OPE inversion and large spin perturbation theory

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Outline

Importance of Conformal Field Theories ($d > 2$)

- Realistic symmetry present in many physical systems, (usually critical systems).
- Ubiquitous in dualities in string and gauge theory, such as AdS/CFT.
- Rich mathematical structures, particularly visible in planar theories. (non-transcendental behavior).

In general we would like to study CFT's in non-perturbative regimes (not more love from Feynman :( ).

Conformal Bootstrap: Constraining observables by imposing physical conditions.

- Conformal symmetry
- Crossing symmetry
- OPE expansion (“locality”)
- Unitarity
- Causality
- Global symmetries

In essence, same as S-matrix program 60 to 70’s: Veneziano amplitudes, Regge Theory, BFKL equation...
Applied successfully in the 80’s to 2d-CFT [Ferrara, Gato, Grillo, Belavin, Polyakov, Zamolodchikov]
Revisited and implemented in $d > 2$ CFT ’08 [Rattazzi, Rychkov, Tonni, Vichi].

This talk focus on 4-points $= \langle \prod_1^4 \mathcal{O}_i(x_i) \rangle$.

- Conformal symm.

$$\frac{1}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)}(x_{34}^2)^{\frac{1}{2}(\Delta_3+\Delta_4)}} \left(\frac{x_{14}^2}{x_{24}^2}\right)^a \left(\frac{x_{14}^2}{x_{13}^2}\right)^b \mathcal{G}(z,\bar{z})$$

- Crossing Symm. (Equal. Scalars $\rightarrow\ \Delta_i = \Delta$)

$$\left(\frac{1-z}{z}\right)^\Delta \mathcal{G}(z,\bar{z}) = \left(\frac{\bar{z}}{1-\bar{z}}\right)^\Delta \mathcal{G}(1-z,1-\bar{z})\,.$$

- OPE expansion

$$\mathcal{G}(z,\bar{z}) = \sum_{\Delta,J} c_{\Delta,J}\, G_{\Delta,J}(z,\bar{z})\,.$$

We will not impose (too heavily), just to keep the most possible generality,

- Unitarity

$$c_{\Delta,J} > 0, \;\; \Delta - J > d - 2\,.$$

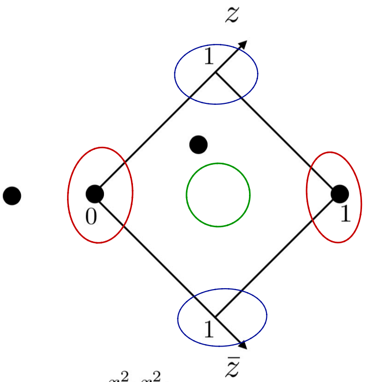
- Causality

$$\langle\prod_{1}^{4}\mathcal{O}_i(x_i)\rangle = 0 \quad x_{ij}^2 < 0 \text{ (Spacelike)}$$

- Global and additional symmetries.

$$\mathcal{F}\left(\langle\prod_{1}^{4}\mathcal{O}_i(x_i)\right)$$

$$\langle \mathcal{O}(0)\mathcal{O}(z,\bar{z})\mathcal{O}(1)\mathcal{O}(\infty)\rangle$$



$$u = z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$$

$$v = (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

- Analytic lightcone - Analytic euclidean - Numerics

Near to lightcone distances $z \to 0$, $G_{\Delta,J}(z,\bar{z}) \to z^{\frac{\tau}{2}}\mathcal{F}_{\Delta,J}(\bar{z})$, ($\tau = \Delta - J$, $\beta = \Delta + J$)

$$z^{-\Delta} + \sum_{\Delta,J} c_{\Delta,J} z^{\frac{\tau-2\Delta}{2}} \mathcal{F}_{\Delta,J}(\bar{z}) \sim z^{-\Delta} = \left(\frac{\bar{z}}{1-\bar{z}}\right)^{\Delta} \mathcal{G}(1-z, 1-\bar{z})\,.$$

such as left hand side is dominated by: $\mathcal{O}_I$. Subsequently taking it near $\bar{z} \to 1$,

$$\begin{aligned} z^{-\Delta} &= \sum_{\Delta,J} c_{\Delta,J}(1-\bar{z})^{\frac{\tau-2\Delta}{2}} \mathcal{F}_{\Delta,J}(1-z) \\ &= \sum_{\Delta,J} c_{\Delta,J}(1-\bar{z})^{\frac{\tau-2\Delta}{2}} \left(\frac{\Gamma(\beta)}{\Gamma(\beta/2)^2}\log(z) + \mathrm{finite}\right)\,. \end{aligned}$$

The leading power singularity on LHS can only come from an infinity sum from the RHS logs. In order to reproduce the LHS, the RHS sum is dominated by the regime $2\beta\sqrt{z} \sim 1$ and by $c_{\Delta,J} = c_{\Delta,J}^{MFT}$ [Zhiboedov, Komargodski, Fitzpatrick, Kaplan,Walters, Alday, Maldacena] which can be seen by a saddle point analysis.

Subleading contributions,

$$\begin{aligned} z^{-\Delta} + c_{\tau_m} z^{\frac{\tau_m - 2\Delta}{2}} \mathcal{F}_{\Delta,J}(\bar{z}) \quad &= z^{-\Delta} + c_{\tau_m} z^{\frac{\tau_m - 2\Delta}{2}} \Big(\log(1-\bar{z}) + \text{fin.} \Big) \\ &= \sum_{\Delta,J} c_{\Delta,J} (1-\bar{z})^{\frac{\tau - 2\Delta}{2}} \mathcal{F}_{\Delta,J}(1-z) \,. \end{aligned}$$

To reproduce the $\log(1-\bar{z})$ we need $\tau = 2\Delta + \gamma_\beta$, with $\gamma_\beta << 1$. This observation plus the dominance of the sum for large β, suggest to look for a series expansion for the anomalous dimension and OPE coefficients in $\frac{1}{\beta}$,

$$\gamma_\beta = \sum_{n=0}^{\infty} \frac{d_n}{\beta^n}, \qquad c_{\Delta,J} = \sum_{n=0}^{\infty} \frac{c_n}{\beta^n} \,.$$

Such perturbation theory has been developed by [Zhiboedov, Alday]: essentially putting in those series in the crossing eq. and solve order by order. We would like instead to study this problem from an alternative perspective.

Partial wave expansion,

$$A(s,t) = \sum_{\ell=0}^{\infty}(2\ell+1) f_\ell(t)\, G_\ell(\cos(\theta))\,,$$

G →Kinematics, f →Dynamics, therefore we want to know f_ℓ:

$$f_\ell(t) = \int_{-1}^{1} dz\, A(z,t)\, G_\ell(z)\,.$$

but

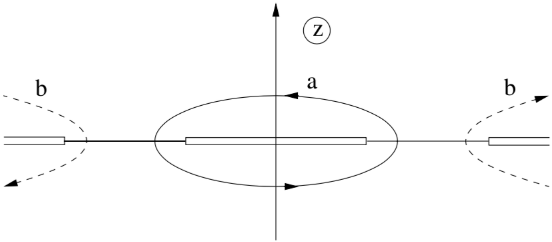
$$G_\ell(z) \sim z^\ell\,.$$

Happily, 2nd sol. Legendre eq.

$$Q_\ell(z+i\epsilon) - Q_\ell(z-i\epsilon) = -i\pi G_\ell(z),\ \&\ Q_\ell(|z|\to\infty) \sim z^{-\ell-1}\,.$$

Froissart-Gribov S-matrix partial wave inversion formula,

$$\begin{aligned} f_\ell(t) &= \oint_a dz\, A(z,t)\, Q_\ell(z) \\ &= \int_b dz\, \mathrm{Disc}[A(z,t)]\, Q_\ell(z)\,. \end{aligned}$$



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1 Taken from Gribov Lectures: Theory of Complex Angular Momenta.

[Simon Caron-Huot] generalized trick above to CFT. Greatly more involved, but the idea is the same. Starting from the partial wave expansion of conformal blocks,

$$\mathcal{G}(z,\bar{z}) = \mathbf{1} + \sum_{J=0} \int_{-i\infty}^{i\infty} \frac{d\Delta}{2\pi i} c_J(\Delta) F_J(z,\bar{z}|\Delta)\,.$$

$$F_J(z,\bar{z}|\Delta) = \frac{1}{2}\Big(K_{\Delta,J} G_{\Delta,J}(z,\bar{z}) + K_{d-\Delta,J} G_{d-\Delta,J}(z,\bar{z})\Big)\,.$$

Why the Analytic continuation?. Unlike $G_{\Delta,J}$, $F_J(\Delta)$ forms a complete orthogonal set, therefore,

$$c_J(\Delta) = N_J(\Delta) \int d^2z\, \mu(z,\bar{z}) F_J(z,\bar{z}|\Delta) \mathcal{G}(z,\bar{z})\,.$$

The function $F_J(z,\bar{z}|\Delta)$ blows up both at infinity and at zero. However, similarly as in the S-matrix, it can be written in terms of a function with the wanted behaviour $G_{J+d-1}(\Delta+1-d)(z,\bar{z})$ and after contour deformation,

$$c_J(\Delta) = \kappa_{\Delta+J} \int_0^1 d^2z \, \mu(z,\bar{z}) G_{J+d-1}(\Delta+1-d)(z,\bar{z}) \mathrm{dDisc}[\mathcal{G}(z,\bar{z})] \, .$$

at leading order in small z ($\beta = \Delta + J$),

$$c^t(J,\Delta) = \int_0^1 \frac{dz}{2z} z^{\frac{\tau}{2}}\, c^t(z,\beta)\,, \tag{1}$$

where the following "generating function" has been defined,

$$c^t(z,\beta) \equiv \int_z^1 \frac{d\bar{z}\,(1-\bar{z})^{a+b}}{\bar{z}^2} \kappa_\beta\, k_\beta(\bar{z}) \text{dDisc}[\mathcal{G}(z,\bar{z})]\,,$$

at small$-z$ one can expand $c^t(z,\beta) = \sum_k c_k\, z^{\tau_k}$, such as,

$$c^t(J,\Delta)\Big|_{\text{poles}} = F(J,\Delta) \int_0^1 \frac{dz}{2z}\, z^{\frac{\tau_k - \tau_0}{2}} = \frac{F(J,\Delta)}{\tau_k - \tau_0}\,.$$

Taking the leading order as

$$c^t(z,\beta)|_{J,\Delta} \sim c_0(\beta) z^{\frac{\tau_0}{2}+\frac{1}{2}\gamma_{12}(\beta)}\,.$$

If the anomalous dimension $\gamma_{12}(\beta)$ is small, then,

$$c^t(z,\beta)|_{J,\Delta} \sim z^{\frac{\tau_0}{2}}\, c_0(\beta)\left(1+\frac{1}{2}\gamma_{12}(\beta)\log(z)\right)\,,$$

$\gamma_{12}(\beta)$ are attached to the $\log(z)$ terms.

Contribution of a single block to the OPE of a large spin operator. Simplest in 4D

$$G_{J,\Delta}(1-z,1-\bar{z}) = \frac{(1-z)(1-\bar{z})}{z-\bar{z}} \quad \Big[k_{\Delta-J-2}(1-z)k_{\Delta+J}(1-\bar{z}) \\ k_{\Delta+J}(1-z)k_{\Delta-J-2}(1-\bar{z})\Big]\,.$$

whose leading log-term around $1-z <<$ is given by,

$$G_{\Delta,J}(1-z,1-\bar{z}) = \frac{1-\bar{z}}{\bar{z}}\log(z) \\ \times\left(\frac{\Gamma(\tau-2)}{\Gamma\left(\frac{\tau-2}{2}\right)^2}k_\beta(1-\bar{z}) - \frac{\Gamma(\beta)}{\Gamma\left(\frac{\beta}{2}\right)^2}k_{\tau-2}(1-\bar{z})\right) + \mathcal{O}(z\log z)\,.$$

$$\gamma_{12}^{J,\Delta}(\beta) = \frac{\Gamma\left(\frac{\beta}{2}\right)^2 \Gamma\left(\Delta_0\right)^2 \Gamma\left(\frac{\beta}{2} - \Delta_0 + 1\right)}{\Gamma(\beta)\Gamma\left(\frac{\beta}{2} + \Delta_0 - 1\right)\Gamma\left(\frac{\Delta-J}{2} - \Delta_0 + 1\right)^2 \Gamma\left(\Delta_0 - \frac{\Delta-J}{2}\right)^2}$$
$$\times \left(\frac{\Gamma(\Delta - J - 2)}{\Gamma\left(\frac{\Delta-J-2}{2}\right)^2}\Omega_{\beta,\Delta+J,\Delta_0-1} - \frac{\Gamma(\Delta+J)}{\Gamma\left(\frac{\Delta+J}{2}\right)^2}\Omega_{\beta,\Delta-J-2,\Delta_0-1}\right).$$

$$\begin{aligned}\Omega_{h,h',p} &= \frac{\Gamma(2h)\Gamma(h'-p+1)^2\Gamma(-h'+h+p-1)}{\Gamma(h)^2\Gamma(h'+h-p+1)} \\ &\quad \times {}_4F_3\left[\begin{matrix} h', h', h'-p+1, h'-p+1 \\ 2h', h'+h-p+1, h'-h-p+2 \end{matrix}; 1\right] \\ &\quad + \frac{\Gamma(2h')\Gamma(h+p-1)^2\Gamma(h'-h-p+1)}{\Gamma(h')^2\Gamma(h'+h-p-1)} \\ &\quad \times {}_4F_3\left[\begin{matrix} h, h, h+p+1, h+p+1 \\ 2h, h'+h+p-1, -h'+h+p \end{matrix}; 1\right],\end{aligned}$$

Unfortunately, in general dimensions life is not that easy. Not closed form for the blocks so far. Best we can do (for simplicitly $z \to 0$) :

$$g_{J,\Delta}(y) = y^{\frac{\Delta-J}{2}} \sum_{k=0} g_k(J,\Delta) y^k \,, \qquad \frac{1-\bar{z}}{\bar{z}} \equiv y$$

Conformal casimir(s)$\to$

$$p_{k-1}(\Delta,J)\, g_{k-1}(J,\Delta) + p_{k-2}(J,\Delta)\, g_{k-2}(J,\Delta) + p_k(J,\Delta)\, g_k(J,\Delta) = 0 \,,$$

with for example

$$\begin{aligned} p_{k-1}(J,\Delta) &= -2\left(d - 2(k+\frac{\tau}{2})\right)^2 \\ \times &\left(d(-\Delta + J + 4(k+\frac{\tau}{2}-1)) + \Delta^2 + (J-2)J - 4(k+\frac{\tau}{2})^2 + 4\right) . \end{aligned}$$

Solving order by order for the blocks, allow us to compute order by order on the anomalous dimension,

$$\gamma^{J,\Delta}(\beta) = \frac{f_{11(J,\Delta)} f_{22(J,\Delta)}}{2\, I^{(0,0)}_{-2\Delta_0}} \kappa_\beta \sum_{k=0} c_k(J, \Delta, \beta)\,,$$

at large β we notice we can write,

$$\frac{c_k(\Delta, J, \beta)}{c_0(\Delta, J, \beta)} \sim g_k(\Delta, J) \left[\left(1 - \frac{J - \Delta + 2\Delta_0}{2}\right)_k\right]^2 \left(\frac{2}{\beta}\right)^{2k} .$$

which matches large spin perturbation theory [Zhiboedov, Alday]

Mellin space is agnostic to space-time dim.

$$G_{J,\Delta}(u,v) = \int dsdt\ \mathcal{M}_{\Delta,J}(s,t)\, u^s v^t\,,$$

$$\begin{aligned}\mathcal{M}_{\Delta,J}(s,t) \quad &= \frac{1}{\gamma_{\lambda_1,a}\gamma_{\bar{\lambda_1},b}}\Gamma(\lambda_2-s)\Gamma(\bar{\lambda}_2-s)\Gamma(-t)\Gamma(-t-a-b)\\ &\quad\times\Gamma(s+t+a)\Gamma(s+t+b)\mathcal{P}_{J,\Delta}(s,t,a,b)\end{aligned}$$

Dual gravity S-Matrix.

$$\gamma_{12}^{J,\Delta}(\beta) = \int_0^1 d\bar{z}\frac{(1-\bar{z})^{2a}}{\bar{z}^2}\kappa_\beta k_\beta(\bar{z})\ \mathrm{dDisc}\left[\lim_{\bar{z}\to 0}\mathcal{G}^t_{J,\Delta}(z,\bar{z})\Big|_{\log z}\right]$$

$$= \frac{\Gamma(\Delta+J)\Gamma(1+\Delta-h)}{(d-2)_J\Gamma(\frac{\Delta+J}{2})^4\Gamma(1-h+\frac{\Delta+J}{2})}\sum_{m=0}^{J}\frac{(-1)^{J-m}A_m(J,\Delta)}{\Gamma(1+m-\frac{\Delta+J}{2})}$$

$$\int ds\ (-1)^s\frac{\Gamma(-s)\Gamma(s+\frac{\Delta-J}{2})^2\Gamma(1+s)\Gamma(1-s-\frac{\Delta-J}{2})}{\Gamma(1+s+\Delta-h)\Gamma(1+s+m-J)}$$

$$\Gamma(1-h+m+s+\frac{\Delta-J}{2})I^{(a,a)}_{\Delta-J+\tau'+2s}(\beta)\,,$$

$$\gamma_{12}^{J,\Delta}(\beta)\sim\sum_{m=0}^{J}\mathcal{N}_m(\Delta,J)\,{}_5F_4\left[\begin{matrix}1,b,c,d,f\\ g,h,i,j\end{matrix};1\right].$$

By using large$-j$ PT [Zhiboedov, Alday] $d = 3,\ \Delta_\epsilon = 1$,

$$\gamma_{12} = -\frac{c_0}{j}\left(1 + \sum_{k=1}^{\infty} \frac{c_k}{j^{2k}}\right),$$

where the coefficients,

$$c_k = -\left(\frac{1}{4}\right)^k \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k+1)}.$$

Plugging $d = 3,\ \Delta_\epsilon = 1$ into ours,

$$\gamma_{12}(\beta) = -2f_0^2\,\frac{\Gamma(\Delta)^2\Gamma\left(\frac{\beta}{2} - \Delta + 1\right)\Gamma\left(\frac{\beta-3}{2} + \Delta\right)}{2\pi^2\Gamma\left(\Delta - \frac{1}{2}\right)^2\Gamma\left(\frac{1}{2}(\beta - 2\Delta + 3)\right)\Gamma\left(\frac{\beta}{2} + \Delta - 1\right)} \tag{1}$$

By further set $\Delta = 1$ and replacing $\beta \to 1 - \sqrt{4j^2 + 1}$ we got,

$$\gamma_{12}(\beta) = \frac{2}{\sqrt{1 + 4j^2}\,\pi^3}\,, \tag{2}$$

By Taylor expand around large j,

$$\gamma_{12} = -\frac{c_0}{j}\left(1 + \sum_{k=1}^{\infty}\frac{c_k}{j^{2k}}\right)\,, \tag{3}$$

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Summary and perspectives

- S-matrix bootstrap techniques translate in a, perhaps non-obvious way, but a natural way to the conformal bootstrap.
- In particular Caron-Huot Inversion Formula works as the CFT equivalent of Froissart-Gribov formula in S-matrix theory.
- As a relevant application we have applied it to the computation of anomalous dimensions of large spin operators.
- We have obtained resumed expression in spin that reproduce the results from large spin perturbative expansion

Thank you all!