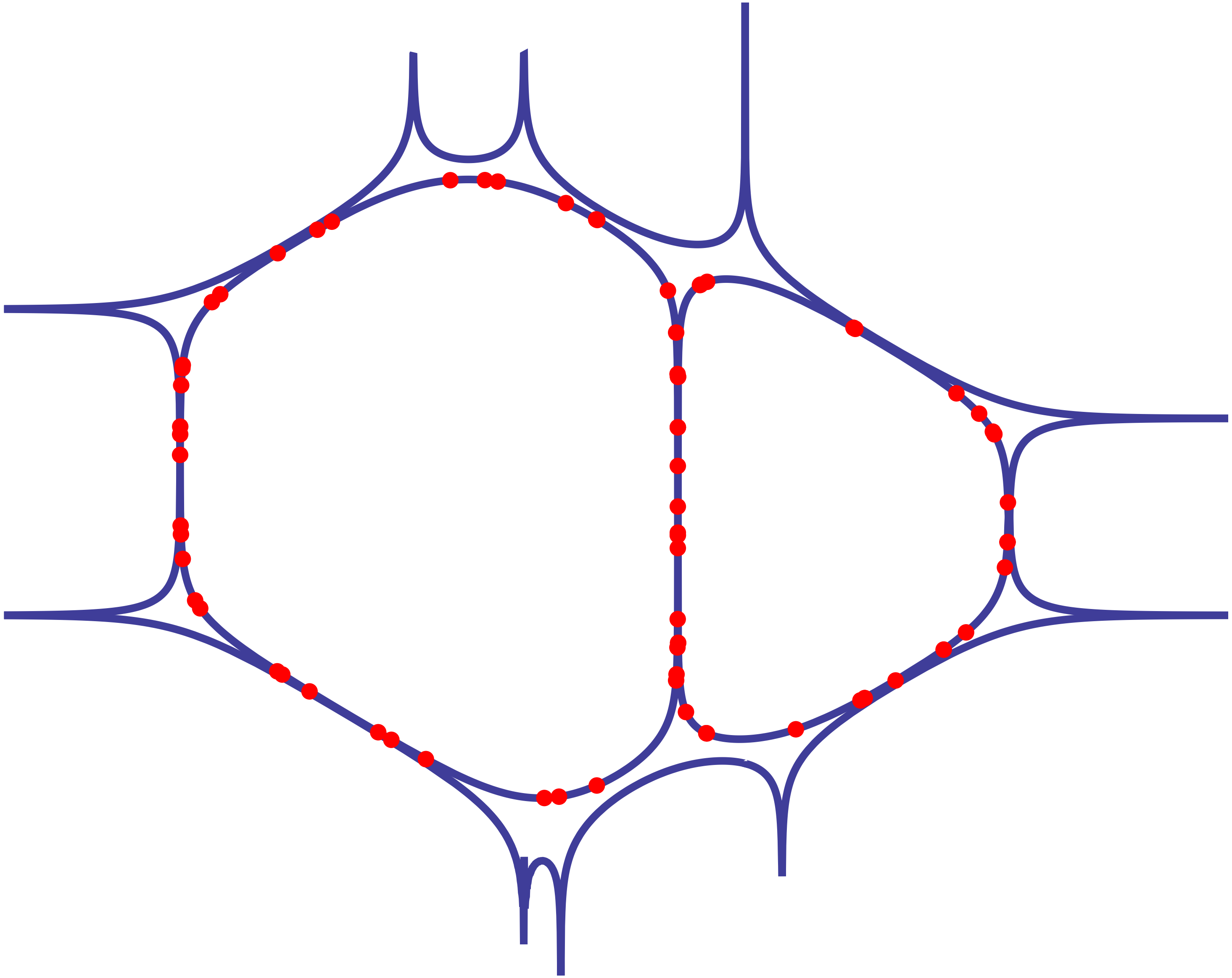
$\to$ In terms of the polynomials $F(x), G(x)$, it takes bi-rational form (**Mumford representation**).

▲ **Example 6.** $N = 2$ case ($q = 1$)

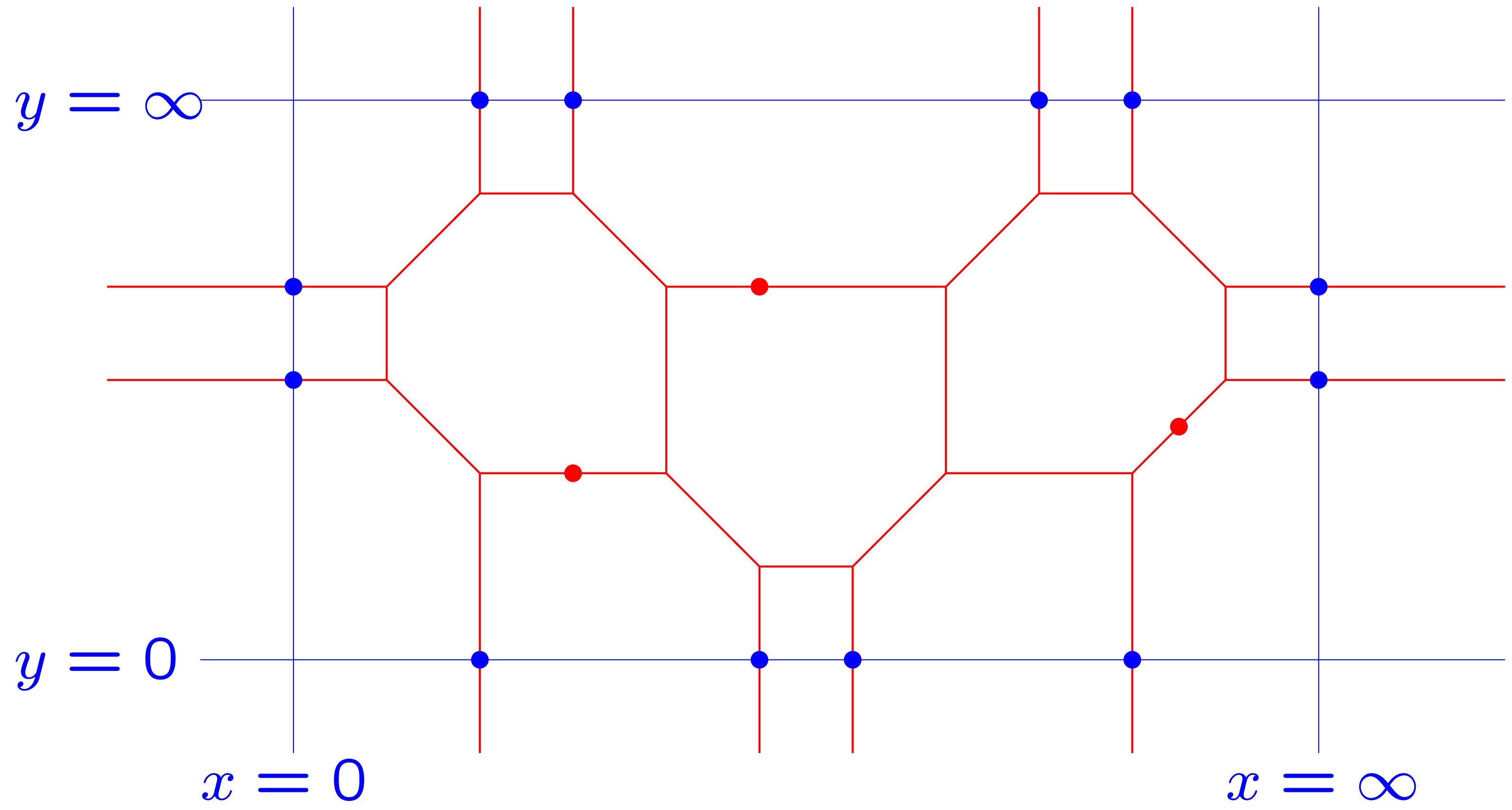
The orbit of the two points $Q_1 = (x_1, y_1), Q_2 = (x_2, y_2)$ is as follows (log-log plot)



- Amoeba of the corresponding spectral curve



• In the **ultra discrete limit**, the spectral curve becomes piecewise linear = **5 brane web**: (following figure is for $N = 3$)



= Spectral curve for periodic BBS [Inoue-Kuniba-Takagi (2011)].

SU(2)-SU(2)-SU(2) $\leftrightarrow$ SU(4) (Base-Fiber duality [Mitev-Pomoni-Taki-Yagi (2014)]).

▲ **Base-Fiber duality** as q-Laplace transformation

- (m,n)**-reduced Lax operator for** q**-KP hierarchy.**

$$\boldsymbol{\Psi}(qz) = \boldsymbol{\mathcal{A}}(z)\boldsymbol{\Psi}(z), \quad \boldsymbol{\mathcal{A}}(z) = DX_m(z)\cdots X_1(z),$$

$$D = \mathrm{diag}(d_1, \cdots, d_n),$$

$$X_i(z) = \begin{bmatrix} x_{i,1} & 1 & & & \\ & x_{i,2} & 1 & & \\ & & \ddots & \ddots & \\ & & & x_{i,n-1} & 1 \\ r_i z & & & & x_{i,n} \end{bmatrix}.$$

- $W(A^{(1)}_{m-1}) \times W(A^{(1)}_{n-1})$ symmetry. [Kajiwara-Noumi-Y (2002)]

- **A duality : n (matrix size) $\leftrightarrow$ m (number of factors)**

(Proof.) We rewrite the (m, n)-reduced equation

$$\mathbf{\Psi}(qz) = \boldsymbol{\mathcal{A}}(z)\mathbf{\Psi}(z) = DX_m \cdots X_2 X_1 \mathbf{\Psi}$$

by putting $\mathbf{\Psi}_1 = \mathbf{\Psi}$, $\mathbf{\Psi}_{i+1} = X_i \mathbf{\Psi}_i$ $(1 \leq i \leq m)$. Then for the components $\psi_{i,j} = (\mathbf{\Psi}_i)_j$, we have

$$\psi_{i+1,j} = x_{i,j}\psi_{i,j} + \psi_{i,j+1},$$
$$\psi_{m+1,j} = d_j^{-1} T_z \psi_{1,j}, \qquad \psi_{i,n+1} = r_i z \psi_{i,1}.$$

These relations are symmetric under the exchange:

$$m \leftrightarrow n,\ \psi_{i,j} \leftrightarrow \psi_{j,i},\ x_{i,j} \leftrightarrow -x_{j,i},\ r_k \leftrightarrow d_k^{-1},\ z \leftrightarrow T_z. \qquad \square$$

- **Two equivalent Lax forms** for q-Garnier system.

(i) $(m,n)=(2,2N+2)$ case:

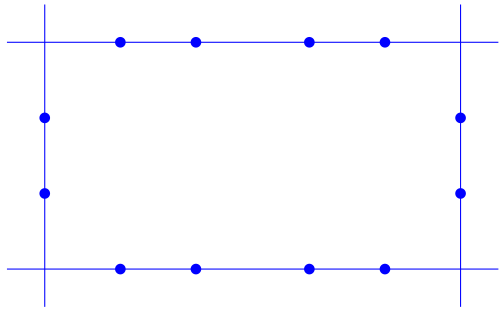
$$\mathcal{A}(z)=\begin{bmatrix} * & * & * & & \\ & * & * & * & \\ & & & \vdots & \\ & & & * & * \\ & & & & * \end{bmatrix}+\begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ * & & & & \\ * & * & & & \end{bmatrix}z.$$

(ii) $(m,n)=(2N+2,2)$ case:

$$\mathcal{A}(z)=\begin{bmatrix} * & * \\ & * \end{bmatrix}+\begin{bmatrix} * & * \\ * & * \end{bmatrix}z+\cdots+\begin{bmatrix} * & * \\ * & * \end{bmatrix}z^N+\begin{bmatrix} * & \\ * & * \end{bmatrix}z^{N+1}.$$

▲ **Various configurations**

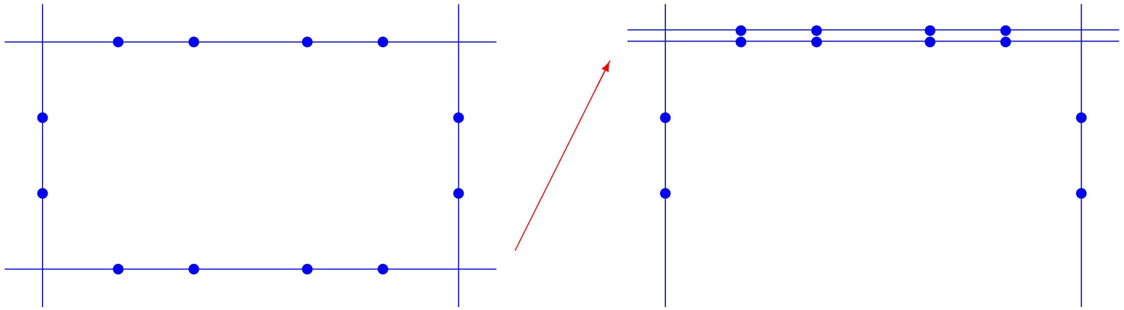
- We have considered:



$\to$ q-**Garnier**

- The most generic case $\to$ **elliptic Garnier**

- A degeneration



$\to$ **differential Garnier**

▲ Summary

(1) Various IMD are formulated by geometric method.

(2) It will be useful for further generalization of IMD and to study their connection to gauge/string theory.

Thank you.

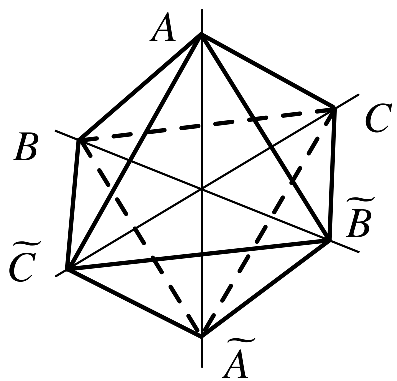
Tau functions

▲ In terms of τ **functions**, discrete/continuous Painlevé equations can be written as bilinear form.

• **Example. Elliptic $E_8^{(1)}$ case** [Ohta-Ramani-Grammaticos (2001)]

For each octahedron (with $(\text{edge})^2 = 2$) on E_8 lattice, we have

$$*\tau_A \tau_{\tilde{A}} + *\tau_B \tau_{\tilde{B}} + *\tau_C \tau_{\tilde{C}} = 0$$



• The system is highly over determined, but consistent!

▲ **Geometric meaning** of the τ-functions [KMNOY (2003)].

• The surface $X = \mathrm{Bl}_9(\mathbb{P}^2) \cong \mathrm{Bl}_8(\mathbb{P}^1 \times \mathbb{P}^1)$ has infinitely many (-1) curves: [Nagata (1960)]

$$\lambda = e_i, \quad \ell - e_i - e_j, \quad 2\ell - e_{i_1} - \cdots - e_{i_5}, \quad \cdots$$
$$\in \mathrm{Pic}(X) = \mathbb{Z}\ell \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_9.$$

Their defining equations $\tau(\lambda) = 0 \to \tau$-functions.

• Bilinear relations: For $\dim|\ell - e_9| = 1$

$$\to [e_2 - e_3][\ell - e_2 - e_3 - e_9]\tau(\ell - e_1 - e_9)\tau(e_1) + (123\ \mathrm{cyc}) = 0,$$

For $\dim|2\ell - e_1 - e_2 - e_3 - e_4| = 1$

$$\to [e_1 - e_2][e_3 - e_4]\tau(\ell - e_1 - e_2)\tau(\ell - e_3 - e_4) + (123\ \mathrm{cyc}) = 0.$$

Quantization

We will consider only the differential cases here.

▲ Since (q, p) are canonical variables, there is a **natural quantization.**

$\rightarrow$ **The duality** $(x, \partial_x) \leftrightarrow (q, \partial_q)$ **becomes manifest**.

▲ Quantum Lax pair for P_{VI}: $\widehat{L}\psi = \widehat{B}\psi = 0$.

$$
\begin{aligned}
\widehat{L} = x(x-1)(x-t)&\left\{\frac{\alpha_0^{(2)}}{x}+\frac{\alpha_1^{(2)}}{x-1}+\frac{\alpha_t^{(2)}}{x-t}-\frac{\epsilon_1-\epsilon_2}{x-q}\right\}\epsilon_1\partial_x \\
&-q(q-1)(q-t)\left\{\frac{\alpha_0^{(1)}}{q}+\frac{\alpha_1^{(1)}}{q-1}+\frac{\alpha_t^{(1)}}{q-t}-\frac{\epsilon_2-\epsilon_1}{q-x}\right\}\epsilon_2\partial_q \\
&+x(x-1)(x-t){\epsilon_1}^2{\partial_x}^2-q(q-1)(q-t){\epsilon_2}^2{\partial_q}^2+C(x-q),
\end{aligned}
$$

$$
\begin{aligned}
\widehat{B} = q(q-1)&\left\{\frac{\alpha_0^{(1)}}{q}+\frac{\alpha_1^{(1)}}{q-1}+\frac{\alpha_t}{q-t}-\frac{\epsilon_2}{q-x}\right\}\epsilon_2\partial_q \\
&+\frac{t(t-1)}{q-t}\epsilon_1\epsilon_2\partial_t+\frac{x(x-1)}{q-x}\epsilon_1\epsilon_2\partial_x+q(q-1){\epsilon_2}^2{\partial_q}^2+C,
\end{aligned}
$$

where $\alpha_i^{(j)} = \alpha_i - \epsilon_j$. The parameters ϵ_1, ϵ_2 play the role of the Planck constants for quantization : $(x, \epsilon_1\partial_x)$ and $(q, \epsilon_2\partial_q)$.

▲ $\hat{L}\psi = \widehat{B}\psi = 0$ are the **BPZ equations** for 6-points block ψ on $\mathbb{P}^1$

$$\psi(x,q,t) = \Big\langle V_{-\epsilon_2}(x)V_{-\epsilon_1}(q)V_{\alpha_0}(0)V_{\alpha_1}(1)V_{\alpha_t}(t)V_{\alpha_\infty}(\infty)\Big\rangle.$$

Where $V_\alpha(z)$ is the Virasoro primary operator (AGT):

$$c = 1 + 6\frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}, \quad \Delta(\alpha) = \frac{\alpha}{2\epsilon_1\epsilon_2}(\epsilon_1+\epsilon_2-\frac{\alpha}{2}).$$

$\rightarrow$ can be extended to **quantum Garnier system**:

$$\psi(x,\{q_i\},t) = \Big\langle V_{-\epsilon_2}(x)\prod_{i=1}^{N} V_{-\epsilon_1}(q_i)\prod_{a=1}^{N+3} V_{\alpha_a}(t_a)\Big\rangle.$$

• **Problem**. Classical (q-)Painlevé/Garnier systems appear at $c = 1(\epsilon_2 = -\epsilon_1)$ and $c = \infty(\epsilon_2 = 0)$. How do they related? ($\mathrm{PSL}(2,\mathbb{Z})$ duality of $W_{L,M,N}[-\frac{\epsilon_2}{\epsilon_1}]$ [Gaiotto-Rapčák (2017)], DIM algebra [Awata-Feigin-Shiraishi (2011)])

Special solutions by Padé method

▲ **Padé problems** (Approximation by a rational function):

(1) Padé approximation (differential):

$$\psi(x) = \frac{P_m(x)}{Q_n(x)} + O(x^{m+n+1}).$$

(2) Padé interpolation:

$$\psi(x) = \frac{P_m(x)}{Q_n(x)}. \quad (x = x_0, x_1, \ldots, x_{m+n})$$

▲ **Main idea.** The functions $P_m(x)$ and $\psi(x)Q_n(x)$ solve the Lax equations for IMD.

▲ **Example.** Padé approximation problem

$$\psi(x) := (1-x)^a(1-\frac{x}{t})^b = \frac{P_m(x)}{Q_n(x)} + O(x^{m+n+1})$$

→ **Special solution for P_{VI}**

$$q = \frac{t(m+n+1)}{(m-n-a-b)} \frac{\tau_{m,n}\tau_{m+1,n+1}}{\tau_{m+1,n}\tau_{m,n+1}},$$

$$\tau_{m,n} = \det\left(p_{m-i+j}\right)_{i,j=1}^{n}, \quad \psi(x) = \sum_{k=0}^{\infty} p_k x^k,$$

associated with the Riemann data:

x	0	1	t	∞	q
exp.	0	0	0	$-m$	0
	$m+n+1$	a	b	$-n-a-b$	2

.

▲ **Some generalizations.**

• $\psi(x) = \prod_{i=1}^{N} (1 - x/t_i)^{a_i}$

(Padé approx.) $\rightarrow$ Garnier system.

• $\psi(x) = \prod_{i=1}^{N} \frac{(xa_i; q)_\infty}{(xb_i; q)_\infty}, \quad (z; q)_\infty = \prod_{i=0}^{\infty} (1 - q^i z)$

(q-grid interpolation) $\rightarrow$ q-Garnier system.

• $\psi(x) = \prod_{i=1}^{N} \frac{\Gamma_{p,q}(xa_i)}{\Gamma_{p,q}(xk/a_i)}, \quad \Gamma_{p,q}(z) = \prod_{i,j=0}^{\infty} \frac{1 - z^{-1} p^{i+1} q^{j+1}}{1 - z p^i q^j}$

(elliptic grid interpolation) $\rightarrow$ elliptic Garnier system.